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Mohand transforms and its application to stability of differential equation

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Abstract

In this paper, we establish the Hyers-Ulam stability of a differential equation of higher order using a new transform technique called Mohand transforms.

Keywords: Hyers-Ulam stability, differential equation, Mohand transform.

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1. Introduction and preliminaries

The stability problem for various forms of a functional equations arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by Ulam [52] in 1940. A simulating and famous talk presented by Ulam [52] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Theorem 1.1 ([52]). *Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$\rho(f(xy), f(x)\, f(y)) < \delta,$$

for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with

$$\rho(f(x), h(x)) < \epsilon$$

for all $x \in G_1$?

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Since then, this question has attracted the attention of many researchers. If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [16] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings, when G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

Theorem 1.2 ([16])**.** *Assume that* G_1 *and* G_2 *are Banach spaces. If a function* $f : G_1 \to G_2$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

for some $\epsilon > 0$ *and for all* $x,\ y \in G_1$*, then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in G_1$ *and* $A : G_1 \to G_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\| \leqslant \epsilon \tag{1.2}$$

for all $x \in G_1$*. Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in G_1$*, then* A *is linear.*

Taking the above fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have *Hyers-Ulam stability* on (G_1, G_2). In the above Theorem, an additive function A satisfying the inequality (1.2) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the Theorem formulated by Hyers was generalized by Aoki [6] and Bourgin [8] for additive mappings.

There is no reason for the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [45] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam stability for the Additive Cauchy Equation. This terminology is justified because the Theorem of Rassias has strongly influenced mathematicians studying stability problems of functional equation. In fact, Rassias proved the following Theorem.

Theorem 1.3 ([45])**.** *Let* X *and* Y *be Banach spaces. Let* $\theta \in (0, \infty)$ *and let* $p \in [0, 1)$*. If a function* $f : X \to Y$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\ (\|x\|^p + \|y\|^p)$$

for all $x,\ y \in X$*, then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{2-2^p}\ \|x\|^p$$

for all $x \in X$*. Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$*, then* A *is linear.*

The findings of Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ualm-Rassias stability of functional equations. In 1991, Gajda [14] answered the question for $p > 1$, which was raised by Rassias [45]. This new concept is known as the Hyers-Ulam-Rassias stability of functional equations. The terminology, Hyers-Ulam-Rassias stability, is originated from these historical backgrounds. The terminology can also be applied to the case of other functional equations. In 1994, a further generalization of Rassias theorem was obtained by Gavruta [15].

The stability concept introduced by Rassias [45] is significantly influenced by a number of Mathematicians to investigate the stability problem for various functional equations and there are many interesting results concerning the Ulam stability problems in ([1–3, 9, 11, 12, 17, 18, 23, 25–27, 30–32, 34, 42, 46, 47, 49, 50]).

Let Y be a normed space and let I be an open interval. Assume that for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(t)y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) + h(t) \right\| \leqslant \epsilon$$

for all $t \in I$ and for some $\epsilon > 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(t)y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) + h(t) = 0$$

such that $\|f(t) - f_0(t)\| \leqslant K(\epsilon)$ for any $x \in I$, where $K(\epsilon)$ is an expression of ϵ only. Then, we say that the above differential equation has the Hyers-Ulam stability.

If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [40, 41]. Thereafter, in 1998, Alsina and Ger [5] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in the following Theorem.

Theorem 1.4 ([5])**.** *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality*

$$\left\| x'(t) - x(t) \right\| \leqslant \epsilon,$$

where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leqslant 3\epsilon$.

This result of Alsina and Ger [5] has been generalized by Takahasi [51]. They proved in [51] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [19–22, 33].

Now a days, the Hyers-Ulam stability of differential equations are investigated by number of authors in [10, 13, 24, 29, 36, 48, 53] and the Hyers-Ulam stability of differential equations has been given attention.

Similarly, many different methods for solving differential equations have been used to study the Hyers-Ulam stability problem for various differential equation. But using transform techniques are also have more significant advantage for solving differential equations with initial conditions. In 2014, Alqifiary and Jung [4] investigated the generalized Hyers-Ulam stability of

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \, x^{(k)}(t) \; = \; f(t),$$

by using the Laplace transform method. In 2020, Murali and Selvan [35] established the different forms of Mittag-Leffler-Hyers-Ulam stability of the first order linear differential equation for both homogeneous and non-homogeneous cases by using Laplace transformation (see also [7]). In 2020, Murali et al. [44] investigated the Hyers-Ulam stability of various differential equations using Fourier transform method (see also [43]). Recently, Jung et al. [28] established the various forms of Hyers-Ulam stability of the first-order linear differential equations with constant coefficients by using Mahgoub integral transform (see also [38]). Very recently, Murali et al. [39] investigated the different forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of second order linear differential equation of the form $u'' + \mu^2 u = q(t)$ by using Abooth transform method (see also [37]).

We may apply these terminologies to other differential equations also. In this paper, we will discuss a new transform technique named Mohand transform and we will study the definition of Mohand transform, inverse Mohand transform, and its convolution. Finally, we demonstrated that the differential equation is Hyers-Ulam stable by Mohand transform. That is, by applying Mohand transform method, we establish the stability of n^{th} order linear differential equation

$$x^{(n)}(r)+\sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(r)=\psi(r)$$

has Hyers-Ulam stability, where a_κ is a scalar, x and ψ are n times continuously differentiable function and of exponential order.

Now, we will introduce some notations, definitions, and preliminaries which are used throughout this paper.

The complex field $\mathbb{C}$ or the real field $\mathbb{R}$ are both referred to as $\mathbb{T}$ in this article. If $|\psi(s)| \leqslant Me^{-\omega r}$ and there exists a positive real constants M for every $s>0$, then a mapping $\psi:(0,\infty)\to\mathbb{T}$ of exponential order. The following is the definition of a Mohand transform of the function $\psi:(0,\infty)\to\mathbb{T}$ is defined by

$$\mathbb{M}(\psi)(\omega)=\omega^2\int_0^\infty \psi(r)\ e^{-\omega r}dr$$

and inverse Mohand transform is defined as follows:

$$\mathbb{M}^{-1}(T)(r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(a+ix)^2}e^{(a+ix)r}T(a+ix)dx.$$

The convolution of ψ and ϕ is denoted by $\psi * \phi$ and is defined as

$$(\psi*\phi)(r)=\int_0^r \psi(r-s)\phi(s)ds,$$

we note that

$$\mathbb{M}(\psi*\phi)=\frac{1}{\omega^2}\mathbb{M}(\psi)\mathbb{M}(\phi).$$

2. Hyers-Ulam stability of linear differential equations

Lemma 2.1. *Let* $P_1(\omega)=\sum\limits_{i=o}^{n} a_i\omega^i$ *and* $P_2(\omega)=\sum\limits_{j=0}^{m} b_j\omega^j$, *where* n *and* m *are non-negative integers with* $n>m$ *and* a_i, b_j *are scalars. Then there exists* $\phi:(0,\infty)\to\mathbb{T}$ *which is an infinitely differentiable mapping, such that*

$$\mathbb{M}(\phi)=\frac{P_2(\omega)}{P_1(\omega)}\quad (R(\omega)>d_q)$$

and

$$\phi^{(i)}(0)=\begin{cases}\frac{b_m}{a_n}, & i=n-m+1,\\ 0, & i=0,1,\dots,n-m,\end{cases}$$

where $d_q=\max\{R(\omega):P_1(\omega)=0\}$.

Proof. Take $\rho=n-m$, we write

$$P_1(\omega)=a_n(\omega-\omega_1)^{n_1}(\omega-\omega_2)^{n_2}\cdots(\omega-\omega_\kappa)^{n_\kappa},$$

where ω_i is a complex number $i=1,2.\dots,\kappa$ and n_j is an integer, with $n=n_1+\dots+n_\kappa$,

$$\frac{P_2(\omega)}{P_1(\omega)}=\sum_{i=1}^{\kappa}\sum_{j=1}^{n_i}\frac{\zeta_{ij}}{(\omega-\omega_i)^j},$$

where ζ_{ij} is a scalar. Let

$$\mu_{ij}(r) = \frac{1}{\omega(j-1)!} * r^{j-1} e^{\omega_i r},$$

where i, j are integers, $1 \leqslant i \leqslant \kappa$ and $1 \leqslant j \leqslant n_i$. Let

$$\phi(r) = \sum_{i=1}^{\kappa} \sum_{j=1}^{n_i} \zeta_{ij} \mu_{ij}(r).$$

Taking Mohand transform to $\phi(r)$ we get

$$\mathbb{M}(\phi) = \frac{P_2(\omega)}{P_1(\omega)}$$

for all ω with $R(\omega) > d_q$, where $d_q = \max\{R(\omega_i) : i = 1, 2, \dots, \kappa\}$. By Maclaurin series, we have

$$\phi(r) = \phi(0) + \phi'(0)r + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!} r^{n-1} + \mu(r),$$

where

$$\mu(r) = \sum_{i=n}^{\infty} \frac{\phi^{(i)}(0)}{i!} r^i.$$

Note that $\mathbb{M}(\mu) = \frac{P(\omega)}{\omega^n}$, where P is a complex function, then

$$\mathbb{M}(\phi) = \phi(0)\omega + \phi'(0) + \frac{\phi''(0)}{\omega} + \cdots + \frac{\phi^{(n-1)}(0)}{\omega^{n-2}} + \frac{P(\omega)}{\omega^{n-1}}.$$

Hence

$$\phi(0)\omega + \phi'(0) + \frac{\phi''(0)}{\omega} + \cdots + \frac{\phi^{(n-1)}(0)}{\omega^{n-2}} + \frac{P(\omega)}{\omega^{n-1}} = \frac{b_0 + b_1\omega + \cdots + b_m u^m}{a_0 + a_1\omega + \cdots + a_{m+\rho}\omega^{m+\rho}}.$$

- If $\rho \geqslant 0$, divide both sides of the above equation by ω and then let us take $\omega \to \infty$, we get $\phi(0) = 0$.
- If $\rho > 0$ and then let $\omega \to \infty$, we get $\phi'(0) = 0$.
- If $\rho > 1$ and multiplying both sides of the above equality by ω and letting $\omega \to \infty$, we get $\phi''(0) = 0$.

Continuing in this way we can reach $\phi(0) = \phi'(0) = \cdots = \phi^{(\rho)}(0) = 0$ and $\phi^{\rho+1}(0) = \frac{b_m}{a_n}$. Hence the proof. □

Lemma 2.2. *Let $n > 1$ be an integer, $\psi : (0, \infty) \to \mathbb{T}$ be a continuous mapping, and $P_1(\omega)$ be a n degree complex polynomial. Then there exists $\mu : (0, \infty) \to T$ which is an n times continuously differentiable function, such that*

$$\mathbb{M}(\mu) = \frac{\mathbb{M}(\psi)}{P_1(\omega)} \quad (R(\omega) > \max\{d_q, d_j\}),$$

where $d_q = \max\{R(\omega) : P_1(\omega) = 0\}$ and d_j is the abscissa of convergence for ψ. In particular, it satisfies that $\mu^{(i)}(0) = 0$ for all $i = 0, 1, 2, \dots, n-1$.

Proof. Let $P_2(\omega) = \omega^2$ and $P_1(\omega) = a_0 + a_1\omega + \cdots + a_n u^n$ in the above Lemma, then applying Mohand transform to $\phi : (0, \infty) \to \mathbb{T}$, we have

$$\mathbb{M}(\phi) = \frac{\omega}{P_1(\omega)} \quad (R(\omega) > d_q)$$

and if $i = 0, 1, 2, \dots, n-2$, then $\phi^{(i)}(0) = 0$, and $\phi^{(n-1)}(0) = \frac{1}{a_n}$. Now we define $\mu = \phi * \psi$, then we obtain $\mathbb{M}(\mu) = \frac{\mathbb{M}(\psi)}{P_1(\omega)}$, and we obtain

$$\mu'(t) = \phi(0)\psi(r) + \int_0^r \phi'(r-s)\psi(s)ds = \int_0^r \phi'(r-s)\psi(s)d(s),$$

then we have

$$\mu^{(i)}(r) = \phi^{(i-1)}(0)\psi(r) + \int_0^r \phi^{(i)}(r-s)\psi(s)ds = \int_0^r \phi^{(i)}(r-s)\psi(s)ds$$

for all $i = 1, 2, \ldots, n-1$. Hence $\mu(0) = \mu'(0) = \cdots = \mu^{(n-1)} = 0$. □

Theorem 2.3. *Let a be a scalar. If a function $x : (0, \infty) \to \mathbb{T}$ satisfies the inequality*

$$|x'(r) + ax(r) - \psi(r)| \leqslant \epsilon \tag{2.1}$$

for every $r > 0$ and $\epsilon > 0$, then there exists a solution $x_a : (0, \infty) \to \mathbb{T}$ of the differential equation

$$x'(r) + ax(r) = \psi(r), \tag{2.2}$$

such that

$$|x_a(r) - x(r)| \leqslant \begin{cases} \frac{(1-e^{-R(a)r})\epsilon}{R(a)}, & R(a) \neq 0, \\ \epsilon r, & R(a) = 0, \end{cases}$$

for all $r > 0$.

Proof. Let $\lambda(r) = x'(r) + ax(r) - \psi(r)$, for all $r > 0$, applying Mohand transform we get

$$M(\lambda) = uM(x) - \frac{x(0)}{u} + aM(x) - M(\psi)$$

and so

$$\mathbb{M}(x) - \frac{x(0)\omega^2 + \mathbb{M}(\psi)}{\omega + a} = \frac{\mathbb{M}(\lambda)}{\omega + a}. \tag{2.3}$$

Let $x_a(r) = x(0)e^{-ar} + (E_{-a} * \psi)r$, where $E_{-a}(r) = e^{-ar}$, so $x_a(0) = x(0)$, then

$$\mathbb{M}(x_a) = \frac{x(0)\omega^2 + \mathbb{M}(\psi)}{\omega + a} = \frac{x_a(0)\omega^2 + \mathbb{M}(\psi)}{\omega + a}. \tag{2.4}$$

Hence

$$\mathbb{M}[x_a'(r) + ax_a(r)] = \mathbb{M}(\psi).$$

Since $\mathbb{M}$ is injective, so $x_a'(r) + ax_a(r) = \psi(r)$. Therefore, x_a is a solution of (2.2). Using (2.3) and (2.4), we have

$$\mathbb{M}[E_{-a} * \lambda] = \frac{\mathbb{M}(\lambda)}{\omega + a}.$$

So, $\mathbb{M}(x) - \mathbb{M}(x_a) = \mathbb{M}[E_{-a} * \lambda]$ and $x(r) - x_a(r) = (E_{-a} * \lambda)(r)$. By (2.1), we get $|\lambda(r)| \leqslant \epsilon$. By using convolution of Mohand transforms, we get

$$|x(r) - x_a(r)| = |(E_{-a} * \lambda)(r)| \leqslant \epsilon e^{-R(a)}r \int_0^r e^{-R(a)s} ds.$$

Hence the proof. □

Theorem 2.4. *Let a_i be scalars, where $i = 0, 1, \ldots, n-1$ and an integer $n > 1$. Then there exists a constant $N > 0$ such that for all mappings $x : (0, \infty) \to \mathbb{T}$ satisfying*

$$\left| x^{(n)}(r) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(r) - \psi(r) \right| \leqslant \epsilon \tag{2.5}$$

for every $r > 0$ and $\epsilon > 0$, there exists $x_a : (0, \infty) \to \mathbb{T}$ which is a solution of the differential equation

$$x^{(n)}(r) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(r) = \psi(r) \tag{2.6}$$

such that

$$|x_a(r) - x(r)| \leqslant \epsilon N \frac{e^{ar}}{a}$$

for every $r > 0$ and $a > \max[0, d_q, d_j]$, where d_q is defined in (2.9).

Proof. Using integration by parts repeatedly, we get

$$\mathbb{M}[x^{(n)}] = \omega^n\ \mathbb{M}(x) - \sum_{j=1}^{n} \omega^{n+2-j} x^{(j-1)}(0).$$

Let $a_n = 1$. So x_0 is a solution of (2.6) if and only if

$$\mathbb{M}(\psi) = \tau_{n,0}(\omega)\mathbb{M}(x_0) - \sum_{j=1}^{n} \tau_{n,j}(\omega) x_0^{(j-1)}(0)\omega^2, \tag{2.7}$$

where $\tau_{n,j}(\omega) = \sum\limits_{\kappa=j}^{n} a_\kappa \omega^{\kappa-j}$ for $j = 0, 1, 2, \ldots, n$. We consider

$$\mu(r) = x^n(r) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(r) - \psi(r)$$

for every $r > 0$. Then

$$\mathbb{M}(\mu) = \tau_{n,0}(\omega)\mathbb{M}(x) - \sum_{j=1}^{n} \tau_{n,j}(\omega) x^{(j-1)}(0)\omega^2 - \mathbb{M}(\psi).$$

Hence we get

$$\mathbb{M}(x) - \frac{1}{\tau_{n,0}(\omega)} \left[\sum_{j=1}^{n} \tau_{n,j}(\omega) x^{(j-1)}(0)\omega^2 + \mathbb{M}(\psi) \right] = \frac{\mathbb{M}(\mu)}{\tau_{n,0}(\omega)}. \tag{2.8}$$

Let d_j be the abscissa of convergence for ψ. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the roots of $\tau_{n,0}$ and let

$$d_p = \max\{R(\omega_\kappa) : \kappa = 1, 2, \ldots, n\}. \tag{2.9}$$

For any ω with $R(\omega) > \max\{d_q, d_j\}$, we define

$$\Omega(\omega) = \frac{1}{\tau_{n,0}(\omega)} \left[\sum_{j=1}^{n} \tau_{n,j}(\omega) x^{(j-1)}(0)\omega^2 + \mathbb{M}(\psi) \right]. \tag{2.10}$$

By Lemma 2.2, we have

$$\mathbb{M}(\psi_0) = \frac{\mathbb{M}(\psi)}{\tau_{n,0}(\omega)}$$

for every ω with $R(\omega) > \max\{d_q, d_j\}$ and $\psi_0(0) = \psi_0'(0) = \cdots = \psi_0^{(n-1)}(0) = 0$ for $j = 1, 2, \ldots, n$. So we have

$$\frac{\tau_{n,j}(\omega)}{\tau_{n,0}(\omega)} = \frac{1}{\omega^j} - \frac{\sum\limits_{\kappa=0}^{j-1} a_\kappa \omega^\kappa}{\omega^j\,(\tau_{n,o}(\omega))}$$

for all ω with $R(\omega) > \max\{0, d_q\}$. By Lemma 2.1, we have

$$P_2(\omega) = \sum_{\kappa=0}^{j-1} a_\kappa \omega^\kappa$$

and $P_1(\omega) = \omega^j \tau_{n,0}(\omega)$. For a differentiable function ϕ_j, we have

$$\mathbb{M}(\phi_j) = \frac{\sum\limits_{\kappa=0}^{j-1} a_\kappa \omega^\kappa}{\omega^j \tau_{n,o}(\omega)}$$

and $\phi_j(0) = \phi_j'(0) = \cdots = \phi_j^{(n-1)} = 0$. Let

$$\psi_j(r) = \frac{r^{j-1}}{(j-1)!} - \phi_j(r)\omega^2 \tag{2.11}$$

for $j = 1, 2, \ldots, n$. Then we get

$$\psi_j^{(i)}(0) = \begin{cases} 0, & i = 0, 1, 2, \ldots, j-2, j, j+1, \ldots, n-1, \\ 1, & i = j-1. \end{cases}$$

If we define

$$x_a(r) = \sum_{j=1}^{n} x^{j-1}(0)\psi_j(r) + \psi_0(r),$$

then we get $x_a^{(i)}(0) = x^{(i)}(0)$ for all $i = 0, 1, 2, \ldots, n-1$. Using (2.10), (2.11) and $\mathbb{M}(x_a) = \Omega(\omega)$, we get

$$\mathbb{M}(x_a) = \frac{1}{\tau_{n,0}(\omega)} \left[\sum_{j=1}^{n} \tau_{n,j}(\omega) x_a^{(j-1)}(0)\omega^2 + \mathbb{M}(\phi) \right] \tag{2.12}$$

for all ω with $R(\omega) > \max\{0, d_q, d_j\}$. Now using (2.7) we get x_0 is a solution of (2.6). Again by considering (2.8) and (2.12), we get

$$\mathbb{M}(x) - \mathbb{M}(x_a) = \frac{\mathbb{M}(\mu)}{\tau_{n,0}(\omega)}$$

and so

$$|x(r) - x_a(r)| = \left| \mathbb{M}^{-1}\left(\frac{\mathbb{M}(\mu)}{\tau_{n,0}(\omega)} \right) \right|$$

for $r > 0$. Using (2.5) and the definition of μ, it satisfies that $|\mu(r)| \leqslant \epsilon$ for all $r > 0$ and so

$$|\mathbb{M}(\mu)| \leqslant \int_0^\infty \left|e^{-\omega r}\right| |\mu(r)| dr \leqslant \frac{\epsilon}{R(\omega)}$$

for all ω with $R(\omega) > 0$. Finally, it follows from the formula for the inverse Mohand transform, we get

$$\begin{aligned} |x(r) - x_a(r)| &= \left| \mathbb{M}^{-1}\left(\frac{\mathbb{M}(\mu)}{\tau_{n,0}(\omega)}\right)\right| \\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \frac{e^{(a+ix)r}\mathbb{M}(\mu)(a+ix)}{(a+ix)^2\tau_{n,0}(a+ix)}dx\right| \\ &\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ar}\frac{\epsilon}{a}\frac{1}{|a+ix|^2|\tau_{n,0}(a+ix)|}dx \\ &\leqslant \frac{\epsilon e^{ar}}{2\pi a}\int_{-\infty}^{\infty}\frac{1}{|a+ix|^2|\tau_{n,0}(a+ix)|}dx \leqslant \epsilon N\frac{e^{ar}}{a} \end{aligned}$$

for every $r > 0$ and any $a > \max\{0, d_q, d_j\}$, where

$$N = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|a+ix|^2|\tau_{n,0}(a+ix)|}dx < \infty$$

because an integer $n > 1$. □

3. Conclusion

In this paper, we established the Hyers-Ulam stability of a differential equation by applying a new transform techniques, namely, Mohand transforms. We established the sufficient criteria for Hyers-Ulam stability of the linear differential equation of higher order by using Mohand transforms. Additionally, this paper also provides another method to study the Hyers-Ulam stability of the higher order linear differential equation and shows that the Mohand transform method is more convenient to study the Hyers-Ulam stability of the linear differential equation.

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