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# Stabilities and instabilities of Euler-Lagrange cubic functional equation

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## Abstract

This article provides the solution and Hyers-Ulam stability results of the following Euler-Lagrange cubic functional equation

$$\begin{aligned}(a-b)\left[(a+b)^3 f\left(\frac{bx+ay}{b+a}\right)+(b-a)^3 f\left(\frac{bx-ay}{b-a}\right)\right]+(a+b)\left[(a+b)^3 f\left(\frac{ax+by}{a+b}\right)+(a-b)^3 f\left(\frac{ax-by}{a-b}\right)\right]\\ = ab(a^2+b^2)[f(x+y)+f(x-y)]+2\left(a^4-b^4\right)f(x),\end{aligned}$$

$a \neq b; a, b \neq 0$, in Banach spaces and paranormed spaces using the direct method with suitable counterexample.

**Keywords:** Cubic functional equation, Banach spaces, paranormed spaces, Hyers-Ulam stability.

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## 1. Introduction

Functional equations occur practically everywhere. Its influence and applications are felt in all fields benefit from their contact, use, and technique. Mathematicians have been working with functional equations for a much longer period of time than the formal discipline has existed.

Functional equations with several unknown functions were subsequently considered by D'Alembert (1747), Cauchy(1821), Abel (1823), Hilbert (1900), Sykora (1904), Stephanos (1904), Hamel (1905), Schimmack (1908), Suto (1913), Wilson (1916), Schweitzer (1918), Sierpinski (1920), Ostrowski (1929), Aczel (1989), and by several other authors.

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A classical question in the theory of functional equations is the following "Whether it is true that a function which approximately satisfies a functional equation $\epsilon$ must be close to an exact solution $\epsilon$? If the problem accepts a solution, can we say that the equation $\epsilon$ is stable".

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability question is: How do the solutions of the inequality differ from those of the given functional equation?

In the fall of 1940, Ulam [54] gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exists $\delta(\epsilon) > 0$ such that if $h : G_1 \to G_2$ satisfies the inequality

$$d\left(h(x * y), h(x) \diamond h(y)\right) < \delta, \quad \forall\, x, y \in G_1,$$

then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

In the next year, Hyers [17] gave a affirmative answer to this question for additive groups under the assumption that groups are Banach spaces. He brilliantly answered the question of Ulam for the case where $G_1$ and $G_2$ are assumed to be Banach spaces. The result of Hyers is stated as follows.

**Theorem 1.1.** *Let $f : E_1 \to E_2$ be a function between Banach spaces such that*

$$\|f(x + y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

*for all $x, y \in E_1$ and $\epsilon > 0$ is a constant. Then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

*exists for each $x \in E_1$ and $A : E_1 \to E_2$ is unique additive mapping satisfying*

$$\|f(x) - A(x)\| \leqslant \epsilon$$

*for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.*

Taking this famous result into consideration, the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is said to have the *Hyers-Ulam stability* on $(E_1, E_2)$ if for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\epsilon \geqslant 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on $E_1$. The method in Theorem 1.1 provided by Hyers which produces the additive function $A$ will be called a *direct method*. This method is the most important and powerful tool to study the stability of various functional equations.

It is possible to prove a stability result similar to Hyers functions that do not have bounded Cauchy difference. Aoki (1950) [4] first generalized the Hyers theorem for unbounded Cauchy difference having sum of norms $(\|x\|^p + \|y\|^p)$. The same result was rediscovered by Rassias [43] in 1978 and proved a generalization of Hyers theorem for additive mappings. This stability result is named *Hyers-Ulam-Rassias stability* or *Hyers-Ulam-Aoki-Rassias stability* for the functional equation. In 1982 Rassias [44], followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\, \|y\|^q$ with $p + q \neq 1$. Later this stability result was called *Ulam-Gavruta-Rassias stability* of functional equation. In 1990, Rassias during the $27^{th}$ International Symposium on Functional Equations asked the question whether such a theorem in [43] can also be proved for value of $p$ greater or equal to 1. In 1991, Gajda [15] provided an affirmative solution to Rassias's question for $p$ strictly greater than one. In 1994, Găvruţa [16] provided a further generalization of Rassias [43] theorem in which he replaced the bound $\epsilon\,(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. This stability result is called *Generalized Hyers-Ulam-Rassias stability* of functional equation.

In 2008, a special case of Găvruţa's theorem for the unbounded Cauchy difference was obtained by Ravi et al. [49] by considering the summation of both the sum and the product of two p-norms in the spirit of Rassias approach and is named *Rassias Stability* of functional equation.

On the other hand, another approach for proving the stability results is also called the method of invariantmeans (Fixed Point Method). In 2003, Radu [42] proposed a new method, successively developed in [10, 11] to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative.

One of the most famous cubic functional equations was introduced by Rassias [46],

$$c(x+2y)+3c(x)=3c(x+y)+c(x-y)+6c(y)$$

and investigated its Ulam stability problem. Also Jun and Kim [21] discussed the generalized Hyers-Ulam-Rassias stability of a cubic functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.2}$$

Baak and Moslenian [8] proved the Hyers-Ulam-Rassias stability of the cubic functional equation (1.2) in orthogonality space. Recently, Karthikeyan et al. [30] established the generalized Ulam-Hyers stability of $(a, b; k > 0)$-cubic functional equation

$$\begin{aligned}\frac{a+\sqrt{k}\,b}{2}f\left(ax+\sqrt{k}\,by\right)+\frac{a-\sqrt{k}\,b}{2}f\left(ax-\sqrt{k}\,by\right)+k(a^2-kb^2)b^2f(y)\\ =k(ab)^2f(x+y)+(a^2-kb^2)a^2f(x),\end{aligned}$$

where $a \neq \pm1, 0; b \neq \pm1, 0; k > 0$ in intuitionistic fuzzy normed space using direct and fixed point methods.

The problem of the stability of various functional and differential equations has been studied in detail by many authors, and there are many interesting results on this problem (see [1–3, 6–9, 12, 18–27, 29, 30, 32–41, 47, 48, 50] and references therein quoted).

Motivated from the above historical developments in the field of functional equations the authors provides the Ulam-Hyers stability results of Euler-Lagrange cubic functional equation

$$\begin{aligned}(a-b)\left[(a+b)^3f\left(\frac{bx+ay}{b+a}\right)+(b-a)^3f\left(\frac{bx-ay}{b-a}\right)\right]+(a+b)\left[(a+b)^3f\left(\frac{ax+by}{a+b}\right)\right.\\ \left.+(a-b)^3f\left(\frac{ax-by}{a-b}\right)\right]=ab(a^2+b^2)[f(x+y)+f(x-y)]+2\left(a^4-b^4\right)f(x),\end{aligned} \tag{1.3}$$

$a \neq b; a, b \neq 0$ in Banach spaces and paranormed spaces using the direct method with suitable counterexample.

## 2. Solution of the cubic functional equation (1.3)

This section deals with the general solution of the functional equation (1.3). Throughout this section, assume that $\mathcal{K}$ and $\mathcal{L}$ are vector spaces.

**Lemma 2.1.** *If a mapping $f : \mathcal{K} \to \mathcal{L}$ satisfies the functional equation (1.3), then the following properties hold:*

(i) $f(0) = 0$;
(ii) $f(2x) = 2^3 f(x)$, *for all* $x \in \mathcal{K}$;
(iii) $f(-y) = -f(y)$, *for all* $y \in \mathcal{K}$; *that is,* $f$ *is an odd function.*

*Proof.* Letting $(x, y)$ by $(0, 0)$ in (1.3), we get $f(0) = 0$. Since $a, b \neq 0$, we arrive at (i). Replacing $(x, y)$ by $(x, x)$ in (1.3), we obtain

$$\begin{aligned}(a-b)\left[(a+b)^3 f(x) + (b-a)^3 f(x)\right] + (a+b)\left[(a+b)^3 f(x) + (a-b)^3 f(x)\right]& \\ = ab\left(a^2+b^2\right) f(2x) - 2\left(a^4 - b^4\right) f(x),& \text{ or} \\ \left(8a^3 b + 8b^3 a\right) f(x) = ab\left(a^2+b^2\right) f(2x),& \text{ or} \\ ab\left(a^2+b^2\right)(8f(x)) = ab\left(a^2+b^2\right) f(2x),& \text{ or} \\ 2^3 f(x) = f(2x),&\end{aligned}$$

for all $x \in \mathcal{K}$. Thus, (ii) holds. Letting $x = 0$ in (1.3), we get

$$\begin{aligned}(a-b)&\left[(a+b)^3 f\left(\frac{ay}{b+a}\right) + (b-a)^3 f\left(\frac{-ay}{b-a}\right)\right] \\ &+ (a+b)\left[(a+b)^3 f\left(\frac{by}{a+b}\right) + (a-b)^3 f\left(\frac{-by}{a-b}\right)\right] = ab(a^2+b^2)[f(y) + f(-y)].\end{aligned} \tag{2.1}$$

Using (ii) in (2.1), we get

$$\begin{aligned}&(a-b)a^3\left[f(y) + f(-y)\right] + (a+b)b^3\left[f(y) + f(-y)\right] = ab(a^2+b^2)[f(y) + f(-y)], \\ &\left[(a-b)a^3 + (a+b)b^3 - ab(a^2+b^2)\right](f(y) + f(-y)) = 0, \quad f(y) = -f(-y),\end{aligned}$$

since $(a-b)a^3 + (a+b)b^3 - ab(a^2+b^2) \neq 0$ for all $y \in \mathcal{K}$. Finally, (iii) holds, since $a, b \neq 0$. Thus $f$ is an odd function. Hence the proof is complete. □

**Proposition 2.2.** *For a mapping* $f : \mathcal{K} \to \mathcal{L}$ *that satisfies the functional equation* (1.3)*, then the following properties hold:*

(i) $f$ *satisfies* (1.2)*;*

(ii) $f$ *fulfills*

$$f(ax + by) + f(ax - by) = ab^2[f(x+y) + f(x-y)] + 2a(a^2 - b^2)f(x), \tag{2.2}$$

*where* $a, b$ *are fixed integers with* $a \pm b \neq 0$*;*

(iii) $f$ *satisfies* (1.3).

*Proof.*

(i)⇔(ii) Refer to Theorem 2.2 and Remark 2.3 of [9].

(ii)⇒(iii) It is easily verified that

$$f(ax) = a^3 f(x), \quad f(bx) = b^3 f(x), \quad f((a-b)x) = (a-b)^3 f(x), \quad \text{and} \quad f((a+b)x) = (a+b)^3 f(x), \tag{2.3}$$

for all $x \in \mathcal{K}$. Replacing $(x, y)$ by $(bx, ay)$ in (2.2) and using (2.3), we obtain

$$f(bx + ay) + f(bx - ay) = a^2 b[f(x+y) + f(x-y)] + 2b(b^2 - a^2)f(x), \tag{2.4}$$

for all $x, y \in \mathcal{K}$. Multiplying (2.2) and (2.4) by $(a+b)$ and $(a-b)$, respectively, and then adding both resultants, we reach to

$$\begin{aligned}(a-b)[f(bx+ay) + f(bx-ay)] + (a+b)[f(ax+by) + f(ax-by)]& \\ = ab(a^2+b^2)[f(x+y) + f(x-y)] + 2\left(a^4 - b^4\right) f(x),&\end{aligned}$$

for all $x, y \in \mathcal{K}$. Now, the relations in (2.3) complete this implication. □

## 3. Stability results in Banach space

This section provides the Ulam-Hyers stability of the functional equation (1.3) in Banach space using direct method. Throughout this section, assume that $\mathcal{M}$ be a normed space and $\mathcal{N}$ be a Banach space. Define a mapping $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ by

$$\begin{aligned}\Omega_{(a,b)}(x,y) = (a-b)\left[(a+b)^3 f\left(\frac{bx+ay}{b+a}\right) + (b-a)^3 f\left(\frac{bx-ay}{b-a}\right)\right] \\ + (a+b)\left[(a+b)^3 f\left(\frac{ax+by}{a+b}\right) + (a-b)^3 f\left(\frac{ax-by}{a-b}\right)\right] \\ - ab(a^2+b^2)[f(x+y)+f(x-y)] - 2\left(a^4-b^4\right) f(x),\end{aligned}$$

where $a, b \neq 0$ for all $x, y \in \mathcal{M}$.

### 3.1. *Banach spaces: direct method*

This subsection gives the stability of the cubic functional equation (1.3) in Banach spaces by using direct method.

**Theorem 3.1.** *Let $\tau = \pm 1$ and $\Gamma : \mathcal{M}^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\sum_{s=0}^{\infty} \frac{\Gamma\left(2^{\tau s}x, 2^{\tau s}y\right)}{2^{3\tau s}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{s\to\infty} \frac{\Gamma\left(2^{\tau s}x, 2^{\tau s}y\right)}{2^{3\tau s}} = 0$$

*for all $x, y \in \mathcal{M}$. Let $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping satisfying the inequality*

$$\left\|\Omega_{(a,b)}(x,y)\right\| \leqslant \Gamma(x,y) \tag{3.1}$$

*for all $x, y \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$, which satisfies (1.3) and*

$$\|f(x) - \mathcal{C}(x)\| \leqslant \frac{1}{2^3 k} \sum_{s=\frac{1-\tau}{2}}^{\infty} \frac{\Gamma\left(2^{\tau s}x, 2^{\tau s}x\right)}{2^{3\tau s}}, \tag{3.2}$$

*where $\mathcal{C}(x)$ is defined by*

$$\mathcal{C}(x) = \lim_{\sigma\to\infty} \frac{f(2^{\sigma\tau}x)}{2^{3\sigma\tau}}$$

*for all $x \in \mathcal{M}$, respectively.*

*Proof.*
Case (i): Assume $\tau = 1$. Changing $(x, y)$ by $(x, x)$ in (3.1), we get

$$\left\|ab(a^2+b^2)f(2x) - ab(a^2+b^2)2^3 f(x)\right\| \leqslant \Gamma(x,x) \tag{3.3}$$

for all $x \in \mathcal{M}$. Above equation (3.3) can be rewritten as

$$\left\|kf(2x) - k2^3 f(x)\right\| \leqslant \Gamma(x,x), \tag{3.4}$$

where $k = ab(a^2+b^2)$, for all $x \in \mathcal{M}$. It follows from (3.4) that

$$\left\|\frac{f(2x)}{2^3} - f(x)\right\| \leqslant \frac{\Gamma(x,x)}{2^3 k} \tag{3.5}$$

for all $x \in \mathcal{M}$. Now replacing $x$ by $2x$ and dividing the result by $2^3$ in (3.5), we have

$$\left\| \frac{f(2^2x)}{2^6} - \frac{f(2x)}{2^3} \right\| \leqslant \frac{\Gamma(2x, 2x)}{2^6 k} \tag{3.6}$$

for all $x \in \mathcal{M}$. From (3.5) and (3.6), we obtain

$$\left\| \frac{f(2^2x)}{2^6} - f(x) \right\| \leqslant \left\{ \left\| \frac{f(2^2x)}{2^6} - \frac{f(2x)}{2^3} \right\| + \left\| \frac{f(2x)}{2^3} - f(x) \right\| \right\} \leqslant \frac{1}{2^3 k} \left[ \Gamma(x, x) + \frac{\Gamma(2x, 2x)}{2^3} \right]$$

for all $x \in \mathcal{M}$. Generalizing, for a positive integer $r$, we obtain

$$\left\| \frac{f(2^r x)}{2^{3r}} - f(x) \right\| \leqslant \frac{1}{2^3 k} \sum_{t=0}^{r-1} \frac{\Gamma(2^t x, 2^t x)}{2^{3t}} \tag{3.7}$$

for all $x \in \mathcal{M}$. Since $\mathcal{N}$ is a Banach space, we have to prove the convergence of the sequence $\left\{ \frac{f(2^r x)}{2^{3r}} \right\}$, replacing $x$ by $2^t x$ and dividing by $2^{3t}$ in (3.7), we get

$$\begin{aligned} \left\| \frac{f(2^{r+t}x)}{2^{3(r+t)}} - \frac{f(2^t x)}{2^{3t}} \right\| &= \frac{1}{2^{3t}} \left\| \frac{f(2^r \cdot 2^t x)}{2^{3r}} - f(2^t x) \right\| \\ &\leqslant \frac{1}{2^{3t}} \frac{1}{2^3 k} \sum_{t=0}^{r-1} \frac{\Gamma(2^r \cdot 2^t x, 2^r \cdot 2^t x)}{2^{3r}} \leqslant \frac{1}{2^3 k} \sum_{t=0}^{\infty} \frac{\Gamma(2^{r+t}x, 2^{r+t}x)}{2^{3(r+t)}} \to 0 \quad \text{as} \quad t \to \infty \end{aligned}$$

for all $x \in \mathcal{M}$. Thus it follows that the sequence $\left\{ \frac{f(2^r x)}{2^{3r}} \right\}$ is a Cauchy in $\mathcal{N}$ and so it converges. Define a mapping $\mathcal{C}(x) : \mathcal{M} \to \mathcal{N}$ by

$$\mathcal{C}(x) = \lim_{\sigma \to \infty} \frac{f(2^\sigma x)}{2^{3\sigma}}$$

for all $x \in \mathcal{M}$. In order to show that $\mathcal{C}$ satisfies (1.3), replacing $(x, y)$ by $(2^r x, 2^r y)$ and dividing by $2^{3r}$ in (3.1), we have

$$\left\| \Omega_{(a,b)} \mathcal{C}(x, y) \right\| = \lim_{r \to \infty} \frac{1}{2^{3r}} \left\| \Omega(2^r x, 2^r y) \right\| \leqslant \lim_{r \to \infty} \frac{1}{2^{3r}} \Gamma(2^r x, 2^r y)$$

for all $x, y \in \mathcal{M}$ and hence the mapping $\mathcal{C}$ is cubic. Taking the limit as $r$ approaches to infinity in (3.7), we find that the mapping $\mathcal{C}$ is a cubic mapping satisfying the inequality (3.2) near the approximate mapping $f : \mathcal{M} \to \mathcal{N}$ of equation (1.3). Hence, $\mathcal{C}$ satisfies (1.3), for all $x, y \in \mathcal{M}$.

To prove that $\mathcal{C}$ is unique, we assume now that there is $\mathcal{C}'$ as another cubic mapping satisfying (1.3) and the inequality (3.2). Then it is easily proved that

$$\mathcal{C}(2^t x) = 2^{3t} \mathcal{C}(x), \qquad \mathcal{C}'(2^t x) = 2^{3t} \mathcal{C}'(x),$$

for all $x \in \mathcal{M}$ and all $t \in \mathbb{N}$. Thus

$$\begin{aligned} \left\| \mathcal{C}(x) - \mathcal{C}'(x) \right\| = \frac{1}{2^{3t}} \left\| \mathcal{C}(2^t x) - \mathcal{C}'(2^t x) \right\| &\leqslant \frac{1}{2^{3t}} \left\{ \left\| \mathcal{C}(2^t x) - f(2^t x) \right\| + \left\| f(2^t x) - \mathcal{C}'(2^t x) \right\| \right\} \\ &\leqslant \frac{2}{2^3 k} \sum_{t=0}^{\infty} \frac{\Gamma(2^{r+t}x, 2^{r+t}x)}{2^{3(r+t)}} \end{aligned}$$

for all $x \in \mathcal{M}$. Therefore, as $r \to \infty$ in the above inequality, we get the uniqueness of $\mathcal{C}$. Hence the theorem holds for $\tau = 1$.

Case (ii): Assume $\tau = -1$. Now replacing $x$ by $\frac{x}{2}$ in (3.4), we get

$$\left\| f(x) - 2^3 f\left(\frac{x}{2}\right) \right\| \leqslant \frac{1}{ab(a^2+b^2)} \Gamma\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in \mathcal{M}$. The rest of the proof is similar to that of case $\tau = 1$. Hence for $\tau = -1$ also the theorem holds. Hence the proof is complete. □

The following corollaries express the instant significance of the Theorem 3.1 concerning the Ulam-Hyers, Hyers- Ulam-Rassias, Ulam-Gavruta-Rassias, and Rassias stability results of the functional equation (1.3).

**Corollary 3.2.** *Let $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping. If there exist real numbers $\epsilon$ and $\sigma$ such that*

$$\left\| \Omega_{(a,b)}(x, y) \right\| \leqslant \epsilon$$

*for all $x, y \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$ such that*

$$\| f(x) - \mathcal{C}(x) \| \leqslant \frac{\epsilon}{k\,|7|},$$

*for all $x \in \mathcal{M}$.*

**Corollary 3.3.** *Let $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping. If there exist real numbers $\epsilon$ and $\sigma$ such that*

$$\left\| \Omega_{(a,b)}(x, y) \right\| \leqslant \epsilon \{ ||x||^\sigma + ||y||^\sigma \}$$

*for all $x, y \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$ such that*

$$\| f(x) - \mathcal{C}(x) \| \leqslant \frac{2\epsilon ||x||^\sigma}{k|2^3 - 2^\sigma|}, \qquad \sigma \neq 3,$$

*for all $x \in \mathcal{M}$.*

**Corollary 3.4.** *Let $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping. If there exist real numbers $\epsilon$ and $\sigma$ such that*

$$\left\| \Omega_{(a,b)}(x, y) \right\| \leqslant \epsilon \{ ||x||^\sigma ||y||^\sigma \}$$

*for all $x, y \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$ such that*

$$\| f(x) - \mathcal{C}(x) \| \leqslant \frac{\epsilon ||x||^{2\sigma}}{k|2^3 - 2^{2\sigma}|}, \qquad 2\sigma \neq 3,$$

*for all $x \in \mathcal{M}$.*

**Corollary 3.5.** *Let $\Omega : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping. If there exist real numbers $\epsilon$ and $\sigma$ such that*

$$\left\| \Omega_{(a,b)}(x, y) \right\| \leqslant \epsilon \left\{ ||x||^\sigma ||y||^\sigma + \left\{ ||x||^{2\sigma} + ||y||^{2\sigma} \right\} \right\}$$

*for all $x, y \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$ such that*

$$\| f(x) - \mathcal{C}(x) \| \leqslant \frac{3\epsilon ||x||^{2\sigma}}{k|2^3 - 2^{2\sigma}|}, \qquad 2\sigma \neq 3,$$

*for all $x \in \mathcal{M}$.*

## 4. Stability results in paranormed spaces

This section deals with the stability results of the functional equation (1.3) in paranormed spaces using direct method. Now, we recall the basic definitions and notations in paranormed space. The concept of statistical convergence for sequences of real numbers was introduced by Fast [13] and Steinhaus [53] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [14, 28, 35, 37, 51]). This notion was defined in normed spaces by Kolk [31].

We recall some basic facts concerning Fréchet spaces.

**Definition 4.1** ([55])**.** Let $X$ be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on $X$ such that

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x + y) \leqslant P(x) + P(y)$ (triangle inequality);
(P4) if $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair $(X, P)$ is called a *paranormed space* if $P$ is a *paranorm* on $X$.

**Definition 4.2** ([55])**.** The paranorm is called total if, in addition, we have

(P5) $P(x) = 0$ implies $x = 0$.

**Definition 4.3** ([55])**.** A Fréchet space is a total and complete paranormed space.

### 4.1. *Paranormed spaces: direct method*

Throughout this section, let $(\mathcal{U}, P)$ be a Fréchet space and $(\mathcal{V}, \|\cdot\|)$ be a Banach space. For the convenience, we define a mapping $\Xi_{(a,b)}(x, y) : \mathcal{U} \longrightarrow \mathcal{V}$ by

$$\begin{aligned}\Xi_{(a,b)}(x,y) = (a-b)&\left[(a+b)^3 f\left(\frac{bx+ay}{b+a}\right) + (b-a)^3 f\left(\frac{bx-ay}{b-a}\right)\right]\\ &+(a+b)\left[(a+b)^3 f\left(\frac{ax+by}{a+b}\right) + (a-b)^3 f\left(\frac{ax-by}{a-b}\right)\right]\\ &-ab(a^2+b^2)[f(x+y)+f(x-y)] - 2\left(a^4-b^4\right) f(x),\end{aligned}$$

where $a, b \neq 0$ for all $x, y \in \mathcal{U}$.

**Theorem 4.4.** *Let $j \in \{-1, 1\}$ be fixed and $\xi : \mathcal{U} \to [0, \infty)$ be a function with the condition*

$$\sum_{n=0}^{\infty} \frac{1}{2^{3nj}} \xi\left(2^{nj}x, 2^{nj}y\right) < +\infty \tag{4.1}$$

*for all $x, y \in \mathcal{U}$. Suppose that a mapping $f : \mathcal{U} \to \mathcal{V}$ satisfies the following inequality*

$$P\left(\Xi_{(a,b)}(x,y)\right) \leqslant \xi(x,y) \tag{4.2}$$

*for all $x, y \in \mathcal{U}$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that*

$$P\left(f(x) - \mathcal{C}(x)\right) \leqslant \sum_{m=\frac{1-j}{2}}^{\infty} \frac{1}{2^{3mj}} \xi\left(2^{mj}x, 2^{mj}y\right) \tag{4.3}$$

*for all $x \in \mathcal{U}$. The mapping $\mathcal{A}(x)$ is defined by*

$$P\left(\lim_{n\to\infty} \frac{1}{2^{3nj}} f\left(2^{nj}x\right) - \mathcal{C}(x)\right) \to 0$$

*for all $x \in \mathcal{U}$.*

*Proof.* Replacing $(x,y)$ by $(x,x)$ in (4.2), we get

$$P\left(ab(a^2+b^2)f(2x)-ab(a^2+b^2)2^3f(x)\right)\leqslant \xi(x,x) \tag{4.4}$$

for all $x\in\mathcal{U}$. Above equation (4.4) can be rewritten as

$$P\left(kf(2x)-2^3kf(x)\right)\leqslant \xi(x,x),$$

where $k=ab(a^2+b^2)$, for all $x\in\mathcal{U}$. For any $m,n>0$, we simplify

$$P\left(\frac{kf(2^m x)}{2^{3m}}-\frac{kf(2^n x)}{2^{3(n-1)}}\right)\leqslant \sum_{\ell=m}^{n-1}\frac{1}{2^{3\ell}}\xi\left(2^\ell x,2^\ell x\right) \tag{4.5}$$

for all $x\in\mathcal{U}$ and all $m,n\geqslant 0$. It follows from (4.5) that the sequence $\left\{\frac{f(2^n x)}{2^{3n}}\right\}$ is Cauchy sequence. Since $\mathcal{V}$ is complete, there exists a mapping $\mathcal{C}:\mathcal{U}\to\mathcal{V}$ by

$$P\left(\lim_{n\to\infty}\frac{f(2^n x)}{2^{3n}}-\mathcal{C}(x)\right)\to 0$$

for all $x\in\mathcal{U}$. By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty}t_n\frac{f(2^n x)}{2^{3n}}-t\mathcal{C}(x)\right)\to 0$$

for all $x\in\mathcal{U}$. Letting $m=0$ and $n\to\infty$ in (4.5), we see that (4.3) holds for all $x\in\mathcal{U}$. To show that $\mathcal{C}$ satisfies (1.3), replacing $(x,y)$ by $(2^n x,2^n y)$ in (4.2), we get

$$P\left(\frac{1}{2^{3n}}\,\xi\left(2^n x,2^n y\right)\right)\leqslant \frac{1}{2^{3n}}\xi\left(2^n x,2^n y\right)$$

for all $x,y\in\mathcal{U}$. Letting $n\to\infty$ in the above inequality and using the definition of $\mathcal{C}(x)$, we see that

$$P\left(\Xi_{(a,b)}\mathcal{C}\left(x,y\right)\right)=0 \tag{4.6}$$

for all $x,y\in\mathcal{U}$. Using condition (P5) in (4.6), we obtain

$$\begin{aligned}(a-b)&\left[(a+b)^3\mathcal{C}\left(\frac{bx+ay}{b+a}\right)+(b-a)^3\mathcal{C}\left(\frac{bx-ay}{b-a}\right)\right]\\&+(a+b)\left[(a+b)^3\mathcal{C}\left(\frac{ax+by}{a+b}\right)+(a-b)^3\mathcal{C}\left(\frac{ax-by}{a-b}\right)\right]\\&=ab(a^2+b^2)[\mathcal{C}(x+y)+\mathcal{C}(x-y)]+2\left(a^4-b^4\right)\mathcal{C}(x),\end{aligned}$$

where $a,b\neq 0$ for all $x,y\in\mathcal{U}$. Hence $\mathcal{C}$ satisfies (1.3) for all $x,y\in\mathcal{U}$. In order to prove that $\mathcal{C}(x)$ is unique, let $\mathcal{C}'(x)$ be another additive mapping satisfying (1.3) and (4.3). Then

$$\begin{aligned}P(\mathcal{C}(x)-\mathcal{C}'(x))=\left\{P\left(\frac{\mathcal{C}(2^m)}{2^{3m}}-\frac{\mathcal{C}'(2^m)}{2^{3m}}\right)\right\}&\leqslant\left\{P\left(\frac{\mathcal{C}(2^m)}{2^{3m}}-\frac{f(2^m)}{2^{3m}}\right)+P\left(\frac{f(2^m)}{2^{3m}}-\frac{\mathcal{C}'(2^m)}{2^{3m}}\right)\right\}\\&\leqslant\sum_{\ell=0}^{\infty}\frac{1}{2^{\ell+m}}\xi\left(2^{\ell+m}x,2^{\ell+m}x\right)\to 0\ \text{ as }\ m\to\infty\end{aligned}$$

for all $x\in\mathcal{U}$. Thus $P(\mathcal{C}(x)-\mathcal{C}'(x))=0$ for all $x\in\mathcal{U}$. Hence, we have $\mathcal{C}(x)=\mathcal{C}'(x)$. Therefore $\mathcal{C}(x)$ is unique. Thus the mapping $\mathcal{C}:\mathcal{U}\to\mathcal{V}$ is a unique cubic mapping. Thus the theorem holds for $j=1$. For $j=-1$, we can prove the result by a similar method. This completes the proof. □

The following corollaries express the instant significance of the Theorem 4.4 concerning the Ulam-Hyers, Hyers- Ulam-Rassias, Ulam-Gavruta-Rassias, and Rassias stability results of the functional equation (1.3).

**Corollary 4.5.** *Let $\Xi : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers $\sigma$ and $s$ such that*

$$P\left(\Xi_{(a,b)}(x,y)\right) \leqslant \sigma$$

*for all $x, y \in \mathcal{U}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that*

$$P\left(f(x) - \mathcal{C}(x)\right) \leqslant \frac{k\sigma}{|7|}$$

*for all $x \in \mathcal{U}$.*

**Corollary 4.6.** *Let $\Xi : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers $\sigma$ and $s$ such that*

$$P\left(\Xi_{(a,b)}(x,y)\right) \leqslant \sigma\{P(x)^s + P(y)^s\}, \qquad s \neq 3,$$

*for all $x, y \in \mathcal{U}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that*

$$P\left(f(x) - \mathcal{C}(x)\right) \leqslant \frac{2k\sigma P(x)^s}{|2^3 - 2^s|}$$

*for all $x \in \mathcal{U}$.*

**Corollary 4.7.** *Let $\Xi : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers $\sigma$ and $s$ such that*

$$P\left(\Xi_{(a,b)}(x,y)\right) \leqslant \sigma\, P(x)^s P(y)^s, \quad s \neq \tfrac{3}{2},$$

*for all $x, y \in \mathcal{U}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that*

$$P\left(f(x) - \mathcal{C}(x)\right) \leqslant \frac{k\sigma P(x)^{2s}}{|2^3 - 2^{2s}|}$$

*for all $x \in \mathcal{U}$.*

**Corollary 4.8.** *Let $\Xi : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers $\sigma$ and $s$ such that*

$$P\left(\Xi_{(a,b)}(x,y)\right) \leqslant \sigma\left\{P(x)^s P(y)^s + \left\{P(x)^{2s} + P(y)^{2s}\right\}\right\}, \quad s \neq \tfrac{3}{2},$$

*for all $x, y \in \mathcal{U}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{U} \to \mathcal{V}$ such that*

$$P\left(f(x) - \mathcal{C}(x)\right) \leqslant \frac{3k\sigma P(x)^{2s}}{|2^3 - 2^{2s}|}$$

*for all $x \in \mathcal{U}$.*

## 5. Counter example for non stability cases

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $s = 3$ in Corollary 3.3.

**Example 5.1.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\rho(x) = \begin{cases} \mu x^3, & \text{if } |x| < 1, \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\rho(2^k x)}{2^{3k}} \quad \text{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left|Df_{(a,b}(x,y)\right| \leqslant 32\left(a^2 - b^2\right)\mu\left(|x|^3 + |y|^3\right) \tag{5.1}$$

for all $x, y \in \mathbb{R}$. Then, there does not exists a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - \mathcal{C}(x)| \leqslant \beta|x|^3 \quad \text{for all} \quad x \in \mathbb{R}. \tag{5.2}$$

*Proof.* Now

$$|f(x)| \leqslant \sum_{n=0}^{\infty} \frac{|\rho(2^k x)|}{|2^{3k}|} = \sum_{k=0}^{\infty} \frac{\mu}{2^{3k}} = \frac{8\mu}{7}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (5.1). If $x = y = 0$, then (5.1) is trivial. If $|x|^3 + |y|^3 \geqslant \frac{1}{2^3}$, then the left hand side of (5.1) is less than $32\left(a^2 - b^2\right)\mu$. Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{2^3}$. Then there exists a positive integer $\ell$ such that

$$\frac{1}{2^{3(\ell+1)}} \leqslant |x|^3 + |y|^3 < \frac{1}{2^{3\ell}}, \tag{5.3}$$

so that $2^{3(\ell-1)}|x|^3 \leqslant \frac{1}{2^3}$, $2^{3(\ell-1)}|y|^3 \leqslant \frac{1}{2^3}$, and consequently

$$\begin{gathered}2^{3(\ell+1)}\left(\frac{bx+ay}{b+a}\right), 2^{3(\ell+1)}\left(\frac{bx-ay}{b-a}\right), 2^{3(\ell+1)}\left(\frac{ax+by}{a+b}\right),\\ 2^{3(\ell+1)}\frac{ax-by}{a-b}, -2^{3(\ell+1)}(x+y), -2^{3(\ell+1)}(x-y), -2^{3(\ell+1)}x \in (-1,1).\end{gathered}$$

Therefore for each $k = 0, 1, \ldots, \ell-1$, we have

$$\begin{gathered}2^{3k}\left(\frac{bx+ay}{b+a}\right), 2^{3k}\left(\frac{bx-ay}{b-a}\right), 2^{3k}\left(\frac{ax+by}{a+b}\right),\\ 2^{3k}\left(\frac{ax-by}{a-b}\right), -2^{3k}(x+y), -2^{3k}(x-y), -2^{3k}x \in (-1,1),\end{gathered}$$

and

$$\begin{aligned}(a-b)&\left[(a+b)^3\rho\left(2^{3k}\left(\frac{bx+ay}{b+a}\right)\right) + (b-a)^3\rho\left(2^{3k}\left(\frac{bx-ay}{b-a}\right)\right)\right]\\ &+ (a+b)\left[(a+b)^\rho\left(2^{3k}\left(\frac{ax+by}{a+b}\right)\right) + (a-b)^3\rho\left(2^{3k}\left(\frac{ax-by}{a-b}\right)\right)\right]\\ &- ab(a^2+b^2)\rho\left(2^{3k}(x+y)\right) - ab(a^2+b^2)\rho\left(2^{3k}(x-y)\right) - 2\left(a^4 - b^4\right)\rho(2^{3k}x) = 0\end{aligned}$$

for $k = 0, 1, \ldots, \ell-1$. From the definition of f and (5.3), we obtain that

$$\begin{aligned}\left|\Omega_{(a,b)}(x,y)\right| \leqslant \sum_{k=0}^{\infty} \frac{1}{2^{3k}}\Big|(a-b)&\left[(a+b)^3\rho\left(2^{3k}\left(\frac{bx+ay}{b+a}\right)\right) + (b-a)^3\rho\left(2^{3k}\left(\frac{bx-ay}{b-a}\right)\right)\right]\\ &+ (a+b)\left[(a+b)^\rho\left(2^{3k}\left(\frac{ax+by}{a+b}\right)\right) + (a-b)^3\rho\left(2^{3k}\left(\frac{ax-by}{a-b}\right)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&-ab(a^2+b^2)\rho\left(2^{3k}(x+y)\right)-ab(a^2+b^2)\rho\left(2^{3k}(x-y)\right)-2\left(a^4-b^4\right)\rho(2^{3k}x)\Big|\\
&\leqslant \sum_{k=\ell}^{\infty}\frac{1}{2^{3k}}\Big|(a-b)\left[(a+b)^3\rho\left(2^{3k}\left(\frac{bx+ay}{b+a}\right)\right)+(b-a)^3\rho\left(2^{3k}\left(\frac{bx-ay}{b-a}\right)\right)\right]+\\
&\quad+(a+b)\left[(a+b)^\rho\left(2^{3k}\left(\frac{ax+by}{a+b}\right)\right)+(a-b)^3\rho\left(2^{3k}\left(\frac{ax-by}{a-b}\right)\right)\right]\\
&\quad-ab(a^2+b^2)\rho\left(2^{3k}(x+y)\right)-ab(a^2+b^2)\rho\left(2^{3k}(x-y)\right)-2\left(a^4-b^4\right)\rho(2^{3k}x)\Big|\\
&\leqslant 32\left(a^2-b^2\right)\mu\left(|x|^3+|y|^3\right).
\end{aligned}
$$

Thus $f$ satisfies (5.1) for all $x, y \in \mathbb{R}$ with $0 \leqslant |x|^3+|y|^3 \leqslant \frac{1}{2^3}$. We claim that the cubic functional equation (1.3) is not stable for $s=3$ in Corollary 3.3. Suppose on the contrary that there exist a mapping $\mathcal{C}:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ satisfying (5.2). Since $f$ is bounded and continuous for all $x\in\mathbb{R}$, $\mathcal{C}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, $\mathcal{C}$ must have the form $\mathcal{C}(x)=cx$ for any $x$ in $\mathbb{R}$. Thus we obtain that

$$
|f(x)| \leqslant (\beta+|c|)\,|x|^3. \tag{5.4}
$$

But we can choose a positive integer $m$ with $m\mu > \beta+|c|$. If $x \in \left(0, \frac{1}{2^{3(m-1)}}\right)$, then $2^k x \in (0,1)$ for all $k=0,1,\dots,m-1$. For this $x$, we get

$$
f(x)=\sum_{n=0}^{\infty}\frac{\rho(2^k x)}{2^{3k}} \geqslant \sum_{n=0}^{m-1}\frac{\mu(2^k x)}{2^{3k}} = m\mu x^3 > (\beta+|c|)\,x^3,
$$

which contradicts (5.4). Therefore the cubic functional equation (1.3) is not stable in sense of Ulam-Hyers and Rassias if $s=3$. □

## 6. Conclusion

This article has proved the general solution and Hyers-Ulam, Hyers-Ulam-Rassias, generalized Hyers-Ulam-Rassias, and Rassias stability results of the new Euler-Lagrange cubic functional equation in Banach spaces and paranormed spaces by using the direct method with suitable counterexample.

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