J. Math. Computer Sci., 35 (2024), 35–51

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Stability and dependency of the solution of a nonlinear quadratic functional integral inclusion with distributed delay

Ahmed M. A. El-Sayed[a], Nesreen F. M. El-Haddad[b,*]

[a]*Faculty of Science, Alexandria University, Alexandria, Egypt.*

[b]*Faculty of Science, Damanhour University, Behera, Egypt.*

Abstract

Let $\mathcal{E}$ be a reflexive Banach space. In this paper, we are interested in the solvability of the nonlinear quadratic functional integral inclusion with distributed delay

$$x(t) \in \mathcal{F}(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds), \quad t \in [0, \infty),$$

on the real half-axis. Our thought is found within the space $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of bounded continuous functions on the real half-axis $\mathcal{R}_+$ and taking values in a reflexive Banach space $\mathcal{E}$ beneath the presumption that the multi-valued function $\mathcal{F}$ fulfill Lipschitz condition in $\mathcal{E}$. The main tool applied in this work is the procedure associated with measures of non-compactness in the space $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ by a given norm of continuity and using Darbo's fixed point theorem. The asymptotic stability and the asymptotic dependency of the solution will be studied. We additionally provide an example to demonstrate the effectiveness and value of our results.

Keywords: Multi-valued function, quadratic functional integral inclusion, measure of non-compactness, Lipshitz condition, reflexive Banach space, asymptotic stability, asymptotic dependency.

2020 MSC: 28B20, 54C60, 47H30.

1. Introduction

Let $\mathcal{E}$ be a reflexive Banach space with norm $\|.\|_{\mathcal{E}}$ and let $\mathcal{R}_+ = [0, \infty)$. Denote by $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ the Banach space of all functions defined, continuous, and bounded on the real half-axis $\mathcal{R}_+$ and taking values in a given Banach space $\mathcal{E}$. The norm of $\rho \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ is defined by

$$\|\rho\|_{\mathcal{BC}} = \sup_{t \in \mathcal{R}_+} \|\rho(t)\|_{\mathcal{E}}.$$

*Corresponding author

Email addresses: amasayed@gmail.com and amasayed@alexu.edu.eg (Ahmed M. A. El-Sayed), nesreen_fawzy20@sci.dmu.edu.eg and nesreen_fawzy20@yahoo.com (Nesreen F. M. El-Haddad)

doi: 10.22436/jmcs.035.01.04

Received: 2024-01-20 *Revised:* 2024-02-25 *Accepted:* 2024-03-05

The theory of measures of non-compactness plays an important role in applications to the nonlinear analysis and to the theories of differential and integral equations (see [2, 4, 7–10, 12, 13]). It also applies to control theory and the operator theory (see [2, 9, 15]). Measures of non-compactness are frequently employed in fixed point theory. It is worthwhile mentioning that Darbo's fixed point theorem and the measures of non-compactness create a powerful and convenient procedure that is very applicable in establishing theorems of existence for various types of operator equations (functional integral, integral, and differential). For solvability of bounded domain see [11, 29]. In [1, 6, 8, 20, 21, 30] the authors investigation the integral equations on different spaces of functions on the real half-axis. Moreover, in [19, 38] authors examine the solvability of non-linear 2D Volterra integral equations through Petryshyn fixed point theorem in Banach space and two systems of nonlinear Volterra integral equation and Volterra integro-differential equation through Banach's contraction principle and considered some properties of this solution. In [39] authors delve into a nonlinear integral equation (IEq) with multiple variable time delays and a nonlinear integro-differential equation (IDEq) without delay by the fixed point method using progressive contractions and some properties of solutions of that IEq with multiple variable time delays and IDEq are investigated. A multi-valued functional equations and multi-valued differential equations have been extensively investigated by some authors. There are many interesting results concerning the existence and properties of these problems (see [18, 31–33, 35, 36]). Also, a functional integral inclusion was studied by Dhage and O'Regan, they proved the existence of extremal solutions using Caratheodory's conditions on the multi-valued function (see [5, 16, 22, 25, 28, 34]) and in our article, we establish our results using Lipschitz condition on the multi-valued function. The existence theorems for the inclusions problems are generally obtained under the assumptions that the multi-valued function is either lower or upper semi-continuous on the domain of its definitions (see [3, 17, 34]) and for the discontinuity of the multi-valued function (see [26]). The Lipschitz selections of the multi-valued functions was investigated by a number of authors (see [23, 37]).

Here, we will apply the theory of measure of non-compactness to study the solutions, the asymptotic stability and the asymptotic dependency of the solution for the nonlinear quadratic functional integral inclusion with distributed delay

$$x(t) \in \mathcal{F}(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s)) ds), \quad t \in \mathcal{R}_+, \tag{1.1}$$

in $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ and using Darbo's fixed point theorem, where $\mathcal{F} : \mathcal{R}_+ \times \mathcal{E} \to \Omega(\mathcal{E})$ is a nonlinear multi-valued mapping and $\Omega(\mathcal{E})$ denote the power set of nonempty subsets of the Banach space $\mathcal{E}$, where the function $\varphi(t)$ is the delay function which is continuous function and $\varphi(t) \leqslant t$.

2. Preliminaries

Here, we display a few documentations and assistant comes about that will be required in this work. First, we state the Darbo's fixed point theorem.

Theorem 2.1 ([24]). *Assume that $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}$ is continuous operator and $\mathcal{Q}$ is a nonempty closed bounded convex subset of the space $\mathcal{E}$ with $\mu(\mathcal{A}X) \leqslant \mathcal{K}\mu(X)$ for any nonempty subset X of $\mathcal{Q}$, where the constant $\mathcal{K} \in [0, 1)$. Then $\mathcal{A}$ has a fixed point in the set $\mathcal{Q}$.*

Now, let $\mathcal{E}$ be a Banach space and let $x : \mathcal{I} \to \mathcal{E}$.

Definition 2.2 ([27]). A multi-valued map $\mathcal{F}$ from $\mathcal{I} \times \mathcal{E}$ to the family of all nonempty closed subsets of $\mathcal{E}$ is called Lipschitzian if there exists $\mathcal{L} > 0$ such that for all $t_1, t_2 \in \mathcal{I}$ and all $x_1, x_2 \in \mathcal{E}$, we have

$$\mathcal{H}(\mathcal{F}(t_1, x_1), \mathcal{F}(t_2, x_2)) \leqslant \mathcal{L}\{|t_1 - t_2| + \|x_1 - x_2\|_{\mathcal{E}}\},$$

where $\mathcal{H}(\ell, \jmath)$ is the Hausdorff metric between the two subsets $\ell, \jmath \in \mathcal{I} \times \mathcal{E}$.

Denote $\mathcal{S}_{\mathcal{F}} = \mathrm{Lip}(\mathcal{I}, \mathcal{E})$ be the set of Lipschitz selections of $\mathcal{F}$.

3. Main result

In this area, we present our fundamental result by demonstrating the existence of solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the nonlinear quadratic functional integral inclusion with distributed delay (1.1).

Definition 3.1. By a solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the nonlinear quadratic functional integral inclusion with distributed delay (1.1) in the reflexive Banach space $\mathcal{E}$, we mean a single-valued function, which fulfills (1.1).

Consider now the following assumptions.

(H1) The set $\mathcal{F}(t, y)$ is compact and convex for all $(t, y) \in \mathcal{R}_+ \times \mathcal{E}$.

(H2) The multi-valued map $\mathcal{F}$ is Lipschitzian with a Lipschitz constant $\mathcal{L}_1 > 0$ such that

$$\mathcal{H}(\mathcal{F}(t_1, y), \mathcal{F}(t_2, z)) \leqslant \mathcal{L}_1(|t_1 - t_2| + \|y - z\|_{\mathcal{E}})$$

for all $t_1, t_2 \in \mathcal{R}_+$ and $y, z \in \mathcal{E}$, where $\mathcal{H}(\ell, \jmath)$ is the Hausdorff metric between the two subsets $\ell, \jmath \in \mathcal{R}_+ \times \mathcal{E}$.

(H3) The set of Lipschitz selections $\mathcal{S}_{\mathcal{F}}$ of the multi-valued function $\mathcal{F}$ is nonempty.

(H4) $u : \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{E} \to \mathcal{R}_+$ is continuous function and there exist continuous functions $k(t, s) : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ and $b(s) : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$|u(t, s, x(s))| \leqslant |k(t, s)| + |b(s)| \|x(s)\|_{\mathcal{E}}, \quad \forall t, s \in \mathcal{R}_+,$$

where

$$\sup_{t \in \mathcal{R}_+} \int_0^t |k(t, s)| ds = \mathcal{K}, \qquad \lim_{t \to \infty} \int_0^t |k(t, s)| ds = 0$$

and

$$\sup_{t \in \mathcal{R}_+} \int_0^t |b(s)| ds = \mathcal{B}, \qquad \lim_{t \to \infty} \int_0^t |b(s)| ds = 0.$$

(H5) $g : \mathcal{R}_+ \times \mathcal{E} \to \mathcal{E}$ satisfies Lipschitz condition and there exists a function $a_2(t)$ and a constant $\mathcal{L}_2 > 0$ such that

$$\|g(t, x(t))\|_{\mathcal{E}} \leqslant \|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2 \|x(t)\|_{\mathcal{E}}.$$

(H6) The function $\varphi : \mathcal{I} \to \mathcal{I}$ is continuous function and $\varphi(t) \leqslant t$.

(H7) There exists a positive real number r of the algebraic equation

$$\mathcal{L}_1 \mathcal{L}_2 b r^2 \mathcal{B} + (\mathcal{L}_1 \mathcal{L}_2 \mathcal{K} + \mathcal{L}_1 \|a_2\|_{\mathcal{BC}} \mathcal{B} - 1) r + \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1 \|a_2\|_{\mathcal{BC}} \mathcal{K} = 0.$$

Now, let $X \subseteq \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ and $\mathcal{Q} = \{x : x \in X\}$. Define the following norm of continuity:

$$\omega^{\mathcal{T}}(x, \varepsilon) = \sup\{\|x(t) - x(\tau)\|_{\mathcal{E}} : \ t, \tau \in \mathcal{I}, \ |t - \tau| \leqslant \varepsilon\}, \quad \omega^{\mathcal{T}}(X, \varepsilon) = \sup\{\omega^{\mathcal{T}}(x, \varepsilon), \ x \in X\},$$

and

$$\omega_0^{\mathcal{T}}(X) = \lim_{\varepsilon \to 0} \omega^{\mathcal{T}}(X, \varepsilon), \quad \omega_0(X) = \lim_{\mathcal{T} \to \infty} \omega_0^{\mathcal{T}}(X).$$

In addition for $t \in \mathcal{R}_+$,

$$\operatorname{diam} X(t) = \sup\{\|x(t) - y(t)\|_{\mathcal{E}} : \ x, y \in X\},$$

and

$$\mathcal{C}(X) = \lim_{t \to \infty} \sup \operatorname{diam} X(t).$$

Finally, the measure of non-compactness on $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ is given by

$$\mu(X) = \omega_0(X) + \mathcal{C}(X) = \omega_0(X) + \lim_{t \to \infty} \sup \ \operatorname{diam} X(t).$$

Remark 3.2. From assumptions (H1)-(H3), there exists a Lipschiz selection $f \in \mathcal{S}_{\mathcal{F}}$; $f : \mathcal{R}_+ \times \mathcal{E} \to \mathcal{E}$ such that

$$\|f(t, y(t))\|_{\mathcal{E}} \leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|y(t)\|_{\mathcal{E}},$$

this selection satisfies

$$x(t) = f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds), \quad t \in \mathcal{R}_+. \tag{3.1}$$

Then the solution of the nonlinear quadratic functional integral equation with distributed delay (3.1), if it exists, is a solution of the nonlinear quadratic functional integral inclusion with distributed delay (1.1).

Now, we seek about the existence of solution of the nonlinear quadratic functional integral equation with distributed delay (3.1).

Theorem 3.3. *Let the assumptions* (H1)-(H7) *be satisfied. Then the nonlinear quadratic functional integral equation with distributed delay* (3.1) *has at least one solution* $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$.

Proof. Define the operator $\mathcal{A}$ by

$$\mathcal{A}x(t) = f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds), \quad t \in \mathcal{R}_+.$$

Let the set $\mathcal{Q}_r$ defined by

$$\begin{aligned}
&\mathcal{Q}_r = \{x : \ x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E}), \ \ \|x\|_{\mathcal{BC}(\mathcal{R}_+,\mathcal{E})} \leqslant r\},\\
&r = \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{K} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}} r\mathcal{B} + \mathcal{L}_1\mathcal{L}_2 r^2\mathcal{B}.
\end{aligned}$$

Then, it is clear that it is nonempty, closed, bounded, and convex subset of the space $\mathcal{E}$. Let $x \in \mathcal{Q}_r$ be arbitrary, then

$$\begin{aligned}
\|\mathcal{A}x(t)\|_{\mathcal{E}} &= \|f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds)\|_{\mathcal{E}}\\
&\leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds\|_{\mathcal{E}}\\
&\leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|g(t, x(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |u(t, s, x(s))|ds\\
&\leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|g(t, x(t))\|_{\mathcal{E}} \int_0^{t} |u(t, s, x(s))|ds\\
&\leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\} \int_0^{t} \{|k(t, s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds\\
&\leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\}\int_0^{t} |k(t, s)|ds + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\} \int_0^{t} |b(s)|\|x(s)\|_{\mathcal{E}}\, ds\\
&\leqslant \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x\|_{\mathcal{BC}}\} \sup_{t\in\mathcal{R}_+} \int_0^{t} |k(t, s)|ds\\
&\quad + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x\|_{\mathcal{BC}}\}\|x\|_{\mathcal{BC}} \sup_{t\in\mathcal{R}_+} \int_0^{t} |b(s)|ds\\
&\leqslant \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x\|_{\mathcal{BC}}\}\mathcal{K} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x\|_{\mathcal{BC}}\}\|x\|_{\mathcal{BC}}\mathcal{B}\\
&\leqslant \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\|x\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}}\|x\|_{\mathcal{BC}}\mathcal{B} + \mathcal{L}_1\mathcal{L}_2(\|x\|_{\mathcal{BC}})^2\mathcal{B}.
\end{aligned}$$

Therefore

$$\|\mathcal{A}x\|_{\mathcal{BC}} \leqslant \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{K} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}} r\mathcal{B} + \mathcal{L}_1\mathcal{L}_2 r^2\mathcal{B} = r,$$

where $r = \|a_1\|_{\mathcal{BC}} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{K} + \mathcal{L}_1\|a_2\|_{\mathcal{BC}} r\mathcal{B} + \mathcal{L}_1\mathcal{L}_2 r^2\mathcal{B}$. Then $\|\mathcal{A}x\|_{\mathcal{BC}} \leqslant r$. Hence, $\mathcal{A}x \in \mathcal{Q}_r$, which proves that $\mathcal{A}\mathcal{Q}_r \subset \mathcal{Q}_r$ and $\mathcal{A} : \mathcal{Q}_r \to \mathcal{Q}_r$. Now, we will show that $\mathcal{A}$ is continuous on the ball $\mathcal{Q}_r$. Let $\{x_n\}$ be a sequence in $\mathcal{Q}_r$ that converges to x, $\forall t \in \mathcal{R}_+$ in $\mathcal{Q}_r$, i.e., $x_n \to x$, $\forall t \in \mathcal{R}_+$. Now

$$\|f(t, g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds)\|_{\mathcal{E}} \leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds\|_{\mathcal{E}}$$

and $x_n \to x$, then $f(t, g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds) \to f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds)$. Since

$$\mathcal{A}x_n(t) = f(t, g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds), \quad t \in \mathcal{R}_+,$$

then

$$\begin{aligned}
&\|\mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \\
&= \|f(t, g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds) - f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds)\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds - g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t, x_n(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds - g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x_n(s))ds - g(t, x(t)) \int_0^{\varphi(t)} u(t, s, x(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t, x_n(t)) - g(t, x(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |u(t, s, x_n(s))|ds + \mathcal{L}_1\|g(t, x(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |u(t, s, x_n(s)) - u(t, s, x(s))|ds \\
&\leqslant \mathcal{L}_1\|g(t, x_n(t)) - g(t, x(t))\|_{\mathcal{E}} \int_0^t |u(t, s, x_n(s))|ds + \mathcal{L}_1\|g(t, x(t))\|_{\mathcal{E}} \int_0^t |u(t, s, x_n(s)) - u(t, s, x(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_n(t) - x(t)\|_{\mathcal{E}} \int_0^t |u(t, s, x_n(s))|ds + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\} \int_0^t |u(t, s, x_n(s)) - u(t, s, x(s))|ds \qquad (3.2) \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_n(t) - x(t)\|_{\mathcal{E}} \int_0^t |u(t, s, x(s))|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\} \int_0^t |u(t, s, x(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_n(t) - x(t)\|_{\mathcal{E}} \int_0^t \{|k(t, s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x(t)\|_{\mathcal{E}}\} \int_0^t \{|k(t, s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_n - x\|_{\mathcal{BC}} \sup_{t\in\mathcal{R}_+} \int_0^t |k(t, s)|ds + \mathcal{L}_1\mathcal{L}_2\|x_n - x\|_{\mathcal{BC}}\|x\|_{\mathcal{BC}} \sup_{t\in\mathcal{R}_+} \int_0^t |b(s)|ds \\
&\quad + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2\|x\|_{\mathcal{BC}}\} \int_0^t |k(t, s)|ds + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2\|x\|_{\mathcal{BC}}\}\|x\|_{\mathcal{BC}} \int_0^t |b(s)|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\} \int_0^t |k(t, s)|ds + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\} r \int_0^t |b(s)|ds,
\end{aligned}$$

select $\mathcal{T} > 0$ such that the following inequality holds for $t > \mathcal{T}$,

$$2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\} \int_0^t |k(t, s)|ds \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\} r \int_0^t |b(s)|ds \leqslant \frac{\varepsilon}{2}.$$

Discuss the following two ideas.

(1) If $t \geqslant \mathcal{T}$, we obtain
$$\|\mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \varepsilon = \varepsilon.$$

(2) If $t \leqslant \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = \sup\{|u(t,s,x_n(s)) - u(t,s,x(s))| : t,s \in \mathcal{I},\ x,x_n \in [-r,r],\ \|x_n(s) - x(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $u(t,s,x(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (3.2),

$$\|\mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\mathcal{T}\omega(\varepsilon).$$

Hence

$$\|\mathcal{A}x_n - \mathcal{A}x\|_{\mathcal{BC}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\mathcal{T}\omega(\varepsilon).$$

Consequently, from the two ideas (1) and (2) and the above established facts, we can deduce that $\mathcal{A}x_n \to \mathcal{A}x$, $\forall x_n \to x$, and the operator $\mathcal{A}$ is continuous on $\mathcal{Q}_r$. Now, take a nonempty set $X \subseteq \mathcal{Q}_r$. Then for any $x_1, x_2 \in X$ and fixed $t \geqslant 0$, we obtain

$$\begin{aligned}
&\|\mathcal{A}x_1(t) - \mathcal{A}x_2(t)\|_{\mathcal{E}} \\
&= \|f(t,g(t,x_1(t))\int_0^{\varphi(t)} u(t,s,x_1(s))ds) - f(t,g(t,x_2(t))\int_0^{\varphi(t)} u(t,s,x_2(s))ds)\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t,x_1(t))\int_0^{\varphi(t)} u(t,s,x_1(s))ds - g(t,x_2(t))\int_0^{\varphi(t)} u(t,s,x_2(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t,x_1(t))\int_0^{\varphi(t)} u(t,s,x_1(s))ds - g(t,x_2(t))\int_0^{\varphi(t)} u(t,s,x_1(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x_2(t))\int_0^{\varphi(t)} u(t,s,x_1(s))ds - g(t,x_2(t))\int_0^{\varphi(t)} u(t,s,x_2(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t,x_1(t)) - g(t,x_2(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x_1(s))|ds + \mathcal{L}_1\|g(t,x_2(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x_1(s)) - u(t,s,x_2(s))|ds \\
&\leqslant \mathcal{L}_1\|g(t,x_1(t)) - g(t,x_2(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_1(s))|ds + \mathcal{L}_1\|g(t,x_2(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_1(s)) - u(t,s,x_2(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_1(t) - x_2(t)\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_1(s))|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_2(t)\|_{\mathcal{E}}\}\int_0^{t} |u(t,s,x_2(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_1(t) - x_2(t)\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|x_1(s)\}_{\mathcal{E}}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_2(t)\|_{\mathcal{E}}\}\int_0^{t} \{|k(t,s)| + |b(s)|\|x_2(s)\|_{\mathcal{E}}\}ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x_1(t) - x_2(t)\|_{\mathcal{E}}\int_0^{t} |k(t,s)|ds + \mathcal{L}_1\mathcal{L}_2\|x_1(t) - x_2(t)\|_{\mathcal{E}}\int_0^{t} |b(s)|\|x_1(s)\|_{\mathcal{E}}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_2(t)\|_{\mathcal{E}}\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_2(t)\|_{\mathcal{E}}\}\int_0^{t} |b(s)|\|x_2(s)\|_{\mathcal{E}}ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2 \operatorname{diam}X(t)\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 \operatorname{diam}X(t) r\mathcal{B} + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\} r\int_0^{t} |b(s)|ds \\
&\leqslant (\mathcal{L}_1\mathcal{L}_2\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{B})\operatorname{diam}X(t) + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\} r\int_0^{t} |b(s)|ds.
\end{aligned}$$

Hence, we obtain

$$\operatorname{diam}(\mathcal{A}X)(t) \leqslant (\mathcal{L}_1\mathcal{L}_2\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{B})\operatorname{diam}X(t) + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^t |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\} r\int_0^t |b(s)|ds$$

and

$$\lim_{t\to\infty} \sup \operatorname{diam}(\mathcal{A}X)(t) \leqslant (\mathcal{L}_1\mathcal{L}_2\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{B}) \lim_{t\to\infty} \sup \operatorname{diam}X(t).$$

Then

$$\lim_{t\to\infty} \sup \operatorname{diam}(\mathcal{A}X)(t) \leqslant \mathcal{C} \lim_{t\to\infty} \sup \operatorname{diam}X(t), \tag{3.3}$$

where we take $\mathcal{C}$ as $\mathcal{C} = \mathcal{L}_1\mathcal{L}_2\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{B}$. Let $\mathcal{T} > 0$ and $\varepsilon > 0$ be given. Let $x \in X \subseteq \mathcal{Q}_r$ and $t, \tau \in \mathcal{I}$ such that $\tau \leqslant t$ and $|t-\tau| \leqslant \varepsilon$, then

$$\begin{aligned}
&\|\mathcal{A}x(t) - \mathcal{A}x(\tau)\|_{\mathcal{E}} \\
&= \|f(t, g(t, x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds) - f(\tau, g(\tau, x(\tau))\int_0^{\varphi(t)} u(\tau,s,x(s))ds)\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\{|t-\tau| + \|g(t,x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(\tau,x(\tau))\int_0^{\varphi(\tau)} u(\tau,s,x(s))ds\|_{\mathcal{E}}\} \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(\tau,x(\tau))\int_0^{\varphi(t)} u(t,s,x(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(\tau,x(\tau))\int_0^{\varphi(\tau)} u(\tau,s,x(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\|\int_0^{\varphi(t)} u(t,s,x(s))ds - \int_0^{\varphi(\tau)} u(\tau,s,x(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\|\int_0^{\varphi(\tau)} u(t,s,x(s))ds + \int_{\varphi(\tau)}^{\varphi(t)} u(t,s,x(s))ds - \int_0^{\varphi(\tau)} u(\tau,s,x(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} u(t,s,x(s))ds \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(\tau)} |u(t,s,x(s)) - u(\tau,s,x(s))|ds + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_{\varphi(\tau)}^{\varphi(t)} |u(t,s,x(s))|ds \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\tau} |u(t,s,x(s)) - u(\tau,s,x(s))|ds + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_{\varphi(\tau)}^{\varphi(t)} |u(t,s,x(s))|ds \\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\|g(t,x(t)) - g(t,x(\tau))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds + \mathcal{L}_1\|g(t,x(\tau)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\tau} |u(t,s,x(s)) - u(\tau,s,x(s))|ds + \mathcal{L}_1\|g(\tau,x(\tau))\|_{\mathcal{E}}\int_{\varphi(\tau)}^{\varphi(t)} |u(t,s,x(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\mathcal{L}_2\|x(t)-x(\tau)\|_{\mathcal{E}}\int_0^t \{|k(t,s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds\\
&\quad + \mathcal{L}_1\|g(t,x(t)) - g(\tau,x(\tau))\|_{\mathcal{E}}\int_0^t \{|k(t,s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds\\
&\quad + \mathcal{L}_1\{\|a_2(\tau)\|_{\mathcal{E}} + \mathcal{L}_2\|x(\tau)\|_{\mathcal{E}}\}\int_0^\tau |u(t,s,x(s)) - u(\tau,s,x(s))|ds\\
&\quad + \mathcal{L}_1\{\|a_2(\tau)\|_{\mathcal{E}} + \mathcal{L}_2\|x(\tau)\|_{\mathcal{E}}\}\int_{\varphi(\tau)}^{\varphi(t)} \{|k(t,s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds\\
&\leqslant \mathcal{L}_1|t-\tau| + \mathcal{L}_1\mathcal{L}_2\|x(t)-x(\tau)\|_{\mathcal{E}}\{\int_0^t |k(t,s)|ds + \int_0^t |b(s)|\|x(s)\|_{\mathcal{E}}ds\}\\
&\quad + \mathcal{L}_1\|g(t,x(\tau)) - g(\tau,x(\tau))\|_{\mathcal{E}}\{\int_0^t |k(t,s)|ds + \int_0^t |b(s)|\|x(s)\|_{\mathcal{E}}ds\}\\
&\quad + \mathcal{L}_1\{\|a_2(\tau)\|_{\mathcal{E}} + \mathcal{L}_2\|x(\tau)\|_{\mathcal{E}}\}\int_0^\tau |u(t,s,x(s)) - u(\tau,s,x(s))|ds\\
&\quad + \mathcal{L}_1\{\|a_2(\tau)\|_{\mathcal{E}} + \mathcal{L}_2\|x(\tau)\|_{\mathcal{E}}\}\{\int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + \int_{\varphi(\tau)}^{\varphi(t)} |b(s)|\|x(s)\|_{\mathcal{E}}ds\}\\
&\leqslant \mathcal{L}_1\varepsilon + \mathcal{L}_1\mathcal{L}_2\omega^{\mathcal{T}}(x,\varepsilon)\{\mathcal{K} + r\mathcal{B}\} + \mathcal{L}_1\omega^{\mathcal{T}}(g,\varepsilon)\{\mathcal{K} + r\mathcal{B}\}\\
&\quad + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\omega^{\mathcal{T}}(u,\varepsilon)\mathcal{T} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_{\varphi(\tau)}^{\varphi(t)} |b(s)|ds,
\end{aligned}
$$

where

$$\omega^{\mathcal{T}}(u,\varepsilon) = \sup\{|u(t,s,x(s)) - u(\tau,s,x(s))| : \ t,\tau\in\mathcal{I},\ \ |t-\tau|\leqslant\varepsilon,\ \|x\|_{\mathcal{BC}}\leqslant r\}$$

and

$$\omega^{\mathcal{T}}(g,\varepsilon) = \sup\{\|g(t,x(\tau)) - g(\tau,x(\tau))\|_{\mathcal{E}} : \ t,\tau\in\mathcal{I},\ \ |t-\tau|\leqslant\varepsilon,\ \|x\|_{\mathcal{BC}}\leqslant r\}$$

are norm of continuity of the functions u and g on the interval $\mathcal{I}$. Hence

$$
\begin{aligned}
\omega^{\mathcal{T}}(\mathcal{A}x,\varepsilon) \leqslant\ & \mathcal{L}_1\varepsilon + \mathcal{L}_1\mathcal{L}_2\omega^{\mathcal{T}}(x,\varepsilon)\{\mathcal{K} + r\mathcal{B}\} + \mathcal{L}_1\omega^{\mathcal{T}}(g,\varepsilon)\{\mathcal{K} + r\mathcal{B}\} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}}\\
&+ \mathcal{L}_2 r\}\mathcal{T}\omega^{\mathcal{T}}(u,\varepsilon) + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_{\varphi(\tau)}^{\varphi(t)} |b(s)|ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^{\mathcal{T}}(\mathcal{A}X,\varepsilon) \leqslant\ & \mathcal{L}_1\varepsilon + \mathcal{L}_1\mathcal{L}_2\omega^{\mathcal{T}}(X,\varepsilon)\{\mathcal{K} + r\mathcal{B}\} + \mathcal{L}_1\omega^{\mathcal{T}}(g,\varepsilon)\{\mathcal{K} + r\mathcal{B}\}\\
&+ \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\mathcal{T}\omega^{\mathcal{T}}(u,\varepsilon) + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\{\varepsilon\sup\{|k(t,s)| : \ s\in\mathcal{I}\ \}\}\\
&+ \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\{r\varepsilon\ \sup\{|b(s)| : \ s\in\mathcal{I}\ \}.\}
\end{aligned}
$$

According to the uniform continuity of the function $u(t,s,x(s))$ on the set $\mathcal{I}\times\mathcal{I}\times[-r,r]$ and the function $g(t,x(t))$ on $\mathcal{I}\times[-r,r]$ we deduce that $\omega^{\mathcal{T}}(u,\varepsilon)\to 0$, $\omega^{\mathcal{T}}(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Consequently, we obtain

$$\omega_0^{\mathcal{T}}(\mathcal{A}X) \leqslant \mathcal{L}_1\mathcal{L}_2\omega_0^{\mathcal{T}}(X)\{\mathcal{K} + r\mathcal{B}\} \leqslant \{\mathcal{L}_1\mathcal{L}_2\mathcal{K} + \mathcal{L}_1\mathcal{L}_2 r\mathcal{B}\}\omega_0^{\mathcal{T}}(X).$$

As $\mathcal{T}\to\infty$, we have

$$\omega_0(\mathcal{A}X) \leqslant \mathcal{C}\omega_0(X). \tag{3.4}$$

Now, from the estimations (3.3) and (3.4) and the definition of the measure of noncompactness μ on X, we obtain

$$\mu(\mathcal{A}X) \leqslant \mathcal{C}\mu(X).$$

Since all conditions of Darbo's fixed point Theorem are satisfied, then the operator $\mathcal{A}$ has at least one fixed point $x \in \mathcal{Q}_r$, then there exists at least one solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the nonlinear quadratic functional integral equation with distributed delay (3.1). Consequently, there exists at least one solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the nonlinear quadratic functional integral inclusion with distributed delay (1.1). □

4. Asymptotic stability

Definition 4.1. The solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the nonlinear quadratic functional integral inclusion with distributed delay (1.1) is asymptotically stable, that is, $\forall \varepsilon > 0$, $\exists \mathcal{T}(\varepsilon) > 0$ and $r > 0$ such that if any two solutions to the nonlinear quadratic functional integral inclusion with distributed delay (1.1) are $x, x_1 \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$, then $\|x(t) - x_1(t)\|_{\mathcal{E}} \leqslant \varepsilon$ for $t \geqslant \mathcal{T}(\varepsilon)$.

Theorem 4.2. *Consider the assumptions* (H1)-(H7) *hold, then the solution* $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ *of* (1.1) *is asymptotically stable.*

Proof. Let $\varepsilon > 0$ be given, take $x, x_1 \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ be any two solutions of the inclusion (1.1), then

$$\begin{aligned}
&\|x(t) - x_1(t)\|_{\mathcal{E}} \\
&= \|f(t, g(t, x(t)) \int_0^{\varphi(t)} u(t,s,x(s))ds) - f(t, g(t, x_1(t)) \int_0^{\varphi(t)} u(t,s,x_1(s))ds)\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1 \|g(t,x(t)) \int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x_1(t)) \int_0^{\varphi(t)} u(t,s,x_1(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1 \|g(t,x(t)) \int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x_1(t)) \int_0^{\varphi(t)} u(t,s,x(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1 \|g(t,x_1(t)) \int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x_1(t)) \int_0^{\varphi(t)} u(t,s,x_1(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1 \|g(t,x(t)) - g(t,x_1(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1 \|g(t,x_1(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |u(t,s,x(s)) - u(t,s,x_1(s))|ds \\
&\leqslant \mathcal{L}_1 \|g(t,x(t)) - g(t,x_1(t))\|_{\mathcal{E}} \int_0^{t} |u(t,s,x(s))|ds + \mathcal{L}_1 \|g(t,x_1(t))\|_{\mathcal{E}} \int_0^{t} |u(t,s,x(s)) - u(t,s,x_1(s))|ds \\
&\leqslant \mathcal{L}_1 \mathcal{L}_2 \|x(t) - x_1(t)\|_{\mathcal{E}} \int_0^{t} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1 \{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2 \|x_1(t)\|_{\mathcal{E}}\} \int_0^{t} |u(t,s,x(s)) - u(t,s,x_1(s))|ds \qquad (4.1) \\
&\leqslant \mathcal{L}_1 \mathcal{L}_2 \|x(t) - x_1(t)\|_{\mathcal{E}} \int_0^{t} |u(t,s,x(s))|ds + 2\mathcal{L}_1 \{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2 \|x_1(t)\|_{\mathcal{E}}\} \int_0^{t} |u(t,s,x_1(s))|ds \\
&\leqslant \mathcal{L}_1 \mathcal{L}_2 \|x(t) - x_1(t)\|_{\mathcal{E}} \int_0^{t} \{|k(t,s)| + |b(s)| \|x(s)\|_{\mathcal{E}}\}ds \\
&\quad + 2\mathcal{L}_1 \{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2 \|x_1(t)\|_{\mathcal{E}}\} \int_0^{t} \{|k(t,s)| + |b(s)| \|x_1(s)\|_{\mathcal{E}}\}ds \\
&\leqslant \mathcal{L}_1 \mathcal{L}_2 \|x - x_1\|_{\mathcal{BC}} \sup_{t \in \mathcal{R}_+} \int_0^{t} |k(t,s)|ds + \mathcal{L}_1 \mathcal{L}_2 \|x - x_1\|_{\mathcal{BC}} \|x\|_{\mathcal{BC}} \sup_{t \in \mathcal{R}_+} \int_0^{t} |b(s)|ds \\
&\quad + 2\{\mathcal{L}_1 \|a_2\|_{\mathcal{BC}} + \mathcal{L}_1 \mathcal{L}_2 \|x_1\|_{\mathcal{BC}}\} \int_0^{t} |k(t,s)|ds + 2\{\mathcal{L}_1 \|a_2\|_{\mathcal{BC}} + \mathcal{L}_1 \mathcal{L}_2 \|x_1\|_{\mathcal{BC}}\} \|x\|_{\mathcal{BC}} \int_0^{t} |b(s)|ds
\end{aligned}$$

$$\leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\}\int_0^t |k(t,s)|ds + 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\}r\int_0^t |b(s)|ds,$$

select $\mathcal{T} > 0$ such that the following inequality holds for $t > \mathcal{T}$,

$$2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\}\int_0^t |k(t,s)|ds \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad 2\{\mathcal{L}_1\|a_2\|_{\mathcal{BC}} + \mathcal{L}_1\mathcal{L}_2 r\}r\int_0^t |b(s)|ds \leqslant \frac{\varepsilon}{2}.$$

Now we have two situations.

(i) If $t \geqslant \mathcal{T}$, we obtain

$$\|x(t) - x_1(t)\|_{\mathcal{E}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \varepsilon, \; t \geqslant \mathcal{T}(\varepsilon).$$

Hence

$$\|x - x_1\|_{\mathcal{BC}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \varepsilon = \varepsilon.$$

(ii) If $t \leqslant \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = \sup\{|u(t,s,x(s)) - u(t,s,x_1(s))| : t,s \in \mathcal{I}, \; x, x_1 \in [-r,r], \; \|x(s) - x_1(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $u(t,s,x(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (4.1),

$$\|x(t) - x_1(t)\|_{\mathcal{E}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\omega(\varepsilon)\mathcal{T}, \; t \geqslant \mathcal{T}(\varepsilon).$$

Hence

$$\|x - x_1\|_{\mathcal{BC}} \leqslant \mathcal{L}_1\mathcal{L}_2\varepsilon\mathcal{L}_2 + \mathcal{L}_1\mathcal{L}_2\varepsilon r\mathcal{B} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\omega(\varepsilon)\mathcal{T} = \varepsilon.$$

□

5. Dependency on $\varphi(t)$

Definition 5.1. The solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of (3.1) depends asymptotically on the function φ, if $\forall\varepsilon > 0$, $\exists\delta > 0$, and $\mathcal{T}(\varepsilon) > 0$ such that $|\varphi(t) - \varphi^*(t)| < \delta$, $t > \mathcal{T}(\varepsilon)$ implies $\|x - x^*\|_{\mathcal{BC}} < \varepsilon$, where x, x^* are the two solutions of (3.1) and

$$x^*(t) = f(t, g(t, x^*(t))\int_0^{\varphi(t)} u(t,s,x^*(s))ds), \quad t \in \mathcal{R}_+,$$

respectively.

Theorem 5.2. *Assume that the assumptions* (H1)-(H7) *hold, then the asymptotic dependency of the solution* $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ *of* (3.1) *on* $\varphi(t)$ *yields.*

Proof. Let φ, φ^* be continuous functions such that

$$|\varphi(t) - \varphi^*(t)| < \delta.$$

Then

$$\|x(t) - x^*(t)\|_{\mathcal{E}}$$
$$= \|f(t, g(t, x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds) - f(t, g(t, x^*(t))\int_0^{\varphi(t)} u(t,s,x^*(s))ds)\|_{\mathcal{E}}$$

$$
\begin{aligned}
&\leqslant \mathcal{L}_1\|g(t,x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x^*(t))\int_0^{\varphi^*(t)} u(t,s,x^*(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t,x(t))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x^*(t))\int_0^{\varphi(t)} u(t,s,x(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x^*(t))\int_0^{\varphi(t)} u(t,s,x(s))ds - g(t,x^*(t))\int_0^{\varphi^*(t)} u(t,s,x^*(s))ds\|_{\mathcal{E}} \\
&\leqslant \mathcal{L}_1\|g(t,x(t)) - g(t,x^*(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(t,x^*(t))\|_{\mathcal{E}}\int_0^{\varphi^*(t)} |u(t,s,x(s)) - u(t,s,x^*(s))|ds + \mathcal{L}_1\|g(t,x^*(t))\|_{\mathcal{E}}\int_{\varphi^*(t)}^{\varphi(t)} |u(t,s,x(s))|ds \\
&\leqslant \mathcal{L}_1\|g(t,x(t)) - g(t,x^*(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds \\
&\quad + \mathcal{L}_1\|g(t,x^*(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s)) - u(t,s,x^*(s))|ds + \mathcal{L}_1\|g(t,x^*(t))\|_{\mathcal{E}}\int_{\varphi^*(t)}^{\varphi(t)} |u(t,s,x(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x(t) - x^*(t)\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\} \\
&\quad \times \int_0^{t} |u(t,s,x(s)) - u(t,s,x^*(s))|ds + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_{\varphi^*(t)}^{\varphi(t)} |u(t,s,x(s))|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x(t) - x^*(t)\|_{\mathcal{E}}\int_0^{t} |u(t,s,x(s))|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_1\|x^*(t)\|_{\mathcal{E}}\}\int_0^{t} |u(t,s,x^*(s))|ds \\
&\quad + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_{\varphi^*(t)}^{\varphi(t)} |u(t,s,x(s))|ds \qquad (5.1) \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x(t) - x^*(t)\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_0^{t} \{|k(t,s)| + |b(s)|\|x^*(s)\|_{\mathcal{E}}\}ds \\
&\quad + \{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_{\varphi^*(t)}^{\varphi(t)} \{|k(t,s)| + |b(s)|\|x(s)\|_{\mathcal{E}}\}ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x(t) - x^*(t)\|_{\mathcal{E}}\int_0^{t} |k(t,s)|ds + \mathcal{L}_1\mathcal{L}_2\|x(t) - x^*(t)\|_{\mathcal{E}}\int_0^{t} |b(s)|\|x(s)\|_{\mathcal{E}}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_0^{t} |b(s)|\|x^*(s)\|_{\mathcal{E}}ds \\
&\quad + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_{\varphi^*(t)}^{\varphi(t)} |k(t,s)|ds + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x^*(t)\|_{\mathcal{E}}\}\int_{\varphi^*(t)}^{\varphi(t)} |b(s)|\|x(s)\|_{\mathcal{E}}ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}\|x\|_{\mathcal{BC}}\mathcal{B} \\
&\quad + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x^*\|_{\mathcal{BC}}\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x^*\|_{\mathcal{BC}}\}\|x^*\|_{\mathcal{BC}}\int_0^{t} |b(s)|ds \\
&\quad + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x^*\|_{\mathcal{BC}}\}\int_{\varphi^*(t)}^{\varphi(t)} |k(t,s)|ds + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2\|x^*\|_{\mathcal{BC}}\}\|x\|_{\mathcal{BC}}\int_{\varphi^*(t)}^{\varphi(t)} |b(s)|ds \\
&\leqslant \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}K + \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}r\mathcal{B} + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^{t} |k(t,s)|ds \\
&\quad + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_0^{t} |b(s)|ds + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\delta_1 + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\delta_2,
\end{aligned}
$$

select $\mathcal{T} > 0$ such that the two relations hold for $t > \mathcal{T}$,

$$2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^t |k(t,s)|ds \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_0^t |b(s)|ds \leqslant \frac{\varepsilon}{2}.$$

Now we have two situations.

(i) If $t \geqslant \mathcal{T}$, we obtain

$$\begin{aligned}\|x(t) - x^*(t)\|_{\mathcal{E}} \leqslant\ & \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}} r\mathcal{B} \\ & + \varepsilon + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\delta_1 + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\delta_2.\end{aligned}$$

Then

$$\|x - x^*\|_{\mathcal{BC}} \leqslant \frac{\varepsilon + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\delta_1 + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\delta_2}{1 - \mathcal{L}_1\mathcal{L}_2\mathcal{K} - \mathcal{L}_1\mathcal{L}_2 r\mathcal{B}} = \varepsilon.$$

(ii) If $t \leqslant \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = \sup\{|u(t,s,x(s)) - u(t,s,x^*(s))| : t,s \in \mathcal{I},\ x,x^* \in [-r,r],\ \|x(s) - x^*(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $u(t,s,x(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (5.1),

$$\begin{aligned}\|x(t) - x^*(t)\|_{\mathcal{E}} \leqslant\ & \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}}\mathcal{K} + \mathcal{L}_1\mathcal{L}_2\|x - x^*\|_{\mathcal{BC}} r\mathcal{B} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\omega(\varepsilon)\mathcal{T} \\ & + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\delta_1 + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\delta_2.\end{aligned}$$

Hence

$$\|x - x^*\|_{\mathcal{BC}} \leqslant \frac{\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\omega(\varepsilon)\mathcal{T} + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\delta_1 + \mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\delta_2}{1 - \mathcal{L}_1\mathcal{L}_1\mathcal{K} - \mathcal{L}_1\mathcal{L}_2 r\mathcal{B}} = \varepsilon.$$

That completes the proof. □

Corollary 5.3. *For every $f \in \mathcal{S}_{\mathcal{F}}$ the solution of the equation (3.1) depends asymptotically on the function $\varphi(t)$.*

6. Dependency on $\mathcal{S}_{\mathcal{F}}$

Definition 6.1. The solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the inclusion (1.1) depends asymptotically on the set $\mathcal{S}_{\mathcal{F}}$, if for every $\varepsilon > 0$, $\mathcal{T}(\varepsilon) > 0$, and any two functions $f,\ h \in \mathcal{S}_{\mathcal{F}}$, there exists $\delta > 0$ such that $\|f - h\|_{\mathcal{E}} < \delta$, $t > \mathcal{T}(\varepsilon)$ implies $\|x_f - x_h\|_{\mathcal{BC}} < \varepsilon$.

Theorem 6.2. *Let the assumptions (H1)-(H7) be fulfilled, then the solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the inclusion (1.1) depends asymptotically on $\mathcal{S}_{\mathcal{F}}$.*

Proof. Let $f, h \in \mathcal{S}_{\mathcal{F}}$ such that

$$\|f(t, g(t, x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s)ds)) - h(t, g(t, x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s)ds))\|_{\mathcal{E}} < \delta, \quad \delta > 0,\ t \in \mathcal{R}_+.$$

Then

$$\begin{aligned}&\|x_f(t) - x_h(t)\ \|_{\mathcal{E}} \\ &= \|f(t, g(t, x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t, g(t, x_h(t))\int_0^{\varphi(t)} u((t),s,x_h(s))ds)\|_{\mathcal{E}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|f(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds)\|_{\mathcal{E}} \\
&\quad + \|h(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t,g(t,x_h(t))\int_0^{\varphi(t)} u(t,s,x_h(s))ds)\|_{\mathcal{E}} \\
&\leqslant \|f(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds)\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds - g(t,x_h(t))\int_0^{\varphi(t)} u(t,s,x_h(s))ds\|_{\mathcal{E}} \\
&\leqslant \|f(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds)\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds - g(t,x_h(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x_h(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds - g(t,x_h(t))\int_0^{\varphi(t)} u(t,s,x_h(s))ds\|_{\mathcal{E}} \\
&\leqslant \|f(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds) - h(t,g(t,x_f(t))\int_0^{\varphi(t)} u(t,s,x_f(s))ds)\|_{\mathcal{E}} \\
&\quad + \mathcal{L}_1\|g(t,x_f(t)) - g(t,x_h(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x_f(s))|ds \\
&\quad + \mathcal{L}_1\|g(t,x_h(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |u(t,s,x_f(s)) - u(t,s,x_h(s))|ds \\
&\leqslant \delta + \mathcal{L}_1\|g(t,x_f(t)) - g(t,x_h(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_f(s))|ds \\
&\quad + \mathcal{L}_1\|g(t,x_h(t))\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_f(s)) - u(t,s,x_h(s))|ds \\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f(t) - x_h(t)\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_f(s))|ds \\
&\quad + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\int_0^{t} |u(t,s,x_f(s)) - u(t,s,x_h(s))|ds \qquad (6.1)\\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f(t) - x_h(t)\|_{\mathcal{E}}\int_0^{t} |u(t,s,x_f(s))|ds + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\int_0^{t} |u(t,s,x_h(s))|ds \\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f(t) - x_h(t)\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|x_f(s)\|_{\mathcal{E}}\}ds \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\int_0^{t} \{|k(t,s)| + |b(s)|\|x_h(s)\|_{\mathcal{E}}\}ds \\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f(t) - x_h(t)\|_{\mathcal{E}}\{\int_0^{t} |k(t,s)|ds + \int_0^{t} |b(s)|\|x_f(t)\|_{\mathcal{E}}ds\} \\
&\quad + 2\mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\{\int_0^{t} |k(t,s)|ds + \int_0^{t} |b(s)|\|x_h(t)\|_{\mathcal{E}}ds\} \\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f - x_h\|_{\mathcal{BC}}\{\mathcal{K} + r\mathcal{B}\} + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\{\int_0^{t} |k(t,s)|ds + r\int_0^{t} |b(s)|ds\} \\
&\leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f - x_h\|_{\mathcal{BC}}\{\mathcal{K} + r\mathcal{B}\} + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^{t} |k(t,s)|ds + 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_0^{t} |b(s)|ds,
\end{aligned}
$$

select $T > 0$ such that the two relations hold for $t > \mathcal{T}$,

$$2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}\int_0^t |k(t,s)|ds \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad 2\mathcal{L}_1\{\|a_2\|_{\mathcal{BC}} + \mathcal{L}_2 r\}r\int_0^t |b(s)|ds \leqslant \frac{\varepsilon}{2}.$$

Now we have two situations.

(1) If $t \geqslant \mathcal{T}$, we obtain

$$\|x_f(t) - x_h(t)\|_{\mathcal{E}} \leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f - x_h\|_{\mathcal{BC}}\{\mathcal{K} + r\mathcal{B}\} + \varepsilon.$$

Then

$$\|x_f - x_h\|_{\mathcal{BC}} \leqslant \frac{\delta + \varepsilon}{1 - \mathcal{L}_1\mathcal{L}_2\{\mathcal{K} + r\mathcal{B}\}} = \varepsilon.$$

(2) If $t \leqslant \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = \sup\{|u(t,s,x_f(s)) - u(t,s,x_h(s))| : t,s \in \mathcal{I},\ x_f,\ x_h \in [-r,r],\ \|x_f(s),\ x_h(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $u(t,s,x(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (6.1),

$$\|x_f(t) - x_h(t)\|_{\mathcal{E}} \leqslant \delta + \mathcal{L}_1\mathcal{L}_2\|x_f - x_h\|_{\mathcal{BC}}\{\mathcal{K} + r\mathcal{B}\} + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\omega(\varepsilon)\mathcal{T}.$$

Hence

$$\|x_f(t) - x_h(t)\|_{\mathcal{BC}} \leqslant \frac{\delta + \mathcal{L}_1\{\|a_2(t)\|_{\mathcal{E}} + \mathcal{L}_2\|x_h(t)\|_{\mathcal{E}}\}\omega(\varepsilon)\mathcal{T}}{1 - \mathcal{L}_1\mathcal{L}_2\{\mathcal{K} + r\mathcal{B}\}} = \varepsilon.$$

This complete the prove of our investigation. □

7. An example

In this section our aim is to illustrate the main result contained in Theorem 3.3. Let $\overline{\mathcal{Q}} = \{x \in \mathcal{E} : \|x\|_{\mathcal{E}} \leqslant 1\}$ and $R_+ = [0,\infty)$. Consider the multi-valued function $\mathcal{F} : \mathcal{R}_+ \times \overline{\mathcal{Q}} \to \Omega(\mathcal{E})$ defined by

$$\mathcal{F}(t,y(t)) = (a_1(t) + \mathcal{L}_1 y(t))\overline{\mathcal{Q}}, \quad t \in \mathcal{R}_+.$$

Then $\mathcal{F}$ is Lipschitz. In fact, for the norm in the Banach space we have

$$\begin{aligned}\|\mathcal{F}(t,y(t))\|_{\mathcal{E}} &= \sup\{\|f\|_{\mathcal{E}} :\ f \in \mathcal{F}(t,y(t))\}\\ &= \|(a_1(t) + \mathcal{L}_1 y(t))\overline{\mathcal{Q}}\|_{\mathcal{E}} = \|a_1(t) + \mathcal{L}_1 y(t)\|_{\mathcal{E}} \leqslant \|a_1(t)\|_{\mathcal{E}} + \mathcal{L}_1\|y(t)\|_{\mathcal{E}}.\end{aligned}$$

Now let $f(t,y(t)) = a_1(t) + \mathcal{L}_1 y(t) \in \mathcal{F}(t,y(t))$. Hence, we can apply our results to the nonlinear quadratic functional integral equation with distributed delay

$$x(t) = t + \frac{t^2}{4}x(t)\int_0^{\varphi(t)} \frac{s + 2\pi(t^2+1)(x(s))}{2\pi(t^2+1)s(s+1)}ds, \quad t \in \mathcal{R}_+. \tag{7.1}$$

Here $g(t,x(t)) = \frac{t^2}{4}x(t)$ and $u(t,s,x(s)) = \frac{s+2\pi(t^2+1)(x(s))}{2\pi(t^2+1)s(s+1)}$ $\varphi(t) \leqslant t$. Now

$$\begin{aligned}\|x(t)\|_{\mathcal{E}} &= \|t + \frac{t^2}{4}x(t)\int_0^{\varphi(t)} \frac{s + 2\pi(t^2+1)(x(s))}{2\pi(t^2+1)s(s+1)}ds\|_{\mathcal{E}}\\ &\leqslant t + \frac{t^2}{4}\|x(t)\|_{\mathcal{E}}\int_0^{\varphi(t)} |\frac{s + 2\pi(t^2+1)(x(s))}{2\pi(t^2+1)s(s+1)}|ds\end{aligned}$$

$$\begin{aligned}
&\leqslant t+\frac{t^2}{4}\|x(t)\|_{\mathcal{E}}\int_0^{\varphi(t)}\frac{|s|+2\pi|t^2+1|\|x(s)\|_{\mathcal{E}}}{2\pi|t^2+1||s||s+1|}ds\\
&\leqslant t+\frac{t^2}{4}\|x(t)\|_{\mathcal{E}}\int_0^{t}\frac{|s|+2\pi|t^2+1|\|x(s)\|_{\mathcal{E}}}{2\pi|t^2+1||s||s+1|}ds\\
&\leqslant t+\frac{t^2}{4}\|x(t)\|_{\mathcal{E}}\{\int_0^{t}\frac{1}{2\pi|t^2+1||s+1|}ds+\int_0^{t}\frac{1}{|s||s+1|}\|x(s)\|_{\mathcal{E}}ds\}\\
&\leqslant t+\frac{t^2}{8\pi|t^2+1|}\|x\|_{\mathcal{BC}}\ln|t+1|+\frac{t^2}{4}(\|x\|_{\mathcal{BC}})^2\ln|\frac{t}{t+1}|\\
&\leqslant 1+\frac{1}{16\pi}\ln 2r+\frac{1}{4}\ln\frac{1}{2}r^2.
\end{aligned}$$

Then

$$\|x\|_{\mathcal{BC}}\leqslant 1+\frac{1}{16\pi}\ln 2r+\frac{1}{4}\ln\frac{1}{2}r^2.$$

The assumptions (H1)-(H7) of Theorem 3.3 are satisfied with $a_1(t)=a_2(t)=t,\mathcal{L}_1=\mathcal{L}_2=0.1$. Obviously, the function $g(t,x(t))$ is continuous. Currently, for any $x_1,x_2\in\mathcal{E}$ and $t\in\mathcal{R}_+$, we have

$$\|g(t,x_1(t))-g(t,x_2(t))\|_{\mathcal{E}}=\|\frac{t^2}{4}x_1(t)-\frac{t^2}{4}x_2(t)\|_{\mathcal{E}}\leqslant|\frac{t^2}{4}|\ \|x_1(t)-x_2(t)\|_{\mathcal{E}}\leqslant|\frac{1}{4}|\ \|x_1(t)-x_2(t)\|_{\mathcal{E}}.$$

And

$$\|g(t,x(t))\|_{\mathcal{E}}\leqslant t+0.1\|x(t)\|_{\mathcal{E}}.$$

Further, we also have $u(t,s,x(s))$ fulfills condition (H4) with

$$|u(t,s,x(s))|=|\frac{s+2\pi(t^2+1)(x(s))}{2\pi(t^2+1)s(s+1)}|\leqslant\frac{|s|+2\pi|t^2+1||x(s)|}{2\pi|t^2+1||s||s+1|}\leqslant\frac{1}{2\pi|t^2+1||s+1|}+\frac{1}{|s||s+1|}|x(s)|.$$

This indicates that we can insert $k(t,s)=\frac{1}{2\pi(t^2+1)(s+1)}$ and $b(s)=\frac{1}{s(s+1)}$. To verify the assumption (H4), notice that

$$\lim_{t\to\infty}\int_0^t k(t,s)ds=\lim_{t\to\infty}\int_0^t\frac{1}{2\pi(t^2+1)(s+1)ds}=\lim_{t\to\infty}\frac{\ln(t+1)}{2\pi(t^2+1)}=0$$

and

$$\lim_{t\to\infty}\int_0^t b(s)ds=\lim_{t\to\infty}\int_0^t\frac{1}{s(s+1)}ds=\lim_{t\to\infty}\ln\frac{t}{t+1}=0.$$

Moreover, we have

$$\mathcal{K}=\sup_{t\in\mathcal{R}_+}\int_0^t k(t,s)ds=\sup_{t\in\mathcal{R}_+}\frac{\ln(t+1)}{2\pi(t^2+1)}=\frac{1}{2\pi}$$

and

$$\mathcal{B}=\sup_{t\in\mathcal{R}_+}\int_0^t b(s)ds=\sup_{t\in\mathcal{R}_+}\ln\frac{t}{t+1}=\frac{1}{2}.$$

Finally, let us pay attention to the fact that the inequality of Theorem 3.3 has the form $\mathcal{C}=\mathcal{L}_1\mathcal{L}_2\mathcal{K}+\mathcal{L}_1\mathcal{L}_2\mathcal{B}r<1$. Consequently, all the requirements of Theorem 3.3 have been met. As a result the nonlinear quadratic functional integral equation (7.1) has at least one asymptotically stable solution in the space $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$.

8. Conclusions

In this paper we use a Lipschitz selection for a multi-valued function in the reflexive Banach space $\mathcal{E}$ to establish the solvability of the quadratic functional integral inclusion with distributed delay (1.1). Our investigation is lying in the space of all functions defined, continuous, and bounded on the real half-axis $\mathcal{R}_+$ and taking values in a given reflexive Banach space $\mathcal{E}$.

In the main result we introduced sufficient conditions and studied the existence of solutions $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the quadratic functional integral inclusion with distributed delay (1.1) using the theory of measure of non-compactness and Darbo's fixed point theorem, the asymptotic stability and the asymptotic dependency of the solution were studied. Finally, a numerical example is illustrated.

References

[1] A. Aghajani, R. Allahyari, M. Mursaleen, *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math., **260** (2014), 68–77. 1

[2] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiĭ, *Measure of Noncompactness and Condens-ing Operators*, Birkhäuser Verlag, Basel, (1992). 1

[3] J. P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, (1984). 1

[4] J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, (1977). 1

[5] J. Banaś, *On the superposition operator and integrable solutions of some functional equations*, Nonlinear Anal., **12** (1988), 777–784. 1

[6] J. Banaś, A. Chlebowicz, *On solutions of an infinite system of nonlinear integral equations on the real half-axis*, Banach J. Math. Anal., **13** (2019), 944–968. 1

[7] J. Banaś, A. Chlebowicz, *On solutions of an infinite system of nonlinear integral equations on the real half-axis*, Banach J. Math. Anal., **13** (2019), 944–968. 1

[8] J. Banaś, A. Chlebowicz, W. Woś, *On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space*, J. Math. Anal. Appl., **489** (2020), 25 pages. 1

[9] J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Marcel Dekker, New York, (1980). 1

[10] J. Banaś, M. Jleli, M. Mursaleen, B. Samet, C. Vetro, *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Springer, Singapore, (2017). 1

[11] J. Banaś, J. C. Mena *Some properties of nonlinear Volterra-Stieltjes integral operators*, Comput. Math. Appl., **49** (2015), 1565–1573. 1

[12] J. Banaś, N. Merentes, B. Rzepka, *Measures of noncompactness in the space of continuous and bounded functions defined on the real half-axis*, Springer, Singapore, (2017). 1

[13] J. Banaś, M. Mursaleen, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New Delhi, (2014). 1

[14] J. Banaś, D. O'Regan, *Volterra-Stieltjes integral operators*, Math. Comput. Modelling, **41** (2005), 335–344.

[15] J. Banaś, T. Zajsac, *Well-posed minimization problems via the theory of measures of noncompactness*, Math. Anal. Appl.: Sel. Top., (2018), 553–586. 1

[16] M. Benchohra, S. Hamani, S. K. Ntouyas, *Boundary value problems for differential equations with fractionl order and nonlocal conditions*, Nonlinear Anal., **71** (2009), 2391–2396. 1

[17] A. I. Bulgakov, L. N. Ljapin, *Some properties of the set of solutions of a Volterra-Hammerstein integral inclusion*, Differ. Uravn., **14** (1978), 1465–1472. 1

[18] T. Cardinali, K. Nikodem, F. Papalini, *Some results on stability and on characterization of K-convexity of set-valued functions*, Ann. Polon, Math., **58** (1993), 185–192. 1

[19] H. V. S. Chauhan, B. Singh, C. Tunç, O. Tunç, *On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **116** (2022), 11 pages. 1

[20] A. Chlebowicz, *Solvability of an infinite system of nonlinear integral equations of Volterra-Hammerstein type*, Adv. Nonlinear Anal., **9** (2020), 1187–1204. 1

[21] A. Chlebowicz, *Existence of solutions to infinite systems of nonlinear integral equations on the real half-axis*, Electron. J. Differential Equations, **2021** (2021), 20 pages. 1

[22] M. Cichoń, A. M. El-Sayed, A. H. Hussien, *Existence theorems for nonlinear functional integral equations of fractional orders*, Comment. Math., **41** (2001), 59–67. 1

[23] Ş. Cobzaş, R. Miculescu, A. Nicolae, *Lipschitz functions*, Springer, Cham, (2019). 1

[24] G. Darbo, *Punti uniti in trasformazioni a condominio non compatto*, RendRend. Sem. Mat. Univ. Padova, **24** (1955), 84–92. 2.1

[25] B. C. Dhage, *A functional integral inclusion involving Caratheodories*, Electron. J. Qual. Theory Differ. Equ., **2003** (2003), 18 pages. 1

[26] B. C. Dhage, *A functional integral inclusion involving discontinuities*, Fixed Point Theory, **5** (2004), 53–64. 1
[27] A. M. A. El-Sayed, A. G. Ibrahim,*Multivalued fractional differential equations*, Appl. Math. Comput., **68** (1995), 15–25. 2.2
[28] A. M. A. El-Sayed, A.-G. Ibrahim, *Set-valued integral equations of fractional-orders*, Appl. Math. Comput., **118** (2001), 113–121. 1
[29] L. S. Goldenstein, I. T. Gohberg, A. S. Markus, *Investigations of some properties of bounded linear operators with their q-norms*, Ucen. Zap. Kishinevsk. Univ., **29** (1957), 29–36. 1
[30] B. Hazarika, R. Arab, M. Mursaleen, *Application measure of noncompactness and operator type contraction for solvability of an infinite system of differential equations in lp-Space*, Filomat, **33** (2019), 2181–2189. 1
[31] K. Nikodem, *On quadratic set-valued functions*, Publ. Math. Debrecen, **30** (1984), 297–301. 1
[32] K. Nikodem, *On Jensen's functional equation for set-valued functions*, Rad. Mat., **3** (1987), 23–33.
[33] K. Nikodem, *Set-valued solutions of the Pexider functional equation*, Funkcial. Ekvac., **31** (1988), 227–231. 1
[34] D. O'Regan, *Integral Inclusions of upper semicontinuous or lower semicontinuous type*, Proc. Amer. Math. Soc., **124** (1996), 2391–2399. 1
[35] D. Popa, *Functional inclusions on square symmetric grupoids and Hyers-Ulam stability*, Math. Inequal. Appl., **7** (2004), 419–428. 1
[36] D. Popa, *Aproperty of a functional inclusion connected with Hyers-Ulam stability*, J. Math. Inequal., **3** (2009), 591–598. 1
[37] P. Shvartsman, *Lipschitz selections of set-valued mappings and Helly's theorem*, J. Geom. Anal., **12** (2002), 289–324. 1
[38] O. Tunç, C. Tunç, G. Petrusel, J.-C. Yao, *On the Ulam stabilities of nonlinear integral equations and integro-differential equations*, Math. Methods Appl. Sci., **47** (2024), 4014–4028. 1
[39] O. Tunç, C. Tunç, J.-C. Yao, *Global existence and uniqueness of solutions of integral equations with multiple variable delays and integro-differential equations: progressive contractions*, Mathematics, **12** (2024), 16 pages. 1