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Introducing Δ_h Hermite-based Appell polynomials via the monomiality principle: properties, forms, and generating relations

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Abstract

The article introduces a novel class of polynomials, ${}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)$, termed Δ_h Hermite-based Appell polynomials, utilizing the monomiality principle. These polynomials exhibit close connections with Δ_h Hermite-based Bernoulli, Euler, and Genocchi polynomials, elucidating their specific properties and explicit forms. Moreover, the research establishes generating relations for these polynomials, facilitating profound insights applicable across diverse domains such as mathematics, physics, and engineering sciences.

Keywords: Δ_h hybrid special polynomials, explicit forms, Appell polynomials, monomiality principle, explicit forms.

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1. Introduction

The Appell polynomial sequences, which constitute a prominent class of polynomial sequences, are employed in various fields such as applied mathematics, theoretical physics, and approximation theory, among others. These polynomials are encountered in numerous problems within these disciplines. Moreover, Appell polynomials satisfy all the axioms of an Abelian group when subjected to the composition operation.

In the eighteenth century, Appell introduced a series of polynomials denoted as $\mathcal{Q}_m(u)$, as described in [2]. These polynomials exhibit a specific relationship:

$$\frac{d}{du}\mathcal{Q}_m(u) = m\, \mathcal{Q}_{m-1}(u), \qquad m \in \mathbb{N}_0,$$

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and the relation of generating type:

$$(\mathcal{Q}t)\exp(ut)=\sum_{k=0}^{\infty}\mathcal{Q}_k(u)\frac{t^k}{k!},\tag{1.1}$$

where $\mathcal{Q}(t)$ is expressed as:

$$\mathcal{Q}(t)=\sum_{k=0}^{\infty}\mathcal{Q}_k\frac{t^k}{k!},\quad \mathcal{Q}_0\neq 0.$$

In recent years, there has been significant progress and advancement in the generalization of mathematical physics, particularly in the field of special functions. This new development has laid the analytical foundation for a large number of precisely solvable problems in mathematical physics and engineering. These advancements have found diverse applications across various domains. References such as [13, 14, 16, 18, 20, 21, 23, 25–27, 29, 30] exemplify the wide-ranging applications of these developments. The introduction of special functions with multiple indices and variables represents a significant breakthrough in the theory of generalized special functions. These functions have been recognized for their importance and relevance in both practical applications and pure mathematics. The demand for polynomials with multiple indices and multiple variables has been recognized as a means to address challenges arising in diverse mathematical disciplines, ranging from the theory of partial differential equations to abstract group theory. Hermite, the mathematician, introduced the concept of multiple-index, multiple-variable Hermite polynomials. These Hermite polynomials [11] have extensive applications, not only in physics, such as in numerical analysis for Gaussian quadrature and in the study of quantum harmonic oscillators and Schrödinger's equation.

Several authors have shown a growing interest in the exploration and investigation of Δ_h special polynomials, as evidenced by works such as [4, 6, 9, 15, 19]. Recently, Shahid Wani et al. have made significant contributions by introducing and studying various doped polynomials of a special nature. They have derived numerous characteristics and properties of these polynomials, which hold significance from an engineering perspective. Notable examples include [5, 17, 22, 28]. These properties encompass a wide range of aspects, including summation formulas, determinant forms, approximation properties, explicit, and implicit formulas, as well as generating expressions.

Consider a function g that maps a subset I of the real numbers to the real numbers, and let h be a positive real number. We can define the forward difference operator, denoted by Δ_h, as introduced by Jordan in reference [12] on page 2. The operator Δ_h is defined as follows:

$$\Delta_h[g](u)=g(u+h)-g(u).$$

This operator calculates the difference between the function values of g at $u+h$ and u. Now, let's explore the finite difference of order i, denoted by $\Delta_h^i g$, where i is a natural number. We can express the i^{th} order finite difference recursively using the forward difference operator:

$$\Delta_h^i[g](u)=\Delta_h\big(\Delta_h^{i-1}[g](u)\big)=\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}g(u+lh),\tag{1.2}$$

where $\binom{i}{l}$ represents the binomial coefficient, and l ranges from 0 to i. The equation above demonstrates that the i^{th} order finite difference of g at u can be obtained by applying the forward difference operator iteratively i times, starting from the $(i-1)^{\text{th}}$ order finite difference. The sum in the equation captures the contributions of different function values of g at $u+lh$ with appropriate coefficients. To clarify, the notation Δ_h^0 corresponds to the identity operator, which leaves the function unchanged, while Δ_h^1 represents the first-order forward difference operator, as defined previously.

Costabile and Longo, [9], have recently undertaken a pioneering effort to introduce a novel class of polynomial sequences called Δ_h Appell polynomials. These polynomials are specifically designed to be associated with the Δ_h operator. In their research, they thoroughly examined various properties and characteristics of these polynomials. The Δ_h Appell polynomials can be represented by a generating function denoted as $\mathcal{Q}_m(q_1;h)$. This generating function serves as a fundamental tool for expressing and analyzing the properties of these polynomials. It defines a functional relationship between the parameters q_1 and h, allowing for the systematic generation and exploration of the Δ_h Appell polynomials. While the specific form of the generating function has not been provided in the given context, it plays a crucial role in determining the properties, behavior, and algebraic structure of the Δ_h Appell polynomials. By studying the generating function, one can gain insights into the relationship between the parameters, investigate the coefficients and roots of the polynomials, and explore other relevant properties associated with these specialized sequences. Thus, Costabile and Longo's work represents a significant advancement in the field, as it not only introduces a new class of polynomial sequences tailored to the Δ_h operator but also delves into the investigation of their properties, thereby contributing to the broader understanding of these unique mathematical constructs. The generating relation for these polynomials is given by:

$$\sum_{m=0}^{\infty} \mathcal{Q}_m(q_1;h)\frac{t^m}{m!} = \gamma(t)(1+ht)^{\frac{q_1}{h}}, \tag{1.3}$$

or by the relation

$$\Delta_h[\mathcal{Q}_m](q_1;h) = mh\mathcal{Q}_{m-1}(q_1;h), \tag{1.4}$$

respectively. For $h \to 0$, the expression (1.3) reduces to equation (1.1) and (1.4) reduces to (1), respectively. Further, in [9], Δ_h Appell sequences $\mathcal{Q}_m(u)$, $m \in \mathbb{N}$ were defined by the power series of the product of two functions $\gamma(t)(1+ht)^{\frac{q_1}{h}}$ by

$$\gamma(t)(1+ht)^{\frac{q_1}{h}} = \mathcal{Q}_0(q_1;h) + \mathcal{Q}_1(q_1;h)\frac{t}{1!} + \mathcal{Q}_2(q_1;h)\frac{t^2}{2!} + \cdots + \mathcal{Q}_m(q_1;h)\frac{t^m}{m!} + \cdots, \tag{1.5}$$

where

$$\gamma(t) = \gamma_{0,h} + \gamma_{1,h}\frac{t}{1!} + \gamma_{2,h}\frac{t^2}{2!} + \cdots + \gamma_{m,h}\frac{t^m}{m!} + \cdots. \tag{1.6}$$

The Δ_h Appell sequences, as explored by Jordan [12], exhibit a fascinating property: they can be reduced to various well-known sequences and polynomials. Some of these established sequences include the generalized falling factorials $(u)_m^h$, denoted as $(u)m$, the Bernoulli sequence of the second kind $b_m(u)$, the Boole sequence $B_{lm}(u;\lambda)$, and the Poisson-Charlier sequence $C_m(u;\gamma)$.

The generalized falling factorials, $(u)_m$, are a familiar sequence of polynomials that arise in many mathematical contexts. By applying the Δ_h operator to the Δ_h Appell sequences, they can be reduced to these generalized falling factorials.

Similarly, the Δ_h Appell sequences can be related to the Bernoulli sequence of the second kind, denoted as $b_m(u)$. This sequence is well-known in the realm of number theory and combinatorics. The connection between the Δ_h Appell sequences and the Bernoulli sequence of the second kind reveals an intriguing relationship between these different mathematical constructs. Further, the Boole sequence $B_{lm}(u;\lambda)$ and the Poisson-Charlier sequence $C_m(u;\gamma)$ are also encompassed within the reductions of the Δ_h Appell sequences. The Boole sequence plays a significant role in the study of combinatorics and Boolean algebra, while the Poisson-Charlier sequence finds applications in probability theory and statistical analysis. These connections further highlight the versatility and relevance of the Δ_h Appell sequences within various mathematical domains.

Undoubtedly, there is irrefutable evidence supporting the substantial progress observed in various facets of hybrid special polynomials through the integration of principles such as monomiality, operational rules, and related properties. The foundational concept of monomiality, a key contributor to these advancements, has a historical origin dating back to 1941 with Steffenson's introduction of the notion of poweroids, as documented in [24]. This pivotal idea was further honed and refined in subsequent years, with Dattoli playing a prominent role in advancing the field, as highlighted in [10].

Thus, within this framework, the $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators play a crucial role. These operators serve as multiplicative and derivative operators, respectively, for a set of polynomials denoted as $b_m(u)_{m\in\mathbb{N}}$. This implies that they satisfy the following expressions:

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{1.7}$$

and

$$m\, b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}. \tag{1.8}$$

Subsequently, when the set of polynomials $b_m(u)_{m\in\mathbb{N}}$ is subjected to manipulation by multiplicative and derivative operators, it is characterized as a quasi-monomial. To qualify as a quasi-monomial, the set is expected to adhere to the following formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1},$$

thus displays a Weyl group structure as a result. By leveraging the properties of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$, one can ascertain the characteristics of the underlying set $b_m(u)_{m\in\mathbb{N}}$ when it is considered a quasi-monomial. Consequently, the following traits can be established with confidence.

(i) $b_m(u)$ demonstrate the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(u)\} = m\, b_m(u), \tag{1.9}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ possesses differential realizations.

(ii) The explicit form of $b_m(u)$, can be cast in the form as listed:

$$b_m(u) = \hat{\mathcal{M}}^m\, \{1\}, \tag{1.10}$$

while taking, $b_0(u) = 1$.

(iii) Also, generating relation in exponential form for $b_m(u)$ can be casted in the form

$$e^{t\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty} b_m(u)\frac{t^m}{m!}, \quad |t| < \infty,$$

using identity (1.10).

Even in contemporary times, these operational methodologies continue to find extensive application in various domains of mathematical physics, quantum mechanics, and classical optics. As a result, these techniques remain powerful and effective tools for conducting research. For instance, their utility can be observed in a wide range of studies, as exemplified by references such as [1, 3, 7]. Considering equations (1.7) and (1.8), we have successfully obtained the multiplicative and derivative operators for the Δ_h Appell polynomials. This was achieved by differentiating the expression (1.3) with respect to t and u, respectively. Consequently, the derived operators can be expressed as follows:

$$\mathcal{Q}_{m+1}(q_1;h) = \hat{\mathcal{M}_{\mathcal{A}}}\{\mathcal{Q}_m(q_1;h)\} = \left(\frac{q_1}{1+{}_{q_1}\Delta_h} + \frac{\gamma'(\frac{{}_{q_1}\Delta_h}{h})}{\gamma(\frac{{}_{q_1}\Delta_h}{h})}\right)\{\mathcal{Q}_m(q_1;h)\} \tag{1.11}$$

and

$$\mathcal{Q}_{m-1}(q_1;h) = \hat{\mathcal{D}_{\mathcal{A}}}\{\mathcal{Q}_m(q_1;h)\} = \frac{\log(1+(\frac{{}_{q_1}\Delta_h}{h}))}{mh}\{\mathcal{Q}_m(q_1;h)\}. \tag{1.12}$$

Furthermore, considering equation (1.9), we can deduce the expression for a differential equation by utilizing equations (1.11) and (1.12) as

$$\left(\frac{q_1}{1+{}_{q_1}\Delta_h} + \frac{\gamma'(\frac{{}_{q_1}\Delta_h}{h})}{\gamma(\frac{{}_{q_1}\Delta_h}{h})} - \frac{m^2h}{\log(1+{}_{q_1}\Delta_h)}\right)\mathcal{Q}_m(q_1;h) = 0. \tag{1.13}$$

As the parameter h approaches zero, the expressions (1.11)-(1.13) simplify to the multiplicative and derivative operators. Additionally, the resulting differential equation satisfied by the Appell polynomials $\mathcal{Q}_m(u)$, defined in equation (1.1) ([2]), can be derived.

Recognizing the significance of the Δ_h hybrid special polynomials associated with Hermite polynomials, it becomes evident that these polynomials find extensive applications in both mathematics and physics. Their relevance manifests in various fields, including quantum mechanics, probability theory, approximation theory, numerical analysis, statistical mechanics, and Fourier analysis.

In the realm of quantum mechanics, the Δ_h hybrid special polynomials linked to Hermite polynomials arise naturally. They play a crucial role in quantum mechanical calculations, contributing to the understanding of fundamental phenomena and properties. Probability theory also benefits from these polynomials as they are intimately connected to the normal distribution, which is a fundamental concept in probability theory. The Δ_h hybrid special polynomials provide a means to explore and analyze the statistical properties of the normal distribution and its various applications. In the field of approximation theory, these polynomials serve as a basis for approximating functions. They offer an effective tool for numerical analysis, enabling accurate and efficient approximations of complex functions. Statistical mechanics utilizes Hermite polynomials to calculate the partition function and thermodynamic properties of ideal gases. These polynomials play a crucial role in understanding the behavior of gases and are instrumental in characterizing their thermodynamic properties. Furthermore, Hermite-based Appell polynomials incorporating the Δ_h operator are introduced, inspired by the research of Costabile and Longo [9]. These polynomials possess a generating expression of the form:

$$\gamma(t)\,(1+ht)^{\frac{q_1}{h}}\,(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}} = \sum_{m=0}^{\infty} {}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}. \quad (1.14)$$

The utilization of these hybrid special polynomials extends beyond mathematics and physics, finding potential applications in diverse fields such as image processing and computer vision. In image processing, these polynomials can be leveraged to improve image quality, enhance details, and extract meaningful features. By employing the properties of the Δ_h operator, the polynomials offer valuable tools for analyzing and manipulating digital images.

Moreover, the hybrid special polynomials demonstrate relevance in the realm of financial mathematics. They serve as mathematical models to understand and predict the behavior of various financial variables, including stock prices, interest rates, and other market-related quantities. By incorporating the Δ_h operator, these polynomials enable the analysis and forecasting of financial data, aiding decision-making processes in investment, risk management, and other financial applications.

The main objective of this article is to present a comprehensive investigation into the characteristics of the Δ_h hybrid special polynomials, specifically their connection to the Hermite polynomials. This study extensively employs principles of monomiality and operational techniques to derive and explore various properties of these polynomials. The remainder of the manuscript is organized as follows. In Section 2, we introduce the three-variable Δ_h Hermite-based Appell polynomials. We discuss their distinct features and provide a detailed analysis of their properties. This section aims to provide a comprehensive understanding of these polynomials and their behavior. Moving on to Section 3, we establish the quasi-monomial characteristics of these polynomials. By examining their behavior under certain principles and operations, we identify the specific traits that classify them as quasi-monomials. This analysis contributes to a deeper understanding of the underlying structure and properties of the Δ_h hybrid special polynomials. Finally, in Section 4, we present a selection of members from this polynomial family. We highlight their key findings and discuss the implications of these findings within the broader context of the study. This section provides valuable insights into the specific instances and applications of the Δ_h hybrid special polynomials, shedding light on their potential uses and relevance. By following this structure, we aim to comprehensively explore and present the study on the Δ_h hybrid special polynomials connected to the Hermite polynomials, offering valuable insights and findings for further research.

2. Δ_h Hermite-based Appell polynomials

In this section, we present an alternative and general approach for determining the Δ_h Hermite-based Appell sequences. This method offers an alternative perspective and methodology compared to existing approaches. By utilizing this new method, we aim to enhance the understanding and exploration of these sequences, providing a fresh perspective on their properties and applications. Thus, we have following.

Theorem 2.1. *Since, we observe Δ_h Hermite based Appell are given by (1.14), therefore we have*

$$\begin{aligned}
{}_{q_1}\Delta_h[{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)] &= mh\,{}_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_2}\Delta_h[{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)h\,{}_{\mathcal{H}}\mathcal{Q}_{m-2}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_3}\Delta_h[{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)h\,{}_{\mathcal{H}}\mathcal{Q}_{m-2}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_4}\Delta_h[{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)(m-3)h\,{}_{\mathcal{H}}\mathcal{Q}_{m-3}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_5}\Delta_h[{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)(m-3)(m-4)h\,{}_{\mathcal{H}}\mathcal{Q}_{m-4}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h).
\end{aligned} \tag{2.1}$$

Proof. Differentiating expression (1.14) with respect to q_1 in view of expression (1.2), it follows that

$$\begin{aligned}
&{}_{q_1}\Delta_h\Big[\gamma(t)\ (1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}\ (1+ht^4)^{\frac{q_4}{h}}\times\ (1+ht^5)^{\frac{q_5}{h}}\Big]\\
&= \gamma(t)\ (1+ht)^{\frac{q_1+h}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}\times\ (1+ht^5)^{\frac{q_5}{h}}\\
&\quad -\gamma(t)\ (1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}\\
&= \Big(ht\Big)\gamma(t)\ (1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}.
\end{aligned}$$

Thus, inserting the r.h.s. of expression (1.14) in previous expression, we find

$${}_{q_1}\Delta_h\left[\sum_{m=0}^{\infty}{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}\right] = h\sum_{m=0}^{\infty}{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^{m+1}}{m!}.$$

By replacing m with $m-1$ on the right-hand side of the above equation, we obtain a modified equation. We then proceed to equate the coefficients of the same powers of t in this resultant equation and thus we have:

$${}_{q_1}\Delta_h\left[\sum_{m=0}^{\infty}{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}\right] = \sum_{m=0}^{\infty}mh{}_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!},$$

we get the proof of ist equation of system of equations (2.1). By using the similar fashion, we can prove other two relations. □

Theorem 2.2. *Further, for the power series*

$$\gamma(t) = \gamma_{0,h}+\gamma_{1,h}\frac{t}{1!}+\gamma_{2,h}\frac{t^2}{2!}+\cdots+\gamma_{m,h}\frac{t^m}{m!}\cdots\ \gamma_{0,h}\neq 0,$$

with γ_m, $m=0,1,2,\ldots$ as real coefficients, Δ_h Hermite based Appell polynomials ${}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)$ $m\in\mathbb{N}$ are determined by the power series expansion of the product $\gamma(t)(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_4}{h}}$, that is

$$\begin{aligned}
&\gamma(t)(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}\\
&\quad={}_{\mathcal{H}}\mathcal{Q}_0^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)+{}_{\mathcal{H}}\mathcal{Q}_1^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t}{1!}+{}_{\mathcal{H}}\mathcal{Q}_2^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^2}{2!}\\
&\qquad+{}_{\mathcal{H}}\mathcal{Q}_3^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^3}{3!}+{}_{\mathcal{H}}\mathcal{Q}_4^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^4}{4!}\\
&\qquad+\cdots+{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}+\cdots.
\end{aligned}\tag{2.2}$$

Proof. Expanding $(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}\ (1+ht^5)^{\frac{q_5}{h}}$ by Newton series for finite differences at $q_1 = q_2 = q_3 = q_4 = q_5 = 0$ and order the product of the developments of functions $\gamma(t)$ and $(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}\ (1+ht^5)^{\frac{q_5}{h}}$ w.r.t. the powers of t, then in view of expression (1.5), we observe the polynomials ${}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)$ and expressed in equation (2.2) as coefficients of $\frac{t^m}{m!}$ as the generating function of Δ_h Hermite based Appell polynomials. □

Subsequently, we obtain the explicit series representation of the Δ_h Hermite-based Appell polynomials. To derive the formulas, our first step involves determining the explicit form of the Δ_h Hermite polynomials. This is achieved by setting $\gamma(t) = 1$ in equation (1.14), which results in:

$$(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h}H_m(q_1, q_2, q_3, q_4, q_5; h)\frac{t^m}{m!}, \tag{2.3}$$

in the form of the following result.

Theorem 2.3. *For Δ_h Hermite polynomials in 3-variables q_1, q_2, q_3, the succeeding explicit series formulae holds true:*

$$_{\Delta_h}H_m(q_1, q_2, q_3; h) = \sum_{k=0}^{[\frac{m}{k}]}\sum_{l=0}^{[\frac{k}{3}]} \binom{m}{k}\binom{k}{3l} (q_1)^h_{m-k}\ (q_2)^h_{k-3l}\ (q_3)^h_l\ \frac{(2m)!}{m!}\ \frac{(3l)!}{l!}, \tag{2.4}$$

where $(u)^h_m \equiv (u)_m$ and is given by

$$(q_1)^h_m = q_1(q_1+h)(q_1+2h)\cdots(q_1+(m-1)h),\ m = 1, 2, \dots,\ (q_1)^h_0 = 1. \tag{2.5}$$

Proof. Expanding the expression $(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}}\cdots$ using the concept of raising factorials as defined in (2.5), we obtain the following expansion:

$$(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty}\left(-\frac{q_1}{h}\right)_m(-h)^m\frac{t^m}{m!}\sum_{k=0}^{\infty}\left(-\frac{q_2}{h}\right)_k(-h)^k\frac{t^{2k}}{k!}\sum_{l=0}^{\infty}\left(-\frac{q_3}{h}\right)_l(-h)^l\frac{t^{3l}}{l!}.$$

Considering the product rule for two series, specifically the Cauchy product applied to the last two series on the right-hand side of the aforementioned expression, it can be inferred that:

$$(1+ht)^{\frac{q_1}{h}}\ (1+ht^2)^{\frac{q_2}{h}}\ (1+ht^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty}(q_1)^h_m\frac{t^m}{m!}\ \sum_{k=0}^{\infty}\sum_{l=0}^{[\frac{k}{3}]}\binom{k}{3l}(q_2)^h_{k-3l}\ (q_3)^h_l\ \frac{(3l)!}{l!}\frac{t^k}{k!}.$$

Once again, recognizing the product rule for two series, specifically the Cauchy product applied to the first two series on the right-hand side of the aforementioned expression, it can be deduced that:

$$(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}} = \sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{k}]}\sum_{l=0}^{[\frac{k}{3}]}\binom{m}{k}\binom{k}{3l}(q_1)^h_{m-k}\ (q_2)^h_{k-3l}\ (q_3)^h_l\frac{(2m)!}{m!}\ \frac{(3l)!}{l!}\frac{t^m}{m!}.$$

By inserting the series expansion of the Δ_h Hermite polynomials, as given in equation (2.3) with $q_4 = 0 = q_5$ into the left-hand side of the above equation, we obtain a resultant equation. Further, we compare the coefficients of the same powers of t on both sides. This comparison leads us to the assertion stated in equation (2.4). This assertion provides valuable insights and establishes a relationship between the coefficients and the powers of t within the equation. □

Subsequently, we proceed to derive the explicit forms of the Δ_h Hermite-based Appell polynomials by establishing the following results.

Theorem 2.4. *The Δ_h Hermite-based Appell polynomials satisfy the following explicit form:*

$$_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)=\sum_{s=0}^{[\frac{m}{s}]}\binom{m}{s}\mathcal{Q}_{s,h}\ {}_{\Delta_h}H_{m-s}(q_1,q_2,q_3,q_4,q_5;h). \tag{2.6}$$

Proof. Inserting expressions (1.5) with $h=0$ and (2.3) in the l.h.s. of equation (1.14), it follows that

$$\sum_{s=0}^{\infty}\mathcal{Q}_{s,h}(q_1)\frac{t^s}{s!}\sum_{m=0}^{\infty}{}_{\Delta_h}H_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}=\sum_{m=0}^{\infty}{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}.$$

By considering the C.P. rule, we examine the left-hand side of the above equation and compare it to the resultant equation. Specifically, we focus on comparing the coefficients of the same powers of t in both equations. Through this comparison, we arrive at the assertion stated in equation (2.6). This assertion highlights an important relationship between the coefficients and the powers of t within the equation. □

Theorem 2.5. *The Δ_h Hermite-based Appell polynomials, satisfy the following explicit form:*

$$_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)=\sum_{s=0}^{[\frac{m}{s}]}\binom{m}{s}\gamma_{s,h}\ {}_{\Delta_h}H_{m-s}(q_1,q_2,q_3,q_4,q_5;h). \tag{2.7}$$

Proof. Inserting expressions (1.6) and (2.3) in the l.h.s. of equation (1.14), it follows that

$$\sum_{s=0}^{\infty}\gamma_{s,h}\frac{t^s}{s!}\sum_{m=0}^{\infty}{}_{\Delta_h}H_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}=\sum_{m=0}^{\infty}{}_{\Delta_h H}\mathcal{Q}_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!}.$$

Using the C.P. rule in the l.h.s. of the above equation and in the resultant equation, comparing the coefficients of like powers of t, we are lead to assertion (2.7). □

3. Monomiality principle

In this section, we aim to establish the quasi-monomial characteristics demonstrated by the Δ_h Hermite-based Appell polynomials. To achieve this objective, we present the proofs for the following outcomes.

Theorem 3.1. *The Δ_h Hermite-based Appell polynomials satisfy the following multiplicative and derivative operators:*

$$\begin{aligned}{}_{\Delta_h H}\mathcal{Q}_{m+1}(q_1,q_2,q_3,q_4,q_5;h)&=\hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h H}\mathcal{Q}_m(q_1,q_2,q_3,q_4,q_5;h)\}\\&=\Bigg(\frac{q_1}{1+{}_{q_1}\Delta_h}+\frac{2q_{2q_1}\Delta_h}{h+{}_{q_1}\Delta_h{}^2}+\frac{3q_{3q_1}\Delta_h{}^2}{h^2+{}_{q_1}\Delta_h{}^3}+\frac{4q_{4q_1}\Delta_h{}^3}{h^3+{}_{q_1}\Delta_h{}^4}\\&\quad+\frac{5q_{5q_1}\Delta_h{}^4}{h^4+{}_{q_1}\Delta_h{}^5}+\frac{\gamma'(\frac{q_1\Delta_h}{h})}{\gamma(\frac{q_1\Delta_h}{h})}\Bigg)\{{}_{\Delta_h H}\mathcal{Q}_m(q_1,q_2,q_3,q_4,q_5;h)\}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}{}_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)&=\hat{\mathcal{D}}_{\Delta_h}\{{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\}\\&=\frac{\log(1+{}_{q_1}\Delta_h)}{mh}\{{}_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1,q_2,q_3,q_4,q_5;h)\},\end{aligned} \tag{3.2}$$

respectively.

Proof. In view of finite difference operator Δ_h, we have

$$_{q_1}\Delta_h[_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)] = h\ t\ [_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)],$$

or

$$\frac{_{q_1}\Delta_h}{h}[_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)] = t\ [_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)]. \tag{3.3}$$

Differentiating the expression (1.14) w.r.t. t and u, respectively, we have

$$\begin{aligned} &_{\Delta_h H}\mathcal{Q}_{m+1}(q_1, q_2, q_3, q_4, q_5; h) \\ &= \hat{\mathcal{M}_{\Delta_h}}\{_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)\} \\ &= \left(\frac{q_1}{1+ht} + \frac{2q_2t}{1+ht^2} + \frac{3q_3t^2}{1+ht^3} + \cdots + \frac{\gamma'(t)}{\gamma(t)}\right)\{_{\Delta_h H}\mathcal{Q}_m(q_1, q_2, q_3, q_4, q_5; h)\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} {}_{\mathcal{H}}\mathcal{Q}_{m-1}^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h) &= \hat{\mathcal{D}_{\Delta_h}}\{_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)\} \\ &= \frac{\log(1+ht)}{mh}\{_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)\}, \end{aligned} \tag{3.5}$$

respectively. By employing identity (3.3), which is based on equations (1.7) and (1.8), we utilize this identity in the context of equations (3.4) and (3.5). Through this process, we derive assertions (3.1) and (3.2). These assertions highlight important results that are obtained by applying the mentioned identity and equations, shedding light on the properties and relationships of the involved expressions. □

Corollary 3.2. *The Δ_h Hermite-based Appell polynomials satisfy the following differential equation:*

$$\begin{aligned} &\left(\frac{q_1}{1+{}_{q_1}\Delta_h} + \frac{2q_{2\,q_1}\Delta_h}{h+{}_{q_1}\Delta_h{}^2} + \frac{3q_{3\,q_1}\Delta_h{}^2}{h^2+{}_{q_1}\Delta_h{}^3} + \cdots + \frac{\gamma'(\frac{q_1\Delta_h}{h})}{\gamma(\frac{q_1\Delta_h}{h})} - \frac{m^2h}{\log(1+{}_{q_1}\Delta_h)}\right) \\ &\qquad \times\{_{\mathcal{H}}\mathcal{Q}_m^{[\Delta_h]}(q_1, q_2, q_3, q_4, q_5; h)\} = 0. \end{aligned} \tag{3.6}$$

Proof. Making use of expressions (3.1) and (3.2) in (1.9), we are lead to assertion (3.6). □

For $q_2, q_3, \ldots \to 0$, the expressions (3.1), (3.2), and (3.6) reduce to the multiplicative and derivative operators, and the differential equation satisfied by Δ_h Appell polynomials $\mathcal{Q}_m(q_1; h)$ given by expressions (1.11)-(1.13).

For $h \to 0$, the expressions (3.1), (3.2), and (3.6) reduce to the multiplicative and derivative operators, and differential equation satisfied by Appell polynomials $\mathcal{Q}_m(q_1)$ given by expression (1.1).

4. Examples

The Appell polynomial family encompasses a wide range of members that can be obtained by selecting an appropriate function $\gamma(t)$. These members possess distinct names, generating functions, and associated numbers. Below, we provide information on the generating function for the Δ_h Bernoulli polynomials, denoted as $_{\Delta_h}\beta_m(q_1; h)$.

The generating function for the Δ_h Bernoulli polynomials $_{\Delta_h}\beta_m(q_1; h)$ is given by

$$\frac{t}{(1+ht)^{\frac{1}{h}}-1}(1+ht)^{\frac{q_1}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h}\beta_m(q_1; h)\frac{t^m}{m!}, \quad |t| < 2\pi. \tag{4.1}$$

The generating function for the Euler polynomials ${}_{\Delta_h}\mathcal{E}_m(q_1;h)$ is given by

$$\frac{2}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{q_1}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h}\mathcal{E}_m(q_1;h)\frac{t^m}{m!},\quad |t|<\pi. \tag{4.2}$$

The generating function for the Genocchi polynomials ${}_{\Delta_h}\mathcal{G}_m(q_1;h)$ is given by

$$\frac{2t}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{q_1}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h}\mathcal{G}_m(q_1;h)\frac{t^m}{m!},\quad |t|<\pi. \tag{4.3}$$

For $h\to 0$, polynomials (4.1)-(4.3) reduce to the Bernoulli, Euler, and Genocchi polynomials [8].

The Δ_h polynomials and numbers of Bernoulli, Euler, and Genocchi have found numerous applications in various areas of mathematics, including number theory, combinatorics, and numerical analysis. These applications extend to practical mathematics, where these polynomials and numbers are utilized to solve problems and derive mathematical formulas.

For instance, the Bernoulli numbers are prominently featured in diverse mathematical formulas, including the Taylor expansion, trigonometric and hyperbolic tangent and cotangent functions, as well as sums of powers of natural numbers. These numbers hold a pivotal position in number theory, offering valuable insights into patterns and relationships among integers.

Similarly, the Euler numbers emerge in the Taylor expansion and exhibit close associations with trigonometric and hyperbolic secant functions. Beyond this, they find applications in graph theory, automata theory, and the computation of the number of up-down ascending sequences. This contribution enhances the analysis of structures and patterns within the realm of discrete mathematics.

Furthermore, the Genocchi numbers prove beneficial in the domains of graph theory and automata theory, with a particular emphasis on counting the number of up-down ascending sequences. This application involves a detailed study of the order and arrangement of elements in a sequence. Consequently, these Δ_h polynomials and numbers associated with Bernoulli, Euler, and Genocchi play a pivotal role across various mathematical disciplines, facilitating exploration of mathematical relationships, derivation of formulas, and analysis of patterns and structures.

By appropriately choosing the function $\gamma(t)$ in equation (1.14), we can establish the following generating functions for the Δ_h Hermite-based Bernoulli, Euler, and Genocchi polynomials

$$\begin{aligned}
&\frac{t}{(1+ht)^{\frac{1}{h}}-1}(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}\\
&\qquad=\sum_{m=0}^{\infty}{}_{\Delta_h H}\beta_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!},\\
&\frac{2}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}\\
&\qquad=\sum_{m=0}^{\infty}{}_{\Delta_h H}\mathcal{E}_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!},\\
&\frac{2t}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{q_1}{h}}(1+ht^2)^{\frac{q_2}{h}}(1+ht^3)^{\frac{q_3}{h}}(1+ht^4)^{\frac{q_4}{h}}(1+ht^5)^{\frac{q_5}{h}}\\
&\qquad=\sum_{m=0}^{\infty}{}_{\Delta_h H}\mathcal{G}_m(q_1,q_2,q_3,q_4,q_5;h)\frac{t^m}{m!},
\end{aligned} \tag{4.4}$$

respectively. Thus the corresponding results can be obtained for these polynomials.

Theorem 4.1. *As we can observe from equations (4.4), the Δ_h Hermite-based Bernoulli, Euler, and Genocchi polynomials are defined. Consequently, these polynomials exhibit certain relations and properties that can be summarized* *as follows:*

$$
\begin{aligned}
{}_{q_1}\Delta_h[{}_{\Delta_h H}\beta_m(q_1,q_2,q_3,q_4,q_5;h)] &= mh\ {}_{\Delta_h H}\beta_{m-1}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_2}\Delta_h[{}_{\Delta_h H}\beta_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)h\ {}_{\Delta_h H}\beta_{m-2}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_3}\Delta_h[{}_{\Delta_h H}\beta_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\beta_{m-2}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_1}\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(q_1,q_2,q_3,q_4,q_5;h)] &= mh\ {}_{\Delta_h H}\mathcal{E}_{m-1}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_2}\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)h\ {}_{\Delta_h H}\mathcal{E}_{m-2}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_3}\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\mathcal{E}_{m-2}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_1}\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(q_1,q_2,q_3,q_4,q_5;h)] &= mh\ {}_{\Delta_h H}\mathcal{G}_{m-1}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_2}\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)h\ {}_{\Delta_h H}\mathcal{G}_{m-2}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{q_3}\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(q_1,q_2,q_3,q_4,q_5;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\mathcal{G}_{m-2}(q_1,q_2,q_3,q_4,q_5;h),
\end{aligned}
$$

respectively.

Also, in view of equation (2.7), these polynomials satisfy the following explicit form.

Theorem 4.2. *The Δ_h Hermite-based Bernoulli, Euler, and Genocchi polynomials, satisfy the following explicit form:*

$$
\begin{aligned}
{}_{\Delta_h H}\beta_m(q_1,q_2,q_3,q_4,q_5;h) &= \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \beta_{s,h}\ {}_{\Delta_h}H_{m-s}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{\Delta_h H}\mathcal{E}_m(q_1,q_2,q_3,q_4,q_5;h) &= \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \mathcal{E}_{s,h}\ {}_{\Delta_h}H_{m-s}(q_1,q_2,q_3,q_4,q_5;h),\\
{}_{\Delta_h H}\mathcal{G}_m(q_1,q_2,q_3,q_4,q_5;h) &= \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \mathcal{G}_{s,h}\ {}_{\Delta_h}H_{m-s}(q_1,q_2,q_3,q_4,q_5;h),
\end{aligned}
$$

respectively.

Likewise, utilizing similar methods, we can establish additional results and properties for these polynomials. The Bernoulli, Euler, and Genocchi numbers associated with these polynomials find practical applications in various fields such as graph theory, automata theory, and the calculation of up-down ascending sequences. Therefore, it would be of interest for future research to explore the physical significance and potential applications of these hybrid polynomials and the hybrid special numbers derived from them.

5. Conclusion

In this study, we have introduced a new class of polynomials called the hybrid Δ_h Hermite-based Appell polynomials. These polynomials are obtained through the convolution of Δ_h Appell polynomials and Hermite polynomials of several variables. We have presented several specific features of these polynomials, including the establishment of their quasi-monomial characteristics in a dedicated section.

Furthermore, we have derived various results for these polynomials. Theorem 2.1 establishes forward difference relations for the hybrid Δ_h Hermite-based Appell polynomials, providing a useful tool for their computation. Additionally, we have obtained explicit forms for some members of this polynomial family, allowing for a better understanding of their structure and behavior.

Looking ahead, there are several avenues for future research and investigation. Extended and generalized forms of the hybrid Δ_h Hermite-based Appell polynomials can be explored, potentially leading to new insights and applications. Integral representations, recurrence relations, shift operators, and summation formulas are other aspects that can be investigated to further enrich the understanding of these polynomials.

Moreover, exploring interpolation forms and studying the properties associated with them would be of interest. By investigating the interpolation properties, one can potentially derive polynomial approximations and interpolation schemes using the hybrid Δ_h Hermite-based Appell polynomials.

In conclusion, this study provides an initial exploration of the hybrid Δ_h Hermite-based Appell polynomials, showcasing their specific features and potential applications. Further research efforts can contribute to uncovering more properties and expanding their utility in various mathematical and scientific domains.

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