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On the degenerate higher order Frobenius-Euler polynomials

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Abstract

Degenerate versions of special numbers, originating from the work of L. Carlitz, play a crucial role in various fields, including pure and applied mathematics, com- binatorics, number theory, and mathematical physics. They are actively under investigation by numerous researchers. Recently, [D. S. Kim, T. Kim, J. Math. Anal. Appl., **493** (2021), 21 pages] introduced the λ-umbral calculus as a research tool specifically for degenerate special polynomials, utilizing it to establish connections between special polynomials and their degenerate counterparts.

In this paper, we investigate the relationships between degenerate Frobenius-Euler polynomials and other versions of degenerate special polynomials and numbers. By employing λ-umbral calculus, explicit formulas for Frobenius-Euler polynomials of order r are derived. The presented formulas reveal connections between these polynomials and well-known special numbers and polynomials. Additionally, the distribution patterns of the roots of these polynomials are examined.

Keywords: Degenerate Frobenius-Euler polynomials, umbral calculus, λ-analogue of the Stirling numbers of the first kind, λ-analogue of the Stirling numbers of the second kind, degenerate polyexponential function.

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1. Introduction

For a $\lambda \in \mathbb{R} - \{0\}$, the *degenerate exponential function*, introduced by Carlitz, is defined as follows (see e.g., [4, 11, 13, 14, 19, 20, 22]):

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \tag{1.1}$$

where $\mathbb{R}$ is the real numbers filed. For the special case $x = 1$, $e_\lambda^1(t)$ is denoted by $e_\lambda(t)$.

In [4], Carlitz defined a degenerate exponential function and attempted to generalize Bernoulli and Eulerian numbers using this function. Since then, studies on many degenerate versions of special functions have been defined actively and investigated properties of these functions. Aydin, Acikgoz and Araci [2] defined the degenerate Hurwitz-zeta, modified degenerate Hurwitz-zeta and degenerate digamma functions, and derived some interesting identities for these functions. Kim and Kim [8] generalized the gamma function and Laplace transform which are called degenerate gamma function and degenerate

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Laplace transform, and gave the degenerate sine, cosine, hyperbolic sine and hyperbolic cosine functions. Kim and Kim [11] introduced degenerate polyexponential functions as a de- generate version of the polyexponential functions initially defined by Hardy [5, 6]. In their work, they explored and established various intriguing properties associated with these degenerate polyexponential functions.

For given integers n, k with $n \geqslant k \geqslant 0$, the *Stirling numbers of the first kind* $S_1(n,k)$ and the *second kind* $S_2(n,k)$, respectively, are defined as follows (see, e.g., [15, 21, 23]):

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1.2}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geqslant 1)$.

As a generalization of the Stirling numbers of the first and the second kind, the *λ-analogue of the Stirling numbers of the first kind* and *the second kind* are defined as follows (see [21]):

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_\lambda^{(1)}(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_\lambda^{(2)}(n,k)(x)_{k,\lambda}, \tag{1.3}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geqslant 1)$, which are called the degenerate falling sequences. By (1.3), we see the generating functions of the these numbers as follows (see [12, 21]):

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k = \sum_{n=k}^{\infty} S_\lambda^{(1)}(n,k)\frac{t^n}{n!} \quad \text{and} \quad \frac{1}{k!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^k = \sum_{n=k}^{\infty} S_\lambda^{(2)}(n,k)\frac{t^n}{n!}, \tag{1.4}$$

As degenerate version of the Stirling numbers of the first and the second kind which are another generalization of these numbers, the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *second kind* $S_{2,\lambda}(n,k)$, respectively, are defined by Kim and Kim (see [12, 18]) as

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad \text{and} \quad \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.5}$$

where $\log_\lambda(t)$ is the compositional inverse of $e_\lambda(t)$ with $e_\lambda\left(\log_\lambda(t)\right) = \log_\lambda(e_\lambda(t)) = t$. Note that, by Newton's binomial expansion,

$$\log_\lambda(1+t) = \frac{1}{\lambda}\left((1+t)^\lambda - 1\right) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!}. \tag{1.6}$$

For given positive integer r, the *higher order Frobenius-Euler polynomials* are defined by the generating function to be as follows (see, e.g., [1, 3, 9]):

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\frac{t^n}{n!},$$

where $u \in \mathbb{C} - \{1\}$. When $x = 0$, $H_n^{(r)}(u) = H_n^{(r)}(0|u)$ are called the *Frobenius-Euler numbers*. The Frobenius-Eler polynomials have good applications related to probabilistic theory and normal ordering shift algebra (see [16, 17]).

From now on, we introduce the umbral calculus which are one of the useful tools for special functions. Let $\mathbb{C}$ be the complex numbers field

$$\mathcal{F} = \left\{ \sum_{n=0}^{\infty} a_k \frac{t^n}{n!} \middle| a_k \in \mathbb{C} \right\},$$

and let
$$\mathbb{P} = \left\{ \sum_{k=0}^{\infty} a_l x^k \middle| a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}.$$
Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Then the linear functional $\langle g(t)| \cdot \rangle$ on $\mathbb{P}$ given by $g(t)$, is defined as follows (see [24]):
$$\langle g(t)| x^n \rangle = a_n. \tag{1.7}$$
From (1.7), we get
$$\left\langle t^k \middle| x^n \right\rangle = n!\delta_{n,k}, \ (n, k \geqslant 0), \tag{1.8}$$
where $\delta_{n,k}$ is Kronecker's symbol. For a nonnegative integer k, the differential operator on $\mathbb{P}$ is defined by
$$\left(t^k\right) x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leqslant n, \\ 0, & \text{if } k > n, \end{cases}$$
and thus if $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, then
$$(f(t))\, x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k} \ \text{(see [24])}. \tag{1.9}$$
Furthermore, by (1.7), (1.8), and (1.9), we see that
$$\langle f(t)g(t)| p(x)\rangle = \langle g(t)| (f(t))p(x)\rangle = \langle f(t)| (g(t))p(x)\rangle,$$
for $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$, and
$$\left\langle e^{yt} \middle| p(x)\right\rangle = p(y) \quad \text{and} \quad \left\langle e^{yt} - 1 \middle| p(x)\right\rangle = p(y) - p(0).$$
The *order* $o(f(t))$ of $f(t) \in \mathcal{F} - \{0\}$ is the smallest positive integer k for which the coefficient of t^k is not 0. If $o(f(t)) = 0$, then $f(t)$ is called *invertible* and has the multiplicative inverse $\frac{1}{f(t)}$ of $f(t)$. $f(t)$ is called *delta series* if $o(f(t)) = 1$, and that series has the compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

Let $g(t)$ be an invertible series and let $f(t)$ be a delta series. Then there is the unique sequence $S_n(x)$ of polynomials with $\deg S_n(x) = n$ with
$$\left\langle g(t)\left(f(t)\right)^k \middle| S_n(x) \right\rangle = n!\delta_{n,k}, \quad (n, k \geqslant 0) \ \text{(see [24])},$$
and thus $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and is denoted by $S_n(x) \sim (g(t), f(t))$.

It is well-known fact that the sequence $S_n(x)$ is the Sheffer sequence for $(g(t), f(t))$ if and only if
$$\frac{1}{g\left(\bar{f}(t)\right)} e^{y\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(y) \frac{t^n}{n!} \ \text{(see [24])},$$
for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$. Let $S_n(x) \sim (g(t), f(t))$ and let $h(x) = \sum_{l=0}^{n} a_l S_l(x) \in \mathbb{P}$. Then
$$a_k = \frac{1}{k!} \left\langle g(t)\,(f(t))^k \middle| h(x) \right\rangle. \tag{1.10}$$
In addition, if $s_n \sim (g(t), f(t))$ and $r_n \sim (h(t), l(t))$, then
$$s_n = \sum_{k=0}^{n} c_{n,k} r_k,$$

where

$$c_{n,k}=\frac{1}{k!}\left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)}\left(l\left(\bar{f}(t)\right)\right)^{k}\middle| x^{n}\right\rangle. \tag{1.11}$$

In this paper, we utilize umbral calculus to find relationships between Frobenius-Euler polynomials and other special polynomials, especially Bernoulli polynomials, Bernoulli polynomials of the second kind, Euler polynomials, Daejee polynomials, Bellpolynomials, Changhee polynomials, Mittag-Leffer polynomials, and Lah-Bell polynomials by using umbral calculus. In addition, we investigate the pattern of the root distribution of these polynomials.

2. Representations of the degenerate Frobenius-Euler polynomials of order r to some special polynomials

In viewpoint of (1.1) and the definition of the higher order Frobenius-Euler polynomials, the *degenerate Frobebnius-Euler polynomials of order* r are defined by the generating function to be as follows (see, e.g., [9, 13]):

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^{r}e_\lambda^{x}(t)=\sum_{n=0}^{\infty}H_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $H_{n,\lambda}^{(r)}(u)=H_{n,\lambda}^{(r)}(0|u)$ are called the *degenerate Frobnius-Euler numbers of order* r. By (1.3) and (2.1), we note that

$$\begin{aligned}\left(\frac{1-u}{e_\lambda(t)-u}\right)^{r}e_\lambda^{x}(t)&=\left(\sum_{n=0}^{\infty}H_{n,\lambda}^{(r)}(u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(r)}(u)(x)_{m,\lambda}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_\lambda^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)x^k\right)\frac{t^n}{n!},\end{aligned}$$

and thus we see that

$$H_{n,\lambda}^{(r)}(x|u)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_\lambda^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)x^k. \tag{2.2}$$

In addition,

$$\left(\frac{e^t-u}{1-u}\right)^{r}=\frac{1}{(1-u)^r}\left(e^t-u\right)^{r}=\sum_{a=0}^{r}\binom{r}{a}\frac{(-u)^{r-a}}{(1-u)^r}e^{at}. \tag{2.3}$$

The *Bernoulli polynomials* are defined by the generating function to be as follows (see e.g., [7, 19]):

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{2.4}$$

When $x=0$, $B_n=B_n(0)$ are the *Bernoulli numbers*. By (2.1) and (2.4), we see the Sheffer sequences of the Bernoulli polynomials and the degenerate Frobenius-Euler polynomials are

$$B_n(x)\sim\left(\frac{e^t-1}{t},t\right)\quad\text{and}\quad H_{n,\lambda}^{(r)}(x|u)\sim\left(\left(\frac{e^t-u}{1-u}\right)^{r},\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right). \tag{2.5}$$

Let $H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}a_{n,l}B_l(x)$. By (1.10), (2.2), and (2.5), we get

$$a_{n,l}=\frac{1}{l!}\left\langle \frac{e^t-1}{t}t^l\middle| H_{n,\lambda}^{(r)}(x|u)\right\rangle$$

$$
\begin{aligned}
&= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u) \frac{1}{l!}\left\langle \frac{e^t-1}{t}t^l \middle| x^k \right\rangle \\
&= \sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l} S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u) \left\langle \frac{e^t-1}{t} \middle| x^{k-l} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l} \frac{S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u)}{k-l+1} \left\langle e^t-1 \middle| x^{k-l+1} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l} \frac{S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u)}{k-l+1}.
\end{aligned} \tag{2.6}
$$

In addition, since

$$
\frac{e_\lambda(t)-1}{t} = \frac{1}{t}\sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{(1)_{n+1,\lambda}}{n+1}\frac{t^n}{n!}, \tag{2.7}
$$

by (1.4), (1.11), and (2.7), we have

$$
\begin{aligned}
a_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)-1}{\frac{1}{\lambda}\log(1+\lambda t)}}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^r}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l \middle| x^n \right\rangle \\
&= \sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(1)}(m,l)\left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \frac{\lambda t}{\log(1+\lambda t)}\frac{e_\lambda(t)-1}{t} \middle| x^{n-m} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(1)}(m,l) H_{a,\lambda}^{(r)}(u)\left\langle \frac{\lambda t}{\log(1+\lambda t)}\frac{e_\lambda(t)-1}{t} \middle| x^{n-m-a} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b} S_\lambda^{(1)}(m,l) H_{a,\lambda}^{(r)}(u)\lambda^b\beta_b\left\langle \frac{e_\lambda(t)-1}{t} \middle| x^{n-m-a-b} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b} \frac{(1)_{n-m-a-b+1,\lambda} S_\lambda^{(1)}(m,l) H_{a,\lambda}^{(r)}(u)\lambda^b\beta_b}{n-m-a-b+1},
\end{aligned} \tag{2.8}
$$

where β_n are the Bernoulli numbers of the second kind, which are defined by the generating function to be

$$
\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty}\beta_n\frac{t^n}{n!}.
$$

Conversely, assume that $B_n(x) = \sum_{l=0}^{n} b_{n,l} H_{l,\lambda}^{(r)}(x|u)$. By (1.4), (1.11), and (2.3), we get

$$
\begin{aligned}
b_{n,l} &= \frac{1}{l!}\left\langle \frac{\left(\frac{e^t-u}{1-u}\right)^r}{\frac{e^t-1}{t}}\left(\frac{e^{\lambda t}-1}{t}\right)^l \middle| x^n \right\rangle \\
&= \sum_{m=l}^{n}\binom{n}{m} S_\lambda^{(2)}(m,l)\left\langle \frac{t}{e^t-1}\left(\frac{e^t-u}{1-u}\right)^r \middle| x^{n-m} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} S_\lambda^{(2)}(m,l) B_a\left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| x^{n-m-a} \right\rangle \\
&= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{r}\binom{n}{m}\binom{n-m}{a}\binom{r}{b} \frac{(-u)^{r-b} S_\lambda^{(2)}(m,l) b^{n-m-a} B_a}{(1-u)^r}.
\end{aligned} \tag{2.9}
$$

By (2.6), (2.8), and (2.9), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer* n*, we have*

$$H_{n,\lambda}^{(r)}(x|u) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}\frac{S_\lambda^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)}{k-l+1}\right)B_l(x)$$
$$= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}\right.$$
$$\left.\times\frac{(1)_{n-m-a-b+1,\lambda}\lambda^b S_\lambda^{(2)}(m,l)H_{a,\lambda}^{(r)}(u)\beta_b}{n-m-a-b+1}\right)B_l(x),$$

and

$$B_n(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{r}\binom{n}{m}\binom{n-m}{a}\binom{r}{b}\frac{(-u)^{r-b}S_\lambda^{(2)}(m,l)b^{n-m-a}B_a}{(1-u)^r}\right)H_{l,\lambda}^{(r)}(x|u).$$

For a given positive integer s, the *Euler polynomials of order* s are defined by the generating function to be as follows (see, e.g., [9, 13]):

$$\left(\frac{2}{e^t+1}\right)^s e^{xt} = \sum_{n=0}^{\infty}E_n^{(s)}(x)\frac{t^n}{n!}. \tag{2.10}$$

In particular, if $s = 1$, then $E_n(x) = E_n^{(1)}(x)$ are called the *Euler polynomials*, and $E_n = E_n(0)$ are the *Euler numbers*. By (2.10), we see that

$$E_n(x) \sim \left(\frac{e^t+1}{2}, t\right).$$

Let $H_{n,\lambda}^{(r)}(x|u) = \sum_{l=0}^{n} a_{n,l}E_l(x)$. Since

$$\frac{e_\lambda(t)+1}{2} = 1 + \frac{1}{2}\sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}, \tag{2.11}$$

by (1.4), (1.11), and (2.11), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^r}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l \middle| x^n\right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m}S_\lambda^{(1)}(m,l)\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^r\frac{e_\lambda(t)+1}{2}\middle| x^{n-m}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_\lambda^{(1)}(m,l)H_{a,\lambda}^{(r)}(u)\left\langle\frac{e_\lambda(t)+1}{2}\middle| x^{n-m-a}\right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m}S_\lambda^{(1)}(m,l)H_{n-m,\lambda}^{(r)}(u) + \frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a}S_\lambda^{(1)}(m,l)H_{a,\lambda}^{(r)}(u)(1)_{n-m-a,\lambda}. \end{aligned} \tag{2.12}$$

In addition, by (2.2), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle\frac{e^t+1}{2}t^l\middle| H_{n,\lambda}^{(r)}(x|u)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_\lambda^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)\frac{1}{l!}\left\langle\frac{e^t+1}{2}t^l\middle| x^k\right\rangle \end{aligned} \tag{2.13}$$

$$= \sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)\left\langle \frac{e^t+1}{2}\middle| x^{k-l}\right\rangle$$
$$= \sum_{m=l}^{n}\binom{n}{m}S_{\lambda}^{(1)}(m,l)H_{n-m,\lambda}^{(r)}(u)+\frac{1}{2}\sum_{m=l+1}^{n}\sum_{k=l+1}^{m}\binom{n}{m}\binom{k}{l}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u).$$

Conversely, assume that $E_n(x)=\sum_{l=0}^{n} b_{n,l}H_{l,\lambda}^{(r)}(x|u)$. Then, by (2.3), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{\left(\frac{e^t-u}{1-u}\right)^r}{\frac{e^t+1}{2}}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\middle| x^n\right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m}S_{\lambda}^{(2)}(m,l)\left\langle \frac{2}{e^t+1}\left(\frac{e^t-u}{1-u}\right)^r\middle| x^{n-m}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{\lambda}^{(2)}(m,l)E_a\left\langle \left(\frac{e^t-u}{1-u}\right)^r\middle| x^{n-m-a}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{r}\binom{n}{m}\binom{n-m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{n-m-a}S_{\lambda}^{(2)}(m,l)E_a}{(1-u)^r}. \end{aligned} \tag{2.14}$$

By (2.12), (2.13), and (2.14), we obtain the following theorem.

Theorem 2.2. *For each nonnegative integer n, we have*

$$\begin{aligned} &H_{n,\lambda}^{(r)}(x|u) \\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\binom{n}{m}S_{\lambda}^{(1)}(m,l)H_{n-m,\lambda}^{(r)}(u)+\frac{1}{2}\sum_{m=l}^{n}\sum_{a=0}^{n-m-1}\binom{n}{m}\binom{n-m}{a}S_{\lambda}^{(1)}(m,l)H_{a,\lambda}^{(r)}(u)(1)_{n-m-a,\lambda}\right)E_l(x) \\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\binom{n}{m}S_{\lambda}^{(1)}(m,l)H_{n-m,\lambda}^{(r)}(u)+\frac{1}{2}\sum_{m=l+1}^{n}\sum_{k=l+1}^{m}\binom{n}{m}\binom{k}{l}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)\right)E_l(x), \end{aligned}$$

and

$$E_n(x)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{r}\binom{n}{m}\binom{n-m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{n-m-a}S_{\lambda}^{(2)}(m,l)E_a}{(1-u)^r}\right)H_{l,\lambda}^{(r)}(x|u).$$

The *Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}. \tag{2.15}$$

When $x=0$, $\beta_n=\beta_n(0)$ are called the *Bernoulli numbers of the second kind*. By (2.15), we see that

$$\beta_n(x)\sim\left(\frac{t}{e^t-1},e^t-1\right). \tag{2.16}$$

Let $H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}a_{n,l}\beta_l(x)$. Then, by (1.5), (1.11), and (2.16), we get

$$a_{n,l}=\frac{1}{l!}\left\langle \frac{\frac{\frac{\log(1+\lambda t)}{\lambda}}{e_{\lambda}(t)-1}}{\left(\frac{e_{\lambda}(t)-u}{1-u}\right)^r}(e_{\lambda}(t)-1)^l\middle| x^n\right\rangle$$

$$
\begin{aligned}
&=\sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^r\frac{t}{e_\lambda(t)-1}\frac{\log(1+\lambda t)}{\lambda t}\middle|x^{n-m}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_{2,\lambda}(m,l)B_{a,\lambda}\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^r\frac{\log(2+\lambda t)}{\lambda t}\middle|x^{n-m-a}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}S_{2,\lambda}(m,l)B_{a,\lambda}\lambda^bD_b\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^r\middle|x^{n-m-a-b}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}S_{2,\lambda}(m,l)\lambda^bB_{a,\lambda}D_bH^{(r)}_{n-m-a-b,\lambda}(u),
\end{aligned}
\tag{2.17}
$$

where $B_{n,\lambda}$ and D_n are the degenerate Bernoulli numbers and the Daehee numbers, respectively, which are defined as follows:

$$
\frac{t}{e_\lambda(t)-1}=\sum_{n=0}^{\infty}B_{n,\lambda}\frac{t^n}{n!}\quad\text{and}\quad\frac{\log(1+t)}{t}=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}.
$$

In addition, by (2.2) and (1.10), we get

$$
\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle\frac{t}{e^t-1}\left(e^t-1\right)^l\middle|H^{(r)}_{n,\lambda}(x|u)\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S^{(1)}_\lambda(m,k)H^{(r)}_{n-m,\lambda}(u)\left\langle\frac{t}{e^t-1}\left(e^t-1\right)^l\middle|x^k\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}S^{(1)}_\lambda(m,k)S_2(a,l)H^{(r)}_{n-m,\lambda}(u)\left\langle\frac{t}{e^t-1}\middle|x^{k-a}\right\rangle\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}S^{(1)}_\lambda(m,k)S_2(a,l)H^{(r)}_{n-m,\lambda}(u)B_{k-a}.
\end{aligned}
\tag{2.18}
$$

Conversely, assume that $E_n(x)=\sum_{l=0}^{n}b_{n,l}H^{(r)}_{l,\lambda}(x|u)$. Then, since

$$
\beta_n(x)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m}x^m,
\tag{2.19}
$$

by (1.10), (2.3), and (2.19), we have

$$
\begin{aligned}
b_{n,l}&=\frac{1}{l!}\left\langle\left(\frac{e^t-u}{1-u}\right)^r\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\middle|\beta_n(x)\right\rangle\\
&=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m}\left\langle\left(\frac{e^t-u}{1-u}\right)^r\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\middle|x^m\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=l}^{m}\binom{n}{m}\binom{m}{a}S^{(2)}_\lambda(a,l)\beta_{n-m}\left\langle\left(\frac{e^t-u}{1-u}\right)^r\middle|x^{m-a}\right\rangle\\
&=\sum_{m=l}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{n}{m}\binom{m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{m-a}S^{(2)}_\lambda(a,l)\beta_{n-m}}{(1-u)^r}.
\end{aligned}
\tag{2.20}
$$

By (2.17), (2.18), and (2.20), we obtain the following theorem.

Theorem 2.3. *For each nonnegative integer* n*, we have*

$$H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}S_{2,\lambda}(m,l)\lambda^{b}B_{a,\lambda}D_{b}H_{n-m-a-b,\lambda}^{(r)}(u)\right)\beta_{l}(x)$$
$$=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}S_{\lambda}^{(1)}(m,k)S_{2}(a,l)H_{n-m,\lambda}^{(r)}(u)B_{k-a}\right)\beta_{l}(x),$$

and

$$\beta_{n}(x)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{n}{m}\binom{m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{m-a}S_{\lambda}^{(2)}(a,l)\beta_{n-m}}{(1-u)^{r}}\right)H_{l,\lambda}^{(r)}(x|u).$$

The *Daehee polynomials* are defined as follows by using generating function to be:

$$\frac{\log(1+t)}{t}(1+t)^{x}=\sum_{n=0}^{\infty}D_{n}(x)\frac{t^{n}}{n!}.$$

When $x=0$, $D_n=D_n(0)$ are called the *Daehee numbers*. Note that, by the similar way to (2.2),

$$D_{n}(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}D_{n-m}S_{1}(m,k)x^{k}, \tag{2.21}$$

and the Sheffer sequence of the these polynomials is

$$D_{n}(x)\sim\left(\frac{e^{t}-1}{t},e^{t}-1\right). \tag{2.22}$$

Let $H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}a_{n,l}D_{l}(x)$. By (1.11) and (2.22), we get

$$\begin{aligned}
a_{n,l}&=\frac{1}{l!}\left\langle\frac{\frac{e_{\lambda}(t)-1}{\frac{1}{\lambda}\log(1+\lambda t)}}{\left(\frac{e_{\lambda}(t)-u}{1-u}\right)^{r}}(e_{\lambda}(t)-1)^{l}\middle|x^{n}\right\rangle\\
&=\frac{1}{l!}\left\langle\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}\frac{\lambda t}{\log(1+\lambda t)}\frac{1}{t}(e_{\lambda}(t)-1)^{l+1}\middle|x^{n}\right\rangle\\
&=(l+1)\sum_{m=l+1}^{n}\binom{n}{m}S_{2,\lambda}(m,l+1)\left\langle\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}\frac{\lambda t}{\log(1+\lambda t)}\frac{1}{t}\middle|x^{n-m}\right\rangle\\
&=\sum_{m=l+1}^{n}\binom{n}{m}\frac{(l+1)S_{2,\lambda}(m,l+1)}{n-m+1}\left\langle\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}\frac{\lambda t}{\log(1+\lambda t)}\middle|x^{n-m+1}\right\rangle\\
&=\sum_{m=l+1}^{n}\sum_{a=0}^{n-m+1}\binom{n}{m}\binom{n-m+1}{a}\frac{(l+1)\lambda^{a}S_{2,\lambda}(m,l+1)\beta_{a}}{n-m+1}\left\langle\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}\middle|x^{n-m-a+1}\right\rangle\\
&=\sum_{m=l+1}^{n}\sum_{a=0}^{n-m+1}\binom{n}{m}\binom{n-m+1}{a}\frac{(l+1)\lambda^{a}S_{2,\lambda}(m,l+1)\beta_{a}H_{n-m-a+1,\lambda}^{(r)}(u)}{n-m+1}.
\end{aligned}\tag{2.23}$$

In addition, by (1.10) and (2.2), we have

$$a_{n,l}=\frac{1}{l!}\left\langle\frac{e^{t}-1}{t}(e^{t}-1)^{l}\middle|H_{n,\lambda}^{(r)}(x|u)\right\rangle$$
$$=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)\frac{1}{l!}\left\langle\frac{(e^{t}-1)^{l+1}}{t}\middle|x^{k}\right\rangle \tag{2.24}$$

$$= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)}{k+1}\left\langle \frac{1}{(l+1)!}\left(e^{t}-1\right)^{l+1}\middle| x^{k+1}\right\rangle$$
$$= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)S_{\lambda}^{(1)}(m,k)S_2(k+1,l+1)H_{n-m,\lambda}^{(r)}(u)}{k+1}.$$

Conversely, assume that $D_n(x)=\sum_{l=0}^{n} b_{n,l}H_{l,\lambda}^{(r)}(x|u)$. By (1.10), (2.3), and (2.21), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^{t}-u}{1-u}\right)^{r}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^{l}\middle| D_n(x)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}D_{n-m}S_1(m,k)\frac{1}{l!}\left\langle \left(\frac{e^{t}-u}{1-u}\right)^{r}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^{l}\middle| x^{k}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}S_1(a,k)S_{\lambda}^{(1)}(a,l)D_{n-m}\left\langle \left(\frac{e^{t}-u}{1-u}\right)^{r}\middle| x^{k-a}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{r}\binom{n}{m}\binom{k}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{k-a}S_1(a,k)S_{\lambda}^{(1)}(a,l)D_{n-m}}{(1-u)^{r}}. \end{aligned} \tag{2.25}$$

By (2.23), (2.24), and (2.25), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n, we have*

$$\begin{aligned} H_{n,\lambda}(x|u) &= \sum_{l=0}^{n}\left(\sum_{m=l+1}^{n}\sum_{a=0}^{n-m+1}\binom{n}{m}\binom{n-m+1}{a}\frac{(l+1)\lambda^{a}S_{2,\lambda}(m,l+1)\beta_a H_{n-m-a+1,\lambda}^{(r)}(u)}{n-m+1}\right)D_l(x) \\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\frac{(l+1)S_{\lambda}^{(1)}(m,k)S_2(k+1,l+1)H_{n-m,\lambda}^{(r)}(u)}{k+1}\right)D_l(x), \end{aligned}$$

and

$$D_n(x)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{r}\binom{n}{m}\binom{k}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{k-a}S_1(a,k)S_{\lambda}^{(1)}(a,l)D_{n-m}}{(1-u)^{r}}\right)H_{l,\lambda}^{(r)}(x|u).$$

The *Bell polynomials* are defined by the generating function to be as follows (see [11]):

$$e^{x(e^{t}-1)}=\sum_{n=0}^{\infty}\mathrm{Bel}_n(x)\frac{t^{n}}{n!}. \tag{2.26}$$

When $x=1$, $\mathrm{Bel}_n=\mathrm{Bel}_n(1)$ are called the *Bell numbers*. By (1.2) and (2.26), we see that

$$\mathrm{Bel}_n(x)\sim(1,\log(1+t)) \text{ and } \mathrm{Bel}_n(x)=\sum_{m=0}^{n}S_2(n,m)x^{m}.$$

Let $H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}a_{n,l}\mathrm{Bel}_l(x)$. Since

$$\begin{aligned} \frac{1}{l!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^{l} &= \sum_{m=l}^{\infty}S_1(m,l)\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m} \\ &= \sum_{m=l}^{\infty}S_1(m,l)\sum_{r=m}^{\infty}S_{\lambda}^{(1)}(r,m)\frac{t^{r}}{r!}=\sum_{a=l}^{\infty}\sum_{m=l}^{a}S_1(m,l)S_{\lambda}^{(1)}(a,m)\frac{t^{a}}{a!}, \end{aligned} \tag{2.27}$$

by (1.11) and (2.27), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^r}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \middle| \left(\frac{1}{l!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l\right) x^n \right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S_\lambda^{(1)}(a,m)\left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \middle| x^{n-a}\right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S_\lambda^{(1)}(a,m)H_{n-a,\lambda}^{(r)}(u). \end{aligned} \tag{2.28}$$

Conversely, assume that $\mathrm{Bel}_n(x) = \sum_{l=0}^{n} b_{n,l}H_{l,\lambda}^{(r)}(x|u)$. Since

$$\begin{aligned} \frac{1}{(1-u)^r}\sum_{a=0}^{r}\binom{r}{a}(-u)^{r-a}e^{a(e^t-1)} &= \sum_{a=0}^{r}\binom{r}{a}\frac{(-u)^{r-a}}{(1-u)^r}\sum_{b=0}^{\infty}a^b\frac{1}{b!}(e^t-1)^b \\ &= \sum_{\alpha=0}^{\infty}\sum_{b=0}^{\alpha}\sum_{a=0}^{r}\binom{r}{a}\frac{(-u)^{r-a}a^bS_2(\alpha,b)}{(1-u)^r}\frac{t^\alpha}{\alpha!}, \end{aligned} \tag{2.29}$$

and

$$\frac{1}{l!}\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l = \sum_{m=l}^{\infty}S_\lambda^{(2)}(m,l)\frac{1}{m!}(e^t-1)^m = \sum_{s=l}^{\infty}\sum_{m=l}^{s}S_\lambda^{(2)}(m,l)S_2(s,m)\frac{t^s}{s!}, \tag{2.30}$$

by (1.11), (2.29), and (2.30), we have

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^{(e^t-1)}-u}{1-u}\right)^r\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l \middle| x^n\right\rangle \\ &= \sum_{c=l}^{n}\sum_{m=l}^{c}\binom{n}{c}S_\lambda^{(2)}(m,l)S_2(c,m)\left\langle \left(\frac{e^{(e^t-1)}-u}{1-u}\right)^r \middle| x^{n-c}\right\rangle \\ &= \sum_{c=l}^{n}\sum_{m=l}^{c}\sum_{b=0}^{n-c}\sum_{a=0}^{r}\binom{r}{a}\binom{n}{c}\frac{S_\lambda^{(2)}(m,l)S_2(c,m)S_2(n-c,b)(-u)^{r-a}a^b}{(1-u)^r}. \end{aligned} \tag{2.31}$$

In addition, by (1.10) and (2.3), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^t-u}{1-u}\right)^r\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| \mathrm{Bel}_n(x)\right\rangle \\ &= \sum_{m=0}^{n}S_2(n,m)\left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^m\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=l}^{m}\binom{m}{a}S_2(n,m)S_\lambda^{(2)}(a,l)\left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| x^{m-a}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{r}{b}\binom{m}{a}\frac{S_2(n,m)S_\lambda^{(2)}(a,l)(-u)^{r-b}b^{m-a}}{(1-u)^r}. \end{aligned} \tag{2.32}$$

By (2.28), (2.31), and (2.32), we obtains the following theorem.

Theorem 2.5. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$H^{(r)}_{n,\lambda}(x|u) = \sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_1(m,l)S^{(1)}_{\lambda}(a,m)H^{(r)}_{n-a,\lambda}(u)\right)\mathrm{Bel}_l(x),$$

and

$$\begin{aligned}\mathrm{Bel}_n(x) &= \sum_{l=0}^{n}\left(\sum_{c=l}^{n}\sum_{m=l}^{c}\sum_{b=0}^{n-c}\sum_{a=0}^{r}\binom{r}{a}\binom{n}{c}\frac{S^{(2)}_{\lambda}(m,l)S_2(c,m)S_2(n-c,b)(-u)^{r-a}a^{b}}{(1-u)^{r}}\right)H^{(r)}_{l,\lambda}(x|u)\\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{r}{b}\binom{m}{a}\frac{S_2(n,m)S^{(2)}_{\lambda}(a,l)(-u)^{r-b}b^{m-a}}{(1-u)^{r}}\right)H^{(r)}_{l,\lambda}(x|u).\end{aligned}$$

The *Changhee polynomials* are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^{x} = \sum_{n=0}^{\infty}\mathrm{Ch}_n(x)\frac{t^n}{n!}. \tag{2.33}$$

When $x = 0$, $\mathrm{Ch}_n = \mathrm{Ch}_n(0)$ are called the *Changhee numbers*. By the similar way to (2.2) and (2.33), we note that

$$\mathrm{Ch}_n(x) \sim \left(\frac{e^t+1}{2}, e^t-1\right) \quad \text{and} \quad \mathrm{Ch}_n(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_1(m,k)\mathrm{Ch}_{n-m}x^{k}. \tag{2.34}$$

Let $H^{(r)}_{n,\lambda}(x|u) = \sum_{l=0}^{n} a_{n,l}\mathrm{Ch}_l(x)$. Since

$$\frac{1}{l!}\left(e^{\left(\frac{1}{\lambda}\log(1+\lambda t)\right)}-1\right)^{l} = \sum_{m=l}^{\infty}S_2(m,l)\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m} = \sum_{a=l}^{\infty}\sum_{m=l}^{a}S_2(m,l)S^{(1)}_{\lambda}(a,m)\frac{t^a}{a!}, \tag{2.35}$$

by (1.11) and (2.35), we get

$$\begin{aligned}a_{n,l} &= \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^{r}}\left(e^{\frac{\log(1+\lambda t)}{\lambda}}-1\right)^{l}\middle| x^n\right\rangle\\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_2(m,l)S^{(1)}_{\lambda}(a,m)\left\langle\left(\frac{1-u}{e_\lambda(t)-u}\right)^{r}\frac{e_\lambda(t)+1}{2}\middle| x^{n-a}\right\rangle\\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=0}^{n-a}\binom{n}{a}\binom{n-a}{b}S_2(m,l)S^{(1)}_{\lambda}(a,m)H^{(r)}_{b,\lambda}(u)\left\langle\frac{e_\lambda(t)+1}{2}\middle| x^{n-a-b}\right\rangle\\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a}S_2(m,l)S^{(1)}_{\lambda}(a,m)H^{(r)}_{n-a,\lambda}(u)\\ &\quad+\frac{1}{2}\sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=0}^{n-a-1}\binom{n}{a}\binom{n-a}{b}(1)_{n-a-b,\lambda}S_2(m,l)S^{(1)}_{\lambda}(a,m)H^{(r)}_{b,\lambda}(u).\end{aligned} \tag{2.36}$$

Conversely, assume that $\mathrm{Ch}_n(x) = \sum_{l=0}^{n} b_{n,l}H^{(r)}_{l,\lambda}(x|u)$. Since

$$\left(\frac{(1-u)+t}{1-u}\right)^{r} = \sum_{a=0}^{r}\frac{(r)_a}{(1-u)^a}\frac{t^a}{a!},$$

by (1.6) and (1.11), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \frac{\left(\frac{(1+t)-u}{1-u}\right)^r}{\frac{t+2}{2}} \left(\frac{e^{\lambda \log(1+t)}-1}{\lambda}\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{2}{t+2}\left(\frac{(1-u)+t}{1-u}\right)^r \middle| \left(\frac{1}{l!}\left(\log_\lambda(1+t)\right)^l\right) x^n \right\rangle \\ &= \sum_{m=l}^{n}\binom{n}{m} S_{1,\lambda}(m,l)\left\langle \frac{2}{t+2}\left(\frac{(1-u)+t}{1-u}\right)^r \middle| x^{n-m}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} S_{1,\lambda}(m,l) Ch_a \left\langle \left(\frac{(1-u)+t}{1-u}\right)^r \middle| x^{n-m-a}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} \frac{S_{1,\lambda}(m,l) Ch_a (r)_{n-m-a}}{(1-u)^{n-m-a}}. \end{aligned} \tag{2.37}$$

In addition, by (1.10), (2.3), and (2.34), we get

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^t-u}{1-u}\right)^r \left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| Ch_n(x)\right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_1(m,k) Ch_{n-m}\left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^k\right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a} S_1(m,k) S_\lambda^{(2)}(a,l) Ch_{n-m}\left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| x^{k-a}\right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{r}\binom{n}{m}\binom{k}{a}\binom{r}{b} \frac{(-u)^{r-b} S_1(m,k) S_\lambda^{(2)}(a,l) b^{k-a} Ch_{n-m}}{(1-u)^r}. \end{aligned} \tag{2.38}$$

By (2.36), (2.37), and (2.38), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer* n, *we have*

$$\begin{aligned} H_{n,\lambda}^{(r)}(x|u) = \sum_{l=0}^{n}\Bigg(&\sum_{a=l}^{n}\sum_{m=l}^{a}\binom{n}{a} S_2(m,l) S_\lambda^{(1)}(a,m) H_{n-a,\lambda}^{(r)}(u) \\ &+ \frac{1}{2}\sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=0}^{n-a-1}\binom{n}{a}\binom{n-a}{b}(1)_{n-a-b,\lambda} S_2(m,l) S_\lambda^{(1)}(a,m) H_{b,\lambda}^{(r)}(u)\Bigg) Ch_l(x), \end{aligned}$$

and

$$\begin{aligned} Ch_n(x) &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}\frac{S_{1,\lambda}(m,l) Ch_a (r)_{n-m-a}}{(1-u)^{n-m-a}}\right) H_{l,\lambda}^{(r)}(x|u) \\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{r}\binom{n}{m}\binom{k}{a}\binom{r}{b}\frac{(-u)^{r-b} S_1(m,k) S_\lambda^{(2)}(a,l) b^{k-a} Ch_{n-m}}{(1-u)^r}\right) H_{l,\lambda}^{(r)}(x|u). \end{aligned}$$

The *Mittag-Leffer polynomials* are defined by the generating function to be

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}.$$

When $x = 1$, $M_n = M_n(1)$ are called the *Mittag-Leffer numbers*. Since

$$\left(\frac{1+t}{1-t}\right)^x = \left(1+\frac{2t}{1-t}\right)^x = \sum_{a=0}^{\infty}\sum_{m=0}^{a}\sum_{r=0}^{m} 2^m L(a,m) S_1(m,r) x^r \frac{t^a}{a!}, \tag{2.39}$$

by the definition of Mittag-Leffer polynomials and (2.39), we see that

$$M_n(x) \sim \left(1, \frac{e^t-1}{e^t+1}\right) \quad \text{and} \quad M_n(x) = \sum_{m=0}^{n}\sum_{r=0}^{m} 2^m L(n,m) S_1(m,r) x^r. \tag{2.40}$$

Let $H_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n} a_{n,l} M_l(x)$. Then

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^r}\left(\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)^l \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \left(\frac{2}{e_\lambda(t)+1}\right)^l \frac{1}{2^l} \middle| \left(\frac{1}{l!}\left(e_\lambda(t)-1\right)^l\right) x^n \right\rangle \\ &= \frac{1}{2^l}\sum_{m=l}^{n}\binom{n}{m} S_{2,\lambda}(m,l) \left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \left(\frac{2}{e_\lambda(t)+1}\right)^l \middle| x^{n-m} \right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} \frac{S_{2,\lambda}(m,l) E_{a,\lambda}^{(l)}}{2^l} \left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^r \middle| x^{n-m-a} \right\rangle \\ &= \sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a} \frac{S_{2,\lambda}(m,l) E_{a,\lambda}^{(l)} H_{n-m-a,\lambda}^{(r)}(u)}{2^l}. \end{aligned} \tag{2.41}$$

In addition, by (1.10) and (2.2), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^t-1}{e^t+1}\right)^l \middle| H_{n,\lambda}^{(r)}(x|u) \right\rangle \\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u) \frac{1}{2^l}\left\langle \left(\frac{2}{e^t)+1}\right)^l \middle| \left(\frac{1}{l!}\left(e^t-1\right)^l\right) x^k \right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a} \frac{S_2(a,k) S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u)}{2^l} \left\langle \left(\frac{2}{e^t+1}\right)^l \middle| x^{k-a} \right\rangle \\ &= \sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a} \frac{S_2(a,l) S_\lambda^{(1)}(m,k) H_{n-m,\lambda}^{(r)}(u) E_{k-a}^{(l)}}{2^l}. \end{aligned} \tag{2.42}$$

Conversely, assume that $M_n(x) = \sum_{l=0}^{n} b_{n,l} H_{l,\lambda}^{(r)}(x|u)$. Then, by (1.10) and (2.40), we have

$$\begin{aligned} b_{n,l} &= \frac{1}{l!}\left\langle \left(\frac{e^t-u}{1-u}\right)^r \left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| M_n \right\rangle \\ &= \sum_{a=0}^{n}\sum_{m=0}^{a}\sum_{b=0}^{m} 2^m L(a,m) S_1(m,b) \left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^b \right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=l}^{m}\sum_{c=l}^{b} 2^m L(a,m) S_1(m,b) S_\lambda^{(2)}(c,l) \left\langle \left(\frac{e^t-u}{1-u}\right)^r \middle| x^{b-c} \right\rangle \\ &= \sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=l}^{m}\sum_{c=l}^{b}\sum_{d=0}^{r}\binom{r}{d} \frac{(-u)^{r-d} 2^m d^{b-c} L(a,m) S_1(m,b) S_\lambda^{(2)}(c,l)}{(1-u)^r}. \end{aligned} \tag{2.43}$$

By (2.41), (2.42), and (2.43), we obtain the following theorem.

Theorem 2.7. *For each nonnegative integer* n, *we have*

$$H_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}\frac{S_{2,\lambda}(m,l)E_{a,\lambda}^{(l)}H_{n-m-a,\lambda}^{(r)}(u)}{2^{l}}\right)M_{l}(x)$$
$$=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}\frac{S_{2}(a,l)S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)E_{k-a}^{(l)}}{2^{l}}\right)M_{l}(x),$$

and

$$M_{n}(x)=\sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{m=l}^{a}\sum_{b=l}^{m}\sum_{c=l}^{b}\sum_{d=0}^{r}\binom{r}{d}\frac{(-u)^{r-d}2^{m}d^{b-c}L(a,m)S_{1}(m,b)S_{\lambda}^{(2)}(c,l)}{(1-u)^{r}}\right)H_{l,\lambda}^{(r)}(x|u).$$

The *Lah number* $L(n,m)$ has the explicit formula

$$L(n,m)=\binom{n-1}{k-1}\frac{n!}{k!}\quad\text{and}\quad\frac{1}{k!}\left(\frac{t}{1-t}\right)^{l}=\sum_{n=l}^{\infty}L(n,l)\frac{t^{n}}{n!}\quad\text{(see [10]).}\tag{2.44}$$

The *Lah-Bell polynomials* are defined by the generating function to be as follows (see [10])

$$e^{\frac{xt}{1-t}}=\sum_{n=0}^{\infty}B_{n}^{L}(x)\frac{t^{n}}{n!}.\tag{2.45}$$

In the special case $x=1$, $B_{n}^{L}=B_{n}^{L}(1)$ are called the *Lah-Bell numbers*. By (2.44) and (2.45), we note that

$$B_{n}^{L}(x)\sim\left(1,\frac{t}{1+t}\right)\quad\text{and}\quad B_{n}^{L}(x)=\sum_{m=0}^{n}L(n,m)x^{m}.\tag{2.46}$$

Let $H_{n,\lambda}^{(r)}(x)=\sum_{l=0}^{n}a_{n,l}B_{l}^{L}(x)$. Since

$$\begin{aligned}\frac{1}{l!}\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{1+\frac{1}{\lambda}\log(1+\lambda t)}\right)^{l}&=\sum_{m=0}^{\infty}\binom{m+l}{l}(-1)^{m}<l>_{m}\frac{1}{(m+l)!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m+l}\\&=\sum_{m=0}^{\infty}\binom{m+l}{l}(-1)^{m}<l>_{m}\sum_{a=m+l}^{\infty}S_{\lambda}^{(1)}(a,m+l)\frac{t^{a}}{a!}\\&=\sum_{a=l}^{\infty}\sum_{m=0}^{a}\binom{m+l}{l}(-1)^{m}<l>_{m}S_{\lambda}^{(1)}(a,m+l)\frac{t^{a}}{a!},\end{aligned}\tag{2.47}$$

by (1.11) and (2.47), we see that

$$\begin{aligned}a_{n,l}&=\frac{1}{l!}\left\langle\frac{1}{\left(\frac{e_{\lambda}(t)-u}{1-u}\right)^{r}}\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{1+\frac{1}{\lambda}\log(1+\lambda t)}\right)^{l}\middle|x^{n}\right\rangle\\&=\sum_{a=l}^{n}\sum_{m=0}^{a}\binom{m+l}{l}\binom{n}{a}(-1)^{m}<l>_{m}S_{\lambda}^{(1)}(a,m+l)\left\langle\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}\middle|x^{n-a}\right\rangle\\&=\sum_{a=l}^{n}\sum_{m=0}^{a}\binom{m+l}{l}\binom{n}{a}(-1)^{m}<l>_{m}S_{\lambda}^{(1)}(a,m+l)H_{n-a,\lambda}^{(r)}(u).\end{aligned}\tag{2.48}$$

In addition, since

$$\left(\frac{t}{1+t}\right)^{l}=\sum_{a=0}^{\infty}(-1)^{a}<l>_{a}\frac{t^{a+l}}{a!},\tag{2.49}$$

by (1.10), (2.2), and (2.49), we have

$$\begin{aligned}a_{n,l}=\frac{1}{l!}\left\langle\left(\frac{t}{1+t}\right)^{l}\middle|H_{n,\lambda}^{(r)}(x|u)\right\rangle&=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)\frac{1}{l!}\left\langle\left(\frac{t}{1+t}\right)^{l}\middle|x^{k}\right\rangle\\&=\sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)(-1)^{k-l}<l>_{k-l}.\end{aligned}\tag{2.50}$$

Conversely, assume that $B_{n}^{L}(x)=\sum_{l=0}^{n}b_{n,l}H_{l,\lambda}^{(r)}(x)$. Then, by (1.4), (1.10), and (2.46), we get

$$\begin{aligned}b_{n,l}&=\frac{1}{l!}\left\langle\left(\frac{e^{t}-u}{1-u}\right)^{r}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^{l}\middle|B_{n}^{L}(x)\right\rangle\\&=\sum_{m=0}^{n}L(n,m)\left\langle\left(\frac{e^{t}-u}{1-u}\right)^{r}\middle|\left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^{l}\right)x^{m}\right\rangle\\&=\sum_{m=0}^{n}\sum_{a=l}^{m}\binom{m}{a}L(n,m)S_{\lambda}^{(2)}(a,l)\left\langle\left(\frac{e^{t}-u}{1-u}\right)^{r}\middle|x^{m-a}\right\rangle\\&=\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{m-a}S_{\lambda}^{(2)}(a,l)}{(1-u)^{r}}.\end{aligned}\tag{2.51}$$

By (2.48), (2.50), and (2.51), we obtain the following theorem.

Theorem 2.8. *For each nonnegative integer* n*, we have*

$$\begin{aligned}H_{n,\lambda}^{(r)}(x|u)&=\sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{m=0}^{a}\binom{m+l}{l}\binom{n}{a}(-1)^{m}<l>_{m}S_{\lambda}^{(1)}(1,m+l)H_{n-a,\lambda}^{(r)}(u)\right)B_{l}^{L}(x)\\&=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}\binom{k}{l}S_{\lambda}^{(1)}(m,k)H_{n-m,\lambda}^{(r)}(u)(-1)^{k-l}<l>_{k-l}\right)B_{l}^{L}(x),\end{aligned}$$

and

$$B_{n}^{L}(x)=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{r}\binom{m}{a}\binom{r}{b}\frac{(-u)^{r-b}b^{m-a}S_{\lambda}^{(2)}(a,l)}{(1-u)^{r}}\right)H_{l,\lambda}^{(r)}(x|u).$$

3. The zeros of degenerate Frobenius-Euler polynomials

It is well known that finding the roots of polynomials is a very important task in applied mathematics. There are no formulas for the roots of polynomials of higher degree, and formulas for the roots of polynomials of lower degree do exist, but their expressions are very complex. Hence, we will investigate the numerical pattern of the roots of the polynomials $H_{n,\lambda}^{(r)}(x|u)$. Using the Mathematica, the polynomial $H_{n,\lambda}^{(r)}(x|u)$ can be expressed explicitly. For example,

$$\begin{aligned}H_{1,\lambda}^{(r)}(x|u)&=\frac{r}{u-1}+x,\\H_{2,\lambda}^{(r)}(x|u)&=\frac{r(\lambda+r+u-\lambda u)}{(u-1)^{2}}+\left(-\lambda+\frac{2r}{u-1}\right)x+x^{2},\end{aligned}$$

$$H^{(r)}_{3,\lambda}(x|u) = \frac{r(r^2+2\lambda^2(u-1)^2+u(1+3r+u)-3\lambda(u-1)(r+u))}{(u-1)^3} + \frac{3r(r+u)+2\lambda^2(u-1)^2-6r\lambda(u-1)}{(u-1)^2}x + \left(\frac{3r}{u-1}-3\lambda\right)x^2+x^3.$$

First, we want to observe the impact of the parameters u and λ on the distribution of the roots of the polynomials. For the aims, we set the degree of the polynomial as $n = 40$. Using the mathematical tool with 100 working precision, the roots of the polynomial $H^{(r)}_{40,\lambda}(x|u)$ are computed. The absolute numerical error is bounded as

$$\sum_{i=1}^{40} |H^{(r)}_{40,\lambda}(x_i|u)| < 10^{-62},$$

where x_i denotes the roots of the polynomial. Hence, the numerical roots are reliable. We compute the numerical roots of $H^{(r)}_{40,\lambda}(x|u)$ with nine pair of parameters (λ_i, u_j), $\lambda_i \in \{0.1, 1.5, 10\}$, $u_j = \{2, 0, -2\}$, and $r = 2, 3$. The numerical results are plotted in Figs. 1 and 2. As observed in Fig. 1, the roots of the polynomials have four patterns. By comparing the results of both Figs. 1 and 2, the impact of the parameter r on the distribution of the roots of the polynomial is weak. Secondly, we want to investigate the impact of the degree of polynomials on the distribution of roots of the polynomials. We compute the numerical roots of polynomials increasing the degree of polynomials from 1 to 40 and presented in Fig. 4. Finally, we investigate the real roots distribution structure of $H^{(2)}_{n,\lambda}(x|u)$. The numerical results are displayed in Fig. 3.

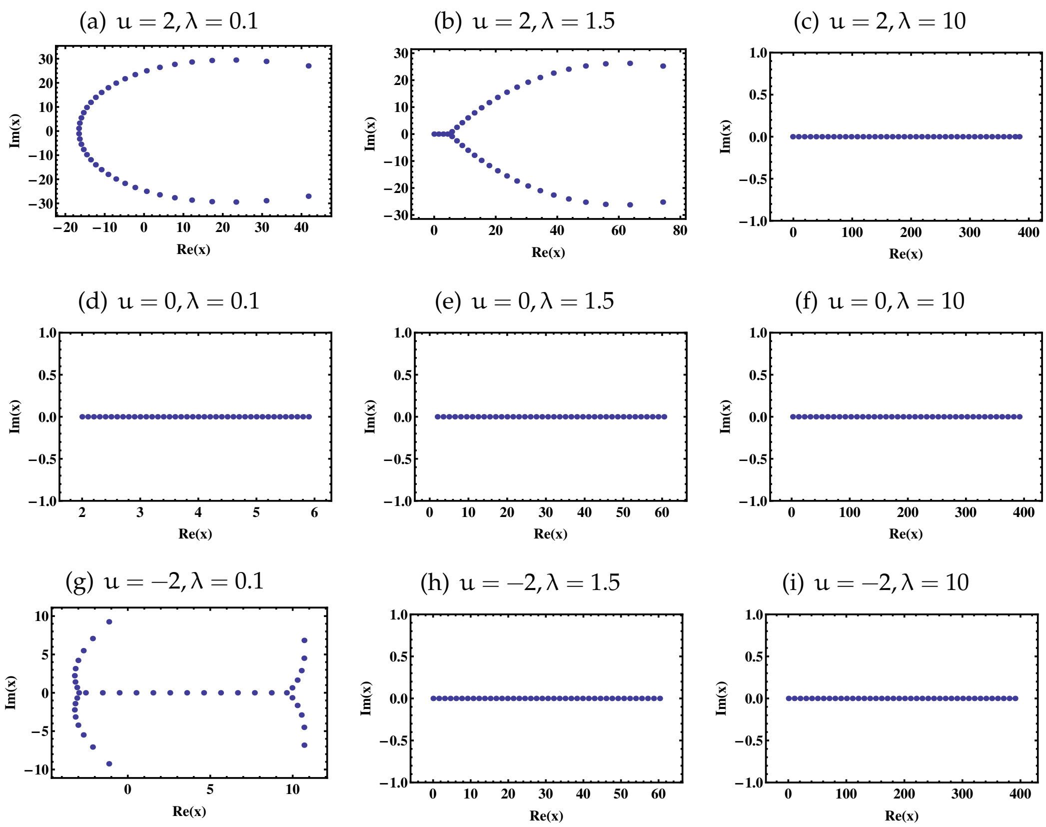

Figure 1: The computed roots of $H^{(2)}_{40,\lambda}(x|u)$.

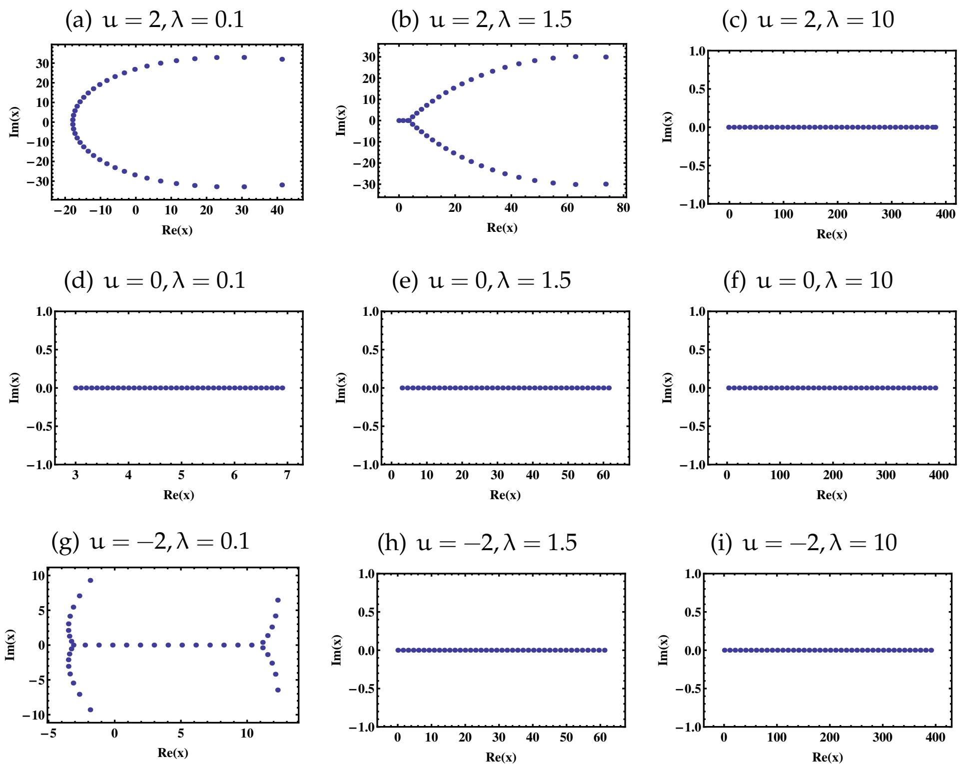

Figure 2: The computed roots of $H^{(3)}_{40,\lambda}(x|u)$.

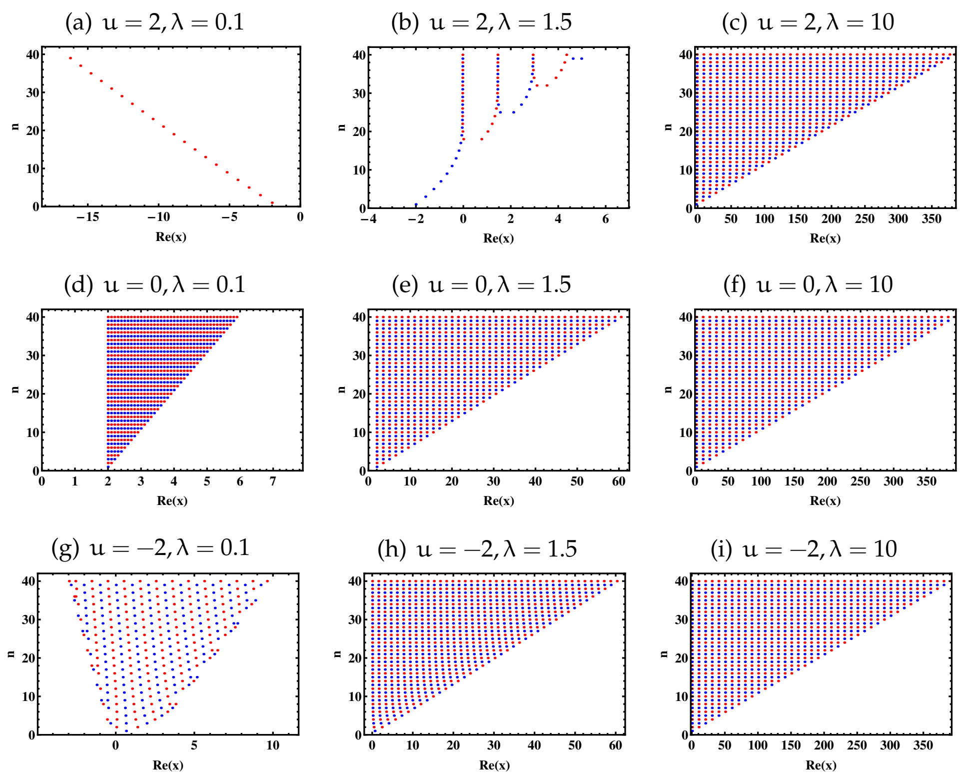

Figure 3: The distribution of real zero of $H^{(2)}_{n,\lambda}(x|u)$ for $\lambda = 0.1, 1.5, 10,\ u = 2, 0, -2$, and $1 \leqslant n \leqslant 40$.

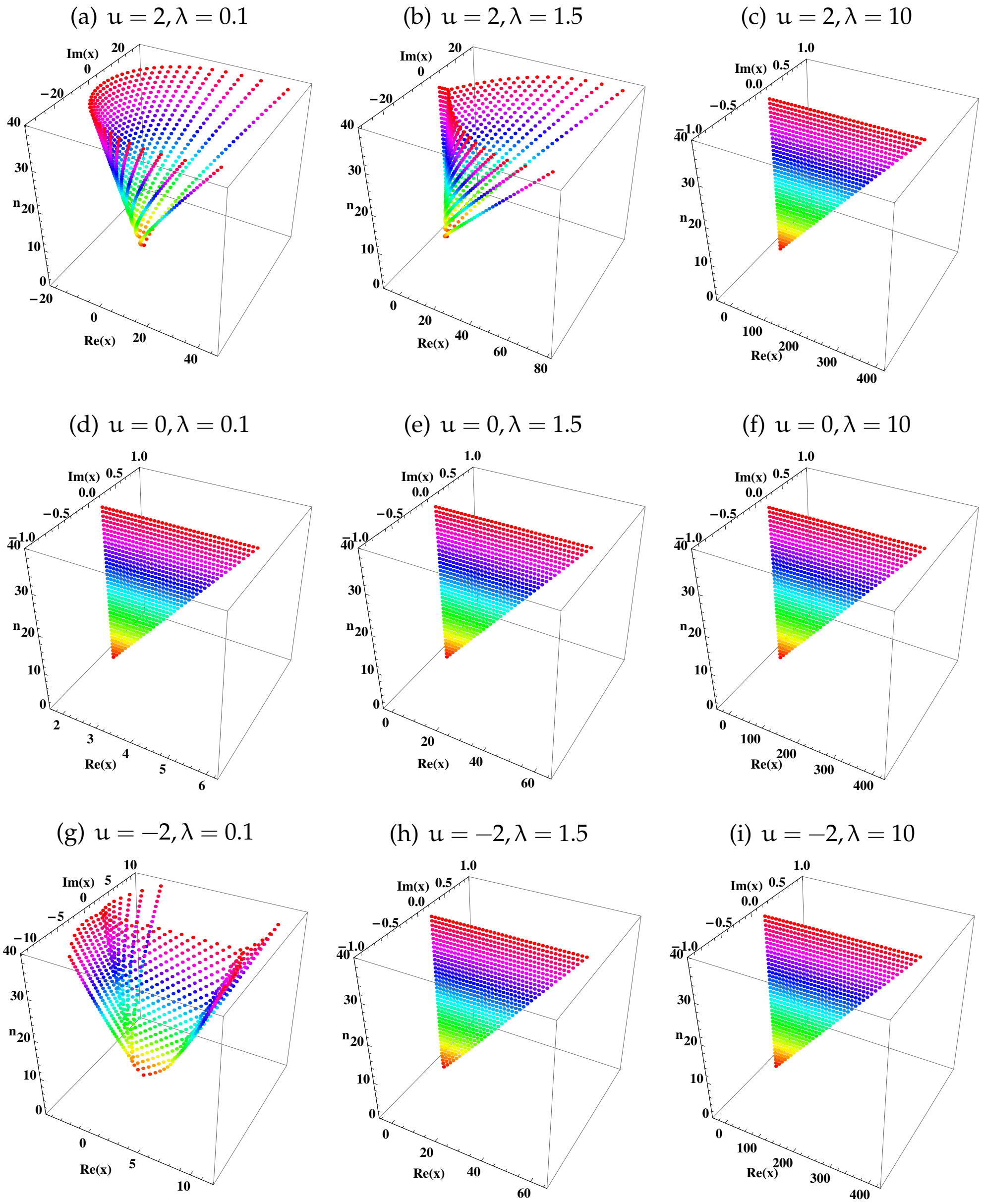

Figure 4: The computed roots of $H^{(2)}_{n,\lambda}(x|u)$ for $\lambda = 0.1, 1.5, 10,\ u = 2, 0, -2$, and $1 \leqslant n \leqslant 40$.

4. Conclusion

The research of special polynomials have been used actively by useful tools in differential equations, algebraic number theory, probability theory, orthogonal polynomials, and special function theory. These researches are utilized by various different tools including generating functions, p-adic analysis, modified umbral calculus, and combinatorial methods.

Recently, degenerate versions of special polynomials and numbers have been studied with λ-analogue or degenerate version of these methods, and their arithmetical and combinatorial properties and relations are also studied by many researchers, and applied in differential equations and probability theories providing new applications.

The aim of this study, we found some relationships between the degenerate higher order Frobenius-Euler polynomials and some others degenerate type of special polynomials expressing them as linear combinations of each other, and present explicit formulas for representations with the help of umbral calculus and vice versa. Moreover, we illustrate the results with some explicit examples. In order to better understanding the polynomials, the distribution of roots are presented.

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Data availability statement

Data sharing is not applicable to this article as no data-sets were generated or analyzed during this study.

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