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Dynamics and general form of the solutions of rational difference equations

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Abstract

The main objective of this article is to find the general solution to some special cases of the fractional recursive equation

$$\Psi_{n+1}=\frac{\alpha\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(\beta+\delta\Psi_{n}\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})},\quad n=0,1,2,\ldots,$$

where α,β and δ are arbitrary real numbers. Furthermore, the solution's qualitative behavior is explored, such as local and global stability. For some situations, we have discovered periodic solutions. We also offered numerical examples to demonstrate our results.

Keywords: Solutions expressions of difference equation, local and global stability, periodic solution.

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1. Introduction

Difference equations, often known as discrete dynamical systems, are one of the most important scientific topics. The study of the qualitative features of rational difference equations has recently attracted a lot of interest (see, for example, [1, 3, 6, 7, 15, 26, 30]). The study of rational difference equations of order larger than one is both demanding and gratifying because the results for rational difference equations serve as prototypes for the creation of the basic theory of the global behavior of nonlinear difference equations of order higher than one. However, no efficient general approaches for dealing with the global behavior of rational difference equations of order greater than one have been developed yet. As a result, the study of rational difference equations is important. So, many disciplines of science and technology have recently seen applications of discrete dynamical systems and difference equations.

Investigating the behavior of solutions to a system of nonlinear differential equations and discussing the local asymptotic stability of their equilibrium points is particularly intriguing (see, for example, [2,

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8, 20, 23, 28, 32]). The technique of determining the general form of the solution for special cases of the equation has been the subject of numerous investigations. Many publications have been written about the systems and behavior of rational difference equations (for more details, check out the references).

Abo-Zeid and Kamal [1] solved and studied the global behavior of all admissible solutions of the two difference equations:

$$\chi_{n+1} = \frac{\chi_n \chi_{n-2}}{\chi_{n-1} - \chi_{n-2}}, \qquad \chi_{n+1} = \frac{\chi_n \chi_{n-2}}{-\chi_{n-1} + \chi_{n-2}}.$$

Aljoufi et al. [5] obtained the forms of the eighteenth-order difference equation:

$$W_{n+1} = \frac{W_{n-17}}{\pm 1 \pm W_{n-2} W_{n-5} W_{n-8} W_{n-11} W_{n-14} W_{n-17}},$$

and investigated the stability,boundedness and the periodic character of these solutions. Alshareef et al. [6] examined the dynamics behavior and periodicity character and gave the general form of the solution of some special cases of the difference equation:

$$V_{n+1} = \xi V_{n-8} + \frac{\varepsilon V_{n-8}^2}{\mu V_{n-8} + \kappa V_{n-17}}.$$

El-Metwally and Alharthi [14] studied the qualitative properties of the solutions for nonlinear difference equation:

$$y_{n+1} = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}.$$

Elsayed and Al-Rakhami investigated the qualitative behavior of the critical point and found the solution for a rational recursive sequence in [19]

$$\Psi_{n+1} = \alpha \Psi_{n-2} + \frac{\beta \Psi_{n-2} \Psi_{n-3}}{\gamma \Psi_{n-3} + \delta \Psi_{n-6}}.$$

In [22], Folly-Gbetoula et al. studied the solution of the rational difference equation:

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}.$$

Abdul Khaliq and Elsayed [31] investigated the asymptotic behavior of the solutions of the following difference equation:

$$\omega_{n+1} = \frac{\omega_{n-2} \omega_{n-7}}{\omega_{n-4} \left(\pm 1 \pm \omega_{n-2} \omega_{n-7}\right)},$$

and gave the solution of some special cases of the difference equation. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations. Tollu et al. [39] solved and studied the next system:

$$x_n = \frac{\alpha}{1 + x_{n-1} y_n}, \qquad y_n = \frac{\beta}{1 + x_{n-1} y_n}.$$

The goal of this paper is to find a general solution to some special cases of the fractional recursive equation:

$$\Psi_{n+1} = \frac{\alpha \Psi_{n-1} \Psi_{n-5}}{\Psi_{n-3}(\beta + \delta \Psi_n \Psi_{n-1} \Psi_{n-4} \Psi_{n-5})}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where α, β, and δ are arbitrary real numbers.

2. The behavior of equilibrium points

In this section, we will find the fixed point and study its behavior. To find the critical points of Eq. (1.1), we can write

$$\overline{\Psi} = \frac{\alpha\overline{\Psi}^2}{\overline{\Psi}(\beta + \delta\overline{\Psi}^4)}.$$

Then, we have

$$\overline{\Psi}^2(\beta + \delta\overline{\Psi}^4) = \alpha\overline{\Psi}^2 \;\Rightarrow\; \overline{\Psi}^2(\beta + \delta\overline{\Psi}^4 - \alpha) = 0.$$

Thus, Eq. (1.1) has two fixed points which are

$$\overline{\Psi} = 0 \quad \text{and} \quad \overline{\Psi} = \sqrt[4]{\frac{\alpha-\beta}{\delta}}, \quad \delta \neq 0, \quad \frac{\alpha-\beta}{\delta} > 0.$$

Assume $\varphi : (0,\infty)^5 \to (0,\infty)$ be a C^1 function defined by

$$\varphi(u, v, w, s, t) = \frac{\alpha v t}{w\,(\beta + \delta u v s t)}. \tag{2.1}$$

In consequence,

$$\begin{aligned}
&\frac{\partial\varphi}{\partial u} = \frac{-\alpha\delta v^2 s t^2}{w\,(\beta+\delta u v s t)^2}, \quad \frac{\partial\varphi}{\partial v} = \frac{\alpha\beta t}{w\,(\beta+\delta u v s t)^2}, \quad \frac{\partial\varphi}{\partial w} = \frac{-\alpha v t}{w^2\,(\beta+\delta u v s t)},\\
&\frac{\partial\varphi}{\partial s} = \frac{-\alpha\delta u v^2 t^2}{w\,(\beta+\delta u v s t)^2}, \quad \frac{\partial\varphi}{\partial t} = \frac{\alpha\beta v}{w\,(\beta+\delta u v s t)^2}.
\end{aligned} \tag{2.2}$$

First, at $\overline{\Psi} = 0$, we see that

$$\begin{aligned}
&\frac{\partial\varphi}{\partial u}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 0 = \gamma_1, \quad \frac{\partial\varphi}{\partial v}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = \frac{\alpha}{\beta} = \gamma_2, \quad \frac{\partial\varphi}{\partial w}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = -\frac{\alpha}{\beta} = \gamma_3,\\
&\frac{\partial\varphi}{\partial s}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 0 = \gamma_4, \quad \frac{\partial\varphi}{\partial t}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = \frac{\alpha}{\beta} = \gamma_5.
\end{aligned} \tag{2.3}$$

The linearized equation of Eq. (1.1) about $\overline{\Psi} = 0$ is

$$Z_{n+1} - \gamma_1 Z_n - \gamma_2 Z_{n-1} - \gamma_3 Z_{n-3} - \gamma_4 Z_{n-4} - \gamma_5 Z_{n-5} = 0.$$

Hence,

$$Z_{n+1} - \frac{\alpha}{\beta} Z_{n-1} + \frac{\alpha}{\beta} Z_{n-3} - \frac{\alpha}{\beta} Z_{n-5} = 0.$$

Theorem 2.1. *The fixed point* $\overline{\Psi} = 0$ *is locally asymptotically stable if* $3\alpha < \beta$.

Proof. By using the values in Eq. (2.3) and by Lemma 1 in [30], it can be ensured that equation (1.1) is asymptotically stable if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| + |\gamma_5| < 1,$$

so,

$$\left|\frac{\alpha}{\beta}\right| + \left|-\frac{\alpha}{\beta}\right| + \left|\frac{\alpha}{\beta}\right| < 1,$$

therefore,

$$3\alpha < \beta. \qquad \square$$

Second, at $\overline{\Psi} = \sqrt[4]{\frac{\alpha-\beta}{\delta}}$, we see that

$$\frac{\partial\varphi}{\partial u}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = -\frac{(\alpha-\beta)}{\alpha} = \gamma_1, \quad \frac{\partial\varphi}{\partial v}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = \frac{\beta}{\alpha} = \gamma_2, \quad \frac{\partial\varphi}{\partial w}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = -1 = \gamma_3,$$
$$\frac{\partial\varphi}{\partial s}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = -\frac{(\alpha-\beta)}{\alpha} = \gamma_4, \quad \frac{\partial\varphi}{\partial t}(\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi},\overline{\Psi}) = \frac{\beta}{\alpha} = \gamma_5.$$

The linearized equation of Eq. (1.1) about $\overline{\Psi} = \sqrt[4]{\frac{\alpha-\beta}{\delta}}$ is

$$Z_{n+1} - \gamma_1 Z_n - \gamma_2 Z_{n-1} - \gamma_3 Z_{n-3} - \gamma_4 Z_{n-4} - \gamma_5 Z_{n-5} = 0.$$

Hence,

$$Z_{n+1} + \frac{(\alpha-\beta)}{\alpha} Z_n - \frac{\beta}{\alpha} Z_{n-1} + Z_{n-3} + \frac{(\alpha-\beta)}{\alpha} Z_{n-4} - \frac{\beta}{\alpha} Z_{n-5} = 0.$$

Theorem 2.2. *The fixed point $\overline{\Psi} = \sqrt[4]{\frac{\alpha-\beta}{\delta}}$ is not locally asymptotically stable.*

Proof. From Lemma 1 in [30], it follows that $\overline{\Psi}$ is asymptotically stable if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| + |\gamma_5| < 1,$$

so,

$$\left|-\frac{(\alpha-\beta)}{\alpha}\right| + \left|\frac{\beta}{\alpha}\right| + |-1| + \left|-\frac{(\alpha-\beta)}{\alpha}\right| + \left|\frac{\beta}{\alpha}\right| < 1,$$

it follows that the fixed point $\overline{\Psi} = \sqrt[4]{\frac{\alpha-\beta}{\delta}}$ is not locally asymptotically stable. □

Theorem 2.3. *The equilibrium point $\overline{\Psi} = 0$ of Eq. (1.1) is a global attractor if $\alpha \neq 0$.*

Proof. Let $[a_1, a_2]$ be an interval of real numbers and $\varphi : [a_1, a_2]^5 \to [a_1, a_2]$ is a continuous function defined by Eq. (2.1). Then, we note that from Eq. (2.2) the function $\varphi(u, v, w, s, t)$ is increasing in v and t and is decreasing in u, w, and s. Assume that whenever (B, b) is a solution of the system

$$B = \varphi(b, B, b, b, B), \qquad b = \varphi(B, b, B, B, b),$$

then, we have

$$B = \frac{\alpha B^2}{b(\beta + \delta B^2 b^2)} \Longrightarrow \alpha B^2 = Bb(\beta + \delta B^2 b^2), \tag{2.4}$$
$$b = \frac{\alpha b^2}{B(\beta + \delta b^2 B^2)} \Longrightarrow \alpha b^2 = bB(\beta + \delta b^2 B^2). \tag{2.5}$$

Substrating Eq. (2.4) from Eq. (2.5) we obtain

$$\alpha(B^2 - b^2) = Bb(\beta + \delta B^2 b^2 - \beta - \delta b^2 B^2).$$

In consequence, $B = b$ if $\alpha \neq 0$. It follows by Theorem 1 in [30], the equilibrium point $\overline{\Psi} = 0$ of Eq. (1.1) is a global attractor. Therefore, the proof is complete. □

3. General solution for special cases

In this section, we will find the general solution for some special cases of Eq. (1.1).

3.1. Case 1

In this subsection, we will find the solution of Eq. (1.1) when $\alpha = \beta = \delta = 1$, so Eq. (1.1) becomes

$$\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(1+\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}, \quad n = 0,1,2,\ldots, \tag{3.1}$$

where the initial conditions $\Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$, and Ψ_0 are positive real numbers.

Theorem 3.1. *Suppose that $\{\Psi_n\}_{n=-5}^{\infty}$ be a solution of Eq. (3.1) . Thus for $n = 0,1,2,\ldots$,*

$$\Psi_{8n-5} = \tau\prod_{i=0}^{n-1}\left(\frac{(1+(8i)\eta\lambda\mu\tau)\,(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-4} = \mu\prod_{i=0}^{n-1}\left(\frac{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+4)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-3} = \zeta\prod_{i=0}^{n-1}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+4)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+5)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-2} = \sigma\prod_{i=0}^{n-1}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+5)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)\,(1+(8i+6)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-1} = \lambda\prod_{i=0}^{n-1}\left(\frac{(1+(8i+4)\eta\lambda\mu\tau)\,(1+(8i+6)\eta\lambda\mu\tau)}{(1+(8i+5)\eta\lambda\mu\tau)\,(1+(8i+7)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n} = \eta\prod_{i=0}^{n-1}\left(\frac{(1+(8i+5)\eta\lambda\mu\tau)\,(1+(8i+7)\eta\lambda\mu\tau)}{(1+(8i+6)\eta\lambda\mu\tau)\,(1+(8i+8)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n+1} = \frac{\lambda\tau}{\zeta\,(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-1}\left(\frac{(1+(8i+6)\eta\lambda\mu\tau)\,(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+7)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n+2} = \frac{\eta\mu\,(1+\eta\lambda\mu\tau)}{\sigma\,(1+2\eta\lambda\mu\tau)}\prod_{i=0}^{n-1}\left(\frac{(1+(8i+7)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)\,(1+(8i+10)\eta\lambda\mu\tau)}\right),$$

where $\Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda$, and $\Psi_0 = \eta$.

Proof. By using mathematical induction, we will prove that the solution is true. First, for $n = 0$, the result holds. Second, we suppose that $n > 0$ and our assumption holds for $n-1$, that is

$$\Psi_{8n-13} = \tau\prod_{i=0}^{n-2}\left(\frac{(1+(8i)\eta\lambda\mu\tau)\,(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-12} = \mu\prod_{i=0}^{n-2}\left(\frac{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+4)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-11} = \zeta\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+4)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+5)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-10} = \sigma\prod_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+5)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)\,(1+(8i+6)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-9} = \lambda\prod_{i=0}^{n-2}\left(\frac{(1+(8i+4)\eta\lambda\mu\tau)\,(1+(8i+6)\eta\lambda\mu\tau)}{(1+(8i+5)\eta\lambda\mu\tau)\,(1+(8i+7)\eta\lambda\mu\tau)}\right),$$

$$\Psi_{8n-8}=\eta\prod_{i=0}^{n-2}\left(\frac{(1+(8i+5)\eta\lambda\mu\tau)\,(1+(8i+7)\eta\lambda\mu\tau)}{(1+(8i+6)\eta\lambda\mu\tau)\,(1+(8i+8)\eta\lambda\mu\tau)}\right),$$
$$\Psi_{8n-7}=\frac{\lambda\tau}{\zeta\,(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+6)\eta\lambda\mu\tau)\,(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+7)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}\right),$$
$$\Psi_{8n-6}=\frac{\eta\mu\,(1+\eta\lambda\mu\tau)}{\sigma\,(1+2\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+7)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)\,(1+(8i+10)\eta\lambda\mu\tau)}\right).$$

Now, we prove that the results hold for n. From Eq. (3.1), it follows that

$$\Psi_{8n-5}=\frac{\Psi_{8n-7}\Psi_{8n-11}}{\Psi_{8n-9}(1+\Psi_{8n-6}\Psi_{8n-7}\Psi_{8n-10}\Psi_{n-11})},$$
$$=\frac{\left[\begin{array}{c}\frac{\lambda\tau}{\zeta(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+6)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+7)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}\right)\\ \zeta\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)(1+(8i+4)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+5)\eta\lambda\mu\tau)}\right)\end{array}\right]}{\lambda\prod_{i=0}^{n-2}\left(\frac{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+6)\eta\lambda\mu\tau)}{(1+(8i+5)\eta\lambda\mu\tau)(1+(8i+7)\eta\lambda\mu\tau)}\right)\left[\begin{array}{c}1+\frac{\eta\mu(1+\eta\lambda\mu\tau)}{\sigma(1+2\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+7)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)\\ \frac{\lambda\tau}{\zeta(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+6)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+7)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}\right)\\ \sigma\prod_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+5)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+6)\eta\lambda\mu\tau)}\right)\\ \zeta\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)(1+(8i+4)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+5)\eta\lambda\mu\tau)}\right)\end{array}\right]},$$
$$\Psi_{8n-5}=\frac{\frac{\tau}{(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}\right)}{\left[1+\frac{\eta\lambda\mu\tau}{(1+2\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+10)\eta\lambda\mu\tau)}\right)\right]},$$
$$\Psi_{8n-5}=\frac{\frac{\tau}{(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}\right)}{\left[1+\frac{\eta\lambda\mu\tau}{(1+(8n-6)\eta\lambda\mu\tau)}\right]},$$
$$\Psi_{8n-5}=\frac{\frac{\tau}{(1+\eta\lambda\mu\tau)}\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}\right)}{\left[\frac{(1+(8n-5)\eta\lambda\mu\tau)}{(1+(8n-6)\eta\lambda\mu\tau)}\right]},$$
$$\Psi_{8n-5}=\frac{\tau}{(1+\eta\lambda\mu\tau)}\left[\frac{(1+(8n-6)\eta\lambda\mu\tau)}{(1+(8n-5)\eta\lambda\mu\tau)}\right]\prod_{i=0}^{n-2}\left(\frac{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+8)\eta\lambda\mu\tau)}{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}\right).$$

Hence, we get

$$\Psi_{8n-5}=\tau\prod_{i=0}^{n-1}\left(\frac{(1+(8i)\eta\lambda\mu\tau)\,(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}\right).$$

Similarly, from Eq. (3.1), we have

$$\Psi_{8n-4}=\frac{\Psi_{8n-6}\Psi_{8n-10}}{\Psi_{8n-8}(1+\Psi_{8n-5}\Psi_{8n-6}\Psi_{8n-9}\Psi_{n-10})},$$

$$=\frac{\left[\begin{array}{c}\frac{\eta\mu(1+\eta\lambda\mu\tau)}{\sigma(1+2\eta\lambda\mu\tau)}\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+7)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)\\ \sigma\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+5)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+6)\eta\lambda\mu\tau)}\right)\end{array}\right]}{\eta\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+5)\eta\lambda\mu\tau)(1+(8i+7)\eta\lambda\mu\tau)}{(1+(8i+6)\eta\lambda\mu\tau)(1+(8i+8)\eta\lambda\mu\tau)}\right)\left[\begin{array}{c}1+\tau\prod\limits_{i=0}^{n-1}\left(\frac{(1+(8i)\eta\lambda\mu\tau)(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+1)\eta\lambda\mu\tau)(1+(8i+3)\eta\lambda\mu\tau)}\right)\\ \frac{\eta\mu(1+\eta\lambda\mu\tau)}{\sigma(1+2\eta\lambda\mu\tau)}\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+7)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)\\ \lambda\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+6)\eta\lambda\mu\tau)}{(1+(8i+5)\eta\lambda\mu\tau)(1+(8i+7)\eta\lambda\mu\tau)}\right)\\ \sigma\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+5)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+6)\eta\lambda\mu\tau)}\right)\end{array}\right]},$$

$$\Psi_{8n-4}=\frac{\frac{\mu(1+\eta\lambda\mu\tau)}{(1+2\eta\lambda\mu\tau)}\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)}{\left[\begin{array}{c}1+\frac{\eta\lambda\mu\tau}{(1+3\eta\lambda\mu\tau)}\prod\limits_{i=1}^{n-1}\left(\frac{(1+(8i)\eta\lambda\mu\tau)(1+(8i+2)\eta\lambda\mu\tau)}{(1+(8i+1)\eta\lambda\mu\tau)(1+(8i+3)\eta\lambda\mu\tau)}\right)\\ \prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+8)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)\end{array}\right]},$$

$$\Psi_{8n-4}=\frac{\frac{\mu(1+\eta\lambda\mu\tau)}{(1+2\eta\lambda\mu\tau)}\prod\limits_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)(1+(8i+10)\eta\lambda\mu\tau)}\right)}{\left[1+\frac{\eta\lambda\mu\tau}{(1+(8n-5)\eta\lambda\mu\tau)}\right]},$$

$$\Psi_{8n-4}=\frac{\mu\,(1+\eta\lambda\mu\tau)}{(1+2\eta\lambda\mu\tau)}\left[\frac{(1+(8n-5)\eta\lambda\mu\tau)}{(1+(8n-4)\eta\lambda\mu\tau)}\right]\prod_{i=0}^{n-2}\left(\frac{(1+(8i+3)\eta\lambda\mu\tau)\,(1+(8i+9)\eta\lambda\mu\tau)}{(1+(8i+4)\eta\lambda\mu\tau)\,(1+(8i+10)\eta\lambda\mu\tau)}\right).$$

So, we obtain

$$\Psi_{8n-4}=\mu\prod_{i=0}^{n-1}\left(\frac{(1+(8i+1)\eta\lambda\mu\tau)\,(1+(8i+3)\eta\lambda\mu\tau)}{(1+(8i+2)\eta\lambda\mu\tau)\,(1+(8i+4)\eta\lambda\mu\tau)}\right).$$

Other expressions can be investigated in the same way. The proof has been completed. □

3.2. *Case 2*

In this subsection, we will find the solution of Eq. (1.1) when $\alpha=\beta=1$ and $\delta=-1$, so the Eq. (1.1) becomes

$$\Psi_{n+1}=\frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(1-\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})},\quad n=0,1,2,\ldots, \tag{3.2}$$

where the initial conditions $\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1}$, and Ψ_0 are positive real numbers and ($\Psi_0\Psi_{-1}\Psi_{-4}\Psi_{-5}$ $\notin\left\{\frac{1}{i}:i=1,2,3,\ldots\right\}$).

Theorem 3.2. *Suppose that* $\{\Psi_n\}_{n=-5}^{\infty}$ *be a solution of Eq.* (3.2) *. Thus for* $n = 0, 1, 2, \ldots,$

$$\Psi_{8n-5} = \tau \prod_{i=0}^{n-1} \left(\frac{(1-(8i)\eta\lambda\mu\tau)\,(1-(8i+2)\eta\lambda\mu\tau)}{(1-(8i+1)\eta\lambda\mu\tau)\,(1-(8i+3)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n-4} = \mu \prod_{i=0}^{n-1} \left(\frac{(1-(8i+1)\eta\lambda\mu\tau)\,(1-(8i+3)\eta\lambda\mu\tau)}{(1-(8i+2)\eta\lambda\mu\tau)\,(1-(8i+4)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n-3} = \zeta \prod_{i=0}^{n-1} \left(\frac{(1-(8i+2)\eta\lambda\mu\tau)\,(1-(8i+4)\eta\lambda\mu\tau)}{(1-(8i+3)\eta\lambda\mu\tau)\,(1-(8i+5)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n-2} = \sigma \prod_{i=0}^{n-1} \left(\frac{(1-(8i+3)\eta\lambda\mu\tau)\,(1-(8i+5)\eta\lambda\mu\tau)}{(1-(8i+4)\eta\lambda\mu\tau)\,(1-(8i+6)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n-1} = \lambda \prod_{i=0}^{n-1} \left(\frac{(1-(8i+4)\eta\lambda\mu\tau)\,(1-(8i+6)\eta\lambda\mu\tau)}{(1-(8i+5)\eta\lambda\mu\tau)\,(1-(8i+7)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n} = \eta \prod_{i=0}^{n-1} \left(\frac{(1-(8i+5)\eta\lambda\mu\tau)\,(1-(8i+7)\eta\lambda\mu\tau)}{(1-(8i+6)\eta\lambda\mu\tau)\,(1-(8i+8)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n+1} = \frac{\lambda\tau}{\zeta\,(1-\eta\lambda\mu\tau)} \prod_{i=0}^{n-1} \left(\frac{(1-(8i+6)\eta\lambda\mu\tau)\,(1-(8i+8)\eta\lambda\mu\tau)}{(1-(8i+7)\eta\lambda\mu\tau)\,(1-(8i+9)\eta\lambda\mu\tau)} \right),$$

$$\Psi_{8n+2} = \frac{\eta\mu\,(1-\eta\lambda\mu\tau)}{\sigma\,(1-2\eta\lambda\mu\tau)} \prod_{i=0}^{n-1} \left(\frac{(1-(8i+7)\eta\lambda\mu\tau)\,(1-(8i+9)\eta\lambda\mu\tau)}{(1-(8i+8)\eta\lambda\mu\tau)\,(1-(8i+10)\eta\lambda\mu\tau)} \right),$$

where $\Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda$, *and* $\Psi_0 = \eta$.

Proof. We can use the same steps used to prove Theorem 3.2. □

3.3. *Case 3*

In this subsection, we will find the solution of Eq. (1.1) when $\alpha = 1, \beta = -1$, and $\delta = 1$, so the Eq. (1.1) becomes

$$\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1+\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}, \quad n = 0, 1, 2, \ldots, \tag{3.3}$$

where the initial conditions $\Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$, and Ψ_0 are nonzero positive real numbers and $(\Psi_0\Psi_{-1}\Psi_{-4}\Psi_{-5} \neq 1)$.

Theorem 3.3. *Suppose that* $\{\Psi_n\}_{n=-5}^{\infty}$ *be a solution of Eq.* (3.3) *. Thus Eq.* (3.3) *has an unbounded solution and for* $n = 0, 1, 2, \ldots,$

$$\Psi_{8n-5} = \frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n-4} = \mu\,(-1+\eta\lambda\mu\tau)^{2n},$$

$$\Psi_{8n-3} = \frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n-2} = \sigma\,(-1+\eta\lambda\mu\tau)^{2n},$$

$$\Psi_{8n-1} = \frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n} = \eta\,(-1+\eta\lambda\mu\tau)^{2n},$$

$$\Psi_{8n+1} = \frac{\lambda\tau}{\zeta\,(-1+\eta\lambda\mu\tau)^{2n+1}}, \quad \Psi_{8n+2} = \frac{\eta\mu\,(-1+\eta\lambda\mu\tau)^{2n+1}}{\sigma},$$

where $\Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda$, *and* $\Psi_0 = \eta$.

Proof. By using mathematical induction, we will prove that the solution is true. First, for $n = 0$ the result holds. Second, we suppose that $n > 0$ and our assumption holds for $n-1$, that is

$$\Psi_{8n-13} = \frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n-2}}, \quad \Psi_{8n-12} = \mu(-1+\eta\lambda\mu\tau)^{2n-2},$$
$$\Psi_{8n-11} = \frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n-2}}, \quad \Psi_{8n-10} = \sigma(-1+\eta\lambda\mu\tau)^{2n-2},$$
$$\Psi_{8n-9} = \frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n-2}}, \quad \Psi_{8n-8} = \eta(-1+\eta\lambda\mu\tau)^{2n-2},$$
$$\Psi_{8n-7} = \frac{\lambda\tau}{\zeta(-1+\eta\lambda\mu\tau)^{2n-1}}, \quad \Psi_{8n-6} = \frac{\eta\mu(-1+\eta\lambda\mu\tau)^{2n-1}}{\sigma}.$$

Now, from Eq. (3.3) it follows that

$$\begin{aligned}\Psi_{8n-5} &= \frac{\Psi_{8n-7}\Psi_{8n-11}}{\Psi_{8n-9}(-1+\Psi_{8n-6}\Psi_{8n-7}\Psi_{8n-10}\Psi_{n-11})}\\ &= \frac{\frac{\lambda\tau}{\zeta(-1+\eta\lambda\mu\tau)^{2n-1}}\frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n-2}}}{\left(\frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n-2}}\right)\left(\begin{array}{c}-1+\frac{\eta\mu(-1+\eta\lambda\mu\tau)^{2n-1}}{\sigma}\frac{\lambda\tau}{\zeta(-1+\eta\lambda\mu\tau)^{2n-1}}\\ \sigma(-1+\eta\lambda\mu\tau)^{2n-2}\frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n-2}}\end{array}\right)} = \frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n-1}}\frac{1}{(-1+\eta\lambda\mu\tau)}.\end{aligned}$$

So, we have

$$\Psi_{8n-5} = \frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}.$$

Similarly,

$$\begin{aligned}\Psi_{8n-4} &= \frac{\Psi_{8n-6}\Psi_{8n-10}}{\Psi_{8n-8}(-1+\Psi_{8n-5}\Psi_{8n-6}\Psi_{8n-9}\Psi_{n-10})}\\ &= \frac{\frac{\eta\mu(-1+\eta\lambda\mu\tau)^{2n-1}}{\sigma}\sigma(-1+\eta\lambda\mu\tau)^{2n-2}}{\left(\eta(-1+\eta\lambda\mu\tau)^{2n-2}\right)\left(\begin{array}{c}-1+\frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}\frac{\eta\mu(-1+\eta\lambda\mu\tau)^{2n-1}}{\sigma}\\ \frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n-2}}\sigma(-1+\eta\lambda\mu\tau)^{2n-2}\end{array}\right)} = \frac{\mu(-1+\eta\lambda\mu\tau)^{2n-1}}{\left(\frac{1-\eta\lambda\mu\tau+\eta\lambda\mu\tau}{(-1+\eta\lambda\mu\tau)}\right)}.\end{aligned}$$

Thus, we get

$$\Psi_{8n-4} = \mu(-1+\eta\lambda\mu\tau)^{2n}.$$

Also,

$$\begin{aligned}\Psi_{8n-3} &= \frac{\Psi_{8n-5}\Psi_{8n-9}}{\Psi_{8n-7}(-1+\Psi_{8n-4}\Psi_{8n-5}\Psi_{8n-8}\Psi_{n-9})}\\ &= \frac{\frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}\frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n-2}}}{\left(\frac{\lambda\tau}{\zeta(-1+\eta\lambda\mu\tau)^{2n-1}}\right)\left(\begin{array}{c}-1+\mu(-1+\eta\lambda\mu\tau)^{2n}\frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}\\ \eta(-1+\eta\lambda\mu\tau)^{2n-2}\frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n-2}}\end{array}\right)} = \frac{\frac{1}{(-1+\eta\lambda\mu\tau)^{2n}}\frac{1}{(-1+\eta\lambda\mu\tau)^{2n-2}}}{\frac{1}{\zeta(-1+\eta\lambda\mu\tau)^{2n-2}}}.\end{aligned}$$

Hence, we obtain

$$\Psi_{8n-3} = \frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n}}.$$

Similarly, by using the same method, we can investigate other relations. □

Theorem 3.4. *Equation* (3.3) *has a periodic solution of period eight iff* $\eta\lambda\mu\tau = 2$ *and* $\{\Psi_n\}_{n=-5}^{\infty}$ *will take the form* $\left\{\tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \ldots\right\}$.

Proof. First, assume that there exists a prime period eight solution

$$\tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \ldots$$

of Eq. (3.3); from the form of solution of Eq. (3.3), we can see that

$$\tau = \frac{\tau}{(-1+\eta\lambda\mu\tau)^{2n}}, \quad \mu = \mu\,(-1+\eta\lambda\mu\tau)^{2n}, \quad \zeta = \frac{\zeta}{(-1+\eta\lambda\mu\tau)^{2n}}, \quad \sigma = \sigma\,(-1+\eta\lambda\mu\tau)^{2n},$$
$$\lambda = \frac{\lambda}{(-1+\eta\lambda\mu\tau)^{2n}}, \quad \eta = \eta\,(-1+\eta\lambda\mu\tau)^{2n}, \quad \frac{\lambda\tau}{\zeta} = \frac{\lambda\tau}{\zeta\,(-1+\eta\lambda\mu\tau)^{2n+1}}, \quad \frac{\eta\mu}{\sigma} = \frac{\eta\mu\,(-1+\eta\lambda\mu\tau)^{2n+1}}{\sigma},$$

this means that

$$(-1+\eta\lambda\mu\tau)^{2n} = 1.$$

Thus,

$$\eta\lambda\mu\tau = 2.$$

Second, suppose that $\eta\lambda\mu\tau = 2$. Then, we see from the form of the solution of Eq. (3.3) that

$$\Psi_{8n-5} = \tau, \quad \Psi_{8n-4} = \mu, \quad \Psi_{8n-3} = \zeta, \quad \Psi_{8n-2} = \sigma, \quad \Psi_{8n-1} = \lambda, \quad \Psi_{8n} = \eta, \quad \Psi_{8n+1} = \frac{\lambda\tau}{\zeta}, \quad \Psi_{8n+2} = \frac{\eta\mu}{\sigma}.$$

Thus, we have a periodic solution of period eight and the proof is complete. □

3.4. Case 4

In this subsection, we will find the solution of Eq. (1.1) when $\alpha = 1, \beta = -1$ and $\delta = -1$, so the Eq. (1.1) becomes

$$\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1-\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}, \quad n = 0, 1, 2, \ldots, \tag{3.4}$$

where the initial conditions $\Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$, and Ψ_0 are nonzero positive real numbers and $(\Psi_0\Psi_{-1}\Psi_{-4}\Psi_{-5} \neq -1)$.

Theorem 3.5. *Suppose that $\{\Psi_n\}_{n=-5}^{\infty}$ be a solution of Eq. (3.4) . Then Eq. (3.4) has an unbounded solution and for $n = 0, 1, 2, \ldots$,*

$$\Psi_{8n-5} = \frac{\tau}{(-1-\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n-4} = \mu\,(-1-\eta\lambda\mu\tau)^{2n},$$
$$\Psi_{8n-3} = \frac{\zeta}{(-1-\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n-2} = \sigma\,(-1-\eta\lambda\mu\tau)^{2n},$$
$$\Psi_{8n-1} = \frac{\lambda}{(-1-\eta\lambda\mu\tau)^{2n}}, \qquad \Psi_{8n} = \eta\,(-1-\eta\lambda\mu\tau)^{2n},$$
$$\Psi_{8n+1} = \frac{\lambda\tau}{\zeta\,(-1-\eta\lambda\mu\tau)^{2n+1}}, \quad \Psi_{8n+2} = \frac{\eta\mu\,(-1-\eta\lambda\mu\tau)^{2n+1}}{\sigma},$$

where $\Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda$, and $\Psi_0 = \eta$.

Proof. We can use the same steps used to prove Theorem 3.4. □

Theorem 3.6. *Equation (3.4) has a periodic solution of period eight iff $\eta\lambda\mu\tau = -2$ and $\{\Psi_n\}_{n=-5}^{\infty}$ will take the form $\left\{\tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \tau, \mu, \zeta, \sigma, \lambda, \eta, \frac{\lambda\tau}{\zeta}, \frac{\eta\mu}{\sigma}, \ldots\right\}$.*

Proof. The proof will be the same as the proof of Theorem 3.5. □

4. Numerical examples

In this section, we provide some of the numerical results to demonstrate the solution behavior of Eq. (1.1) for our prior results.

Example 4.1. In numerical simulation, we assume that for Eq. (3.1) the initial values are $\Psi_{-5} = 1.2, \Psi_{-4} = 5.5, \Psi_{-3} = 3.4, \Psi_{-2} = 6.8, \Psi_{-1} = 1.9$, and $\Psi_0 = 7.7$. Then the solution appears in Figure 1.

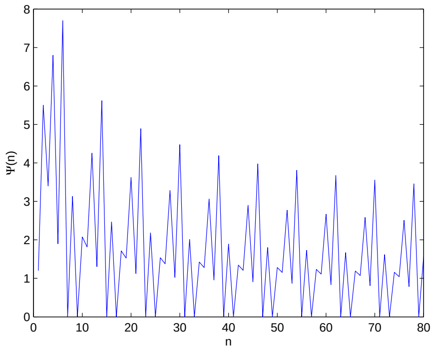

Figure 1: Plotting the solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(1+\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

Example 4.2. Numerically, we take the initial values are $\Psi_{-5} = 0.5, \Psi_{-4} = 0.21, \Psi_{-3} = 0.42, \Psi_{-2} = 0.85, \Psi_{-1} = 0.65$, and $\Psi_0 = 0.12$, the results of Eq. (3.2) are shown in Figure 2.

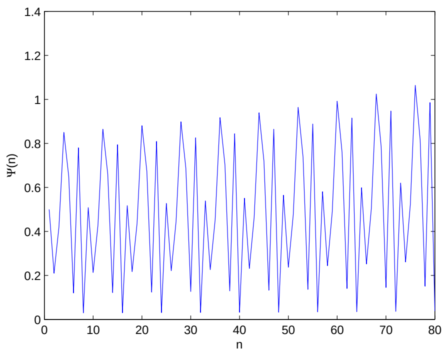

Figure 2: Plotting the solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(1-\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

Example 4.3. Figures 3 and 4 depict the behavior of Eq. (3.3), with Figure 3 indicating that the solution is unbounded where the initial conditions are $\Psi_{-5} = 0.8, \Psi_{-4} = 0.9, \Psi_{-3} = 1.5, \Psi_{-2} = -4, \Psi_{-1} = 1.8$, and $\Psi_0 = 0.4$, and Figure 4 indicating that the solution is periodic when the initial values are $\Psi_{-5} = 1/2, \Psi_{-4} = 5, \Psi_{-3} = 3, \Psi_{-2} = -2.5, \Psi_{-1} = 2$, and $\Psi_0 = 2/5$ in Eq. (3.3) are set to $\Psi_0\Psi_{-1}\Psi_{-4}\Psi_{-5} = 2$.

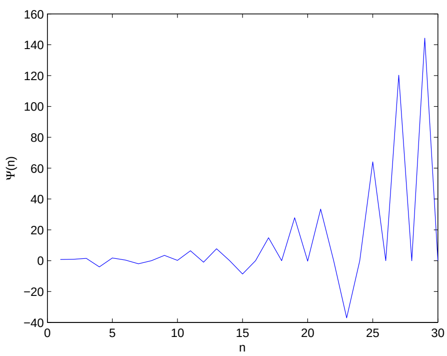

Figure 3: Unbounded solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1+\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

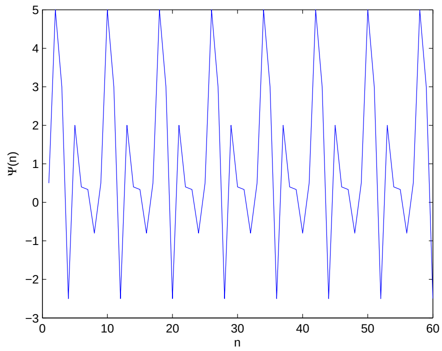

Figure 4: Periodic solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1+\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

Example 4.4. For Eq. (3.4) the initial conditions are set as follows: $\Psi_{-5} = -1.1, \Psi_{-4} = 0.8, \Psi_{-3} = 0.3, \Psi_{-2} = -2, \Psi_{-1} = 1.3$, and $\Psi_0 = 0.1$. And $\Psi_{-5} = 1/3, \Psi_{-4} = -2, \Psi_{-3} = 4.5, \Psi_{-2} = -2.5, \Psi_{-1} = 8$, and $\Psi_0 = 3/8$, and the results are shown in Figures 5 and 6.

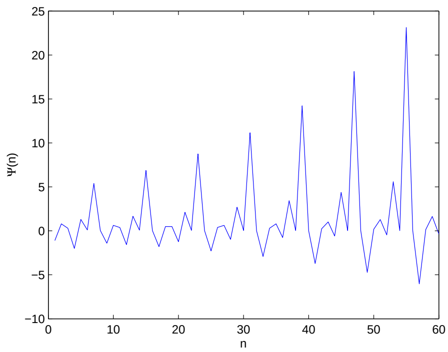

Figure 5: Unbounded solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1-\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

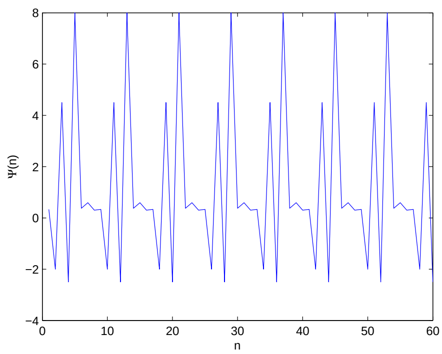

Figure 6: Periodic solution of the difference equation $\Psi_{n+1} = \frac{\Psi_{n-1}\Psi_{n-5}}{\Psi_{n-3}(-1-\Psi_n\Psi_{n-1}\Psi_{n-4}\Psi_{n-5})}$.

5. Conclusion

In this article, we have found the general form of the solutions of rational difference equations and we investigated the existence of positive equilibrium points. In Section 2, we investigated the solution's qualitative behavior, such as local and global stability. In Section 3, we found the solution's expressions to some special cases of the fractional recursive equation (1.1). In Cases 3 and 4, we had a periodic solution of period eight iff $\eta\lambda\mu\tau = 2$ and $\eta\lambda\mu\tau = -2$, respectively. Finally, some illustrative examples are provided to support our theoretical discussion. As future work, we can use the parameters α, β, and δ as a sequence or form a system of the same equation in multiple dimensions.

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