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Supra finite soft-open sets and applications to operators and continuity

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Abstract

In this article, we study the concept of finite open sets in the spaces of supra soft-topologies. We aim by defining this concept to furnish the researchers with a tool they generate new forms of topological concepts via supra soft-topologies such as continuity, compactness, separation axioms, etc. We discuss the main properties of this concept and prove that this class of soft sets is neither infra soft-topology nor supra soft-topology. Also, we reveal its connections with other celebrated generalizations of supra soft-open sets and conclude the conditions they guarantee the equivalence between them. After that, we display some soft operators, like interior and closure, inspired by the classes of supra finite soft-open and supra finite soft-closed sets. Finally, we set forth some types of continuity defined by these classes and describe their main characterizations and behaviors in different cases such as decomposition theorems and transition between the realms of of supra soft-topologies and their parametric supra topologies. The obtained results and implementations are articulated with the aid of some counterexamples.

Keywords: Supra finite soft-open set, supra soft-topology, soft operators, soft continuity.

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1. Introduction

The idea of soft-sets was founded by Molodtsov [53], in 1999, as an efficient approach to transact with suspicion and vagueness. Molodtsov [53] demonstrated some applications of it in diverse fields. This approach is free from the inherent restrictions of the foregoing approaches since it does not require previous procedures such as membership functions and equivalence relations in the theories of fuzzy sets and rough sets, respectively. According to the published monographs, soft set theory has proven to be an invaluable tool for modeling and analyzing various real-world scenarios that appeared in different disciplines such as information theory [51], economics [15], medical science [38, 46], and engineering.

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To build the core concepts and notions of soft-set theory, Maji et al. [50] proposed the definitions of empty and whole sets and the operators of union, intersection, and difference in soft environment. Some of these definitions suffer from some shortcomings, so they were adjusted and reformulated by [8, 43]. In the year 2011, Shabir-Naz [58] pioneered the concept of soft-topology and provided the basic notions of it. The study of classical topological concepts in the frame of soft-topology was subsequently conducted by many authors. Among such studies, soft separation axioms by [3, 52], soft compact and Lindelöf spaces by [35, 44], soft connectedness and soft hyperconnectedness by [31, 55], soft extremally disconnectedness by [34], soft basis by [5], caliber and chain conditions by [6], soft mappings by [45], expandable spaces by [56], soft menger spaces by [25, 47], and soft continuity and homeomorphism by [61, 63]. Also, it has been introduced some generalizations of soft-open sets by following similar techniques to those applied in classical topologies. This leads to insert and generate various forms of soft-topological concepts. The first works in this area were soft semi-open sets by [37], soft α-open by [4], soft somewhere dense [7], soft Q-sets [11]. The practical importance of these generalizations comes from the employment to handle practical problems as illustrated in [18], whereas their theoretical applications are to enable us to create novel versions of topological concepts as introduced in [9, 31, 32]. Some divergences between classical and soft characteristics of some concepts were not taken into account by some authors, which gave rise to some invalid properties as explained by Al-shami [12]. In the last year, Al-shami with some coauthors adopted a fresh way to define generalizations of soft-open sets, which are feebler of their analogous in soft topologies, using the crisp topologies inspired by a soft topology, for more details we refer the readers to the references [21–23, 26, 27, 30, 33].

It was generalized soft-topology in different ways; one of them is to dispense with one of soft-topological terms or more. In classical setting, it was introduced the concept of supra topology by precluding the term of finite intersection. Subsequentally, this structure attracted many topologists they examined topological properties via it, see [10, 13, 17, 42]. In 2014, El-Sheikh and Abd El-latif [40] defined this concept via soft settings and set up main ideas and notions. In the evolutionary path of this field, some researchers and scholars discussed various topics in supra soft-topology like separation axioms [19, 59], covering properties [2], and generalizations of open sets [1, 19, 29]. To contribute to this line, we conduct this work. We see that the structures of supra soft-topology do not get the attention they deserve; especially, after proving that supra topologies have a distinct significance across a broad spectrum of branches such as information systems [20] and digital image processing [48]. Therefore, we hope by this work we provide a novel method to cope with real problems as well as to establish topological concepts that are stronger than their analogous produced by supra soft-open sets and their well-known extensions.

After this introduction, we divide the rest of this article into five sections as follows. The next section is allocated to remember the necessary concepts and facts to understand this content. The main part is Section 3, where we specify to present the concept of "supra finite soft-open sets" which represents a family of supra soft-open sets with the finite region of soft boundary. In Section 4, we define the operators of soft fsso-interior, fsso-closure, fsso-boundary, and fsso-derived and prob the interrelations between them. We introduce in Section 5 three shapes of soft continuity and shed light on their transamination from supra soft-topology to crisp supra topologies they are generated by (2.1). Some counterexamples are built to show these navigations. Ultimately, we summarize the obtained results and suggest a plan for some upcoming work in Section 6.

2. Basic background

The main definitions and properties that are necessary to be familiar with the content of manuscript are recalled in this section.

2.1. Soft set theory

Definition 2.1 ([53])**.** A mapping $\mho : \Lambda \to 2^{U}$ is called a soft-set, denoted by $(\mho, \Lambda)$, in reference to the universal set $U \neq \emptyset$ and a set of parameters $\Lambda \neq \emptyset$. Briefly, we write $\bowtie$-set (resp., $\bowtie$-subset) instead of a soft set (resp., soft subset) and express about it by the following formula.

$$(\mathcal{O},\Lambda)=\{(\lambda,\mathcal{O}(\lambda)):\lambda\in\Lambda \text{ and } \mathcal{O}(\lambda)\in 2^{U}\}.$$

Every image of a parameter $\mathcal{O}(\lambda)$ is named a component of a $\bowtie$-set $(\mathcal{O},\Lambda)$. The family of all $\bowtie$-sets over U with Λ is denoted by $\mathfrak{F}(U,\Lambda)$.

Definition 2.2 ([8])**.** The complement of $(\mathcal{O},\Lambda)$, indicated by $(\mathcal{O},\Lambda)^c$ or $(\mathcal{O}^c,\Lambda)$, is given by

$$\mathcal{O}^c(\lambda)=U-\mathcal{O}(\lambda) \text{ for each } \lambda\in\Lambda.$$

Definition 2.3 ([41, 49, 54])**.** An absolute $\bowtie$-set $(\mathcal{O},\Lambda)$ is defined by $\mathcal{O}(\lambda)=U$ for each $\lambda\in\Lambda$, whereas its complement is named a null $\bowtie$-set. They are symbolized by $\widetilde{U}$ and Φ, respectively. If $(\mathcal{O},\Lambda)$ is given by $\mathcal{O}(\lambda)=u\in U$ and for each $\lambda^*\in\Lambda-\{\lambda\}$ we have $\mathcal{O}(\lambda^*)=\emptyset$, then we call $(\mathcal{O},\Lambda)$ a soft point and it is assigned by u_λ. A $\bowtie$-set $(\mathcal{O},\Lambda)$ is named infinite (resp., uncountable) if $\mathcal{O}(\lambda)$ is infinite (resp., uncountable) for some $\lambda\in\Lambda$; otherwise it is called finite (resp., countable). If for each $\lambda\in\Lambda$ we have either $\mathcal{O}(\lambda)=\emptyset$ or $\mathcal{O}(\lambda)=U$, then we call $(\mathcal{O},\Lambda)$ a pseudo constant $\bowtie$-set. And if $\mathcal{O}(\lambda)=A\subseteq U$ for each $\lambda\in\Lambda$, then we call $(\mathcal{O},\Lambda)$ a stable $\bowtie$-set.

It is easy to see that null and absolute $\bowtie$-sets are pseudo constant and stable $\bowtie$-sets.

Definition 2.4. Let $(\mathcal{O},\Lambda)$ and $(\mathcal{P},\Lambda)$ be $\bowtie$-sets. Then we write

(i) $(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{P},\Lambda)$, if for every $\lambda\in\Lambda$ we have $\mathcal{O}(\lambda)\subseteq\mathcal{P}(\lambda)$ ([43].
(ii) $(\mathcal{O},\Lambda)\sqcup(\mathcal{P},\Lambda)=(\mathcal{Z},\Lambda)$, if for every $\lambda\in\Lambda$ we have $\mathcal{Z}(\lambda)=\mathcal{O}(\lambda)\bigcup\mathcal{P}(\lambda)$ ([49]).
(iii) $(\mathcal{O},\Lambda)\sqcap(\mathcal{P},\Lambda)=(\mathcal{Z},\Lambda)$, if for every $\lambda\in\Lambda$ we have $\mathcal{Z}(\lambda)=\mathcal{O}(\lambda)\bigcap\mathcal{P}(\lambda)$ ([8]).
(iv) $(\mathcal{O},\Lambda)\triangle(\mathcal{P},\Lambda)=(\mathcal{Z},\Lambda)$, if for every $\lambda\in\Lambda$ we have $\mathcal{Z}(\lambda)=\mathcal{O}(\lambda)\backslash\mathcal{P}(\lambda)$ ([8]).
(v) $(\mathcal{O},\Lambda)\times(\mathcal{P},\Lambda)=(\mathcal{Z},\Lambda\times\Lambda)$, where $\mathcal{Z}(\lambda_1,\lambda_2)=\mathcal{O}(\lambda_1)\times\mathcal{P}(\lambda_2)$ for every $(\lambda_1,\lambda_2)\in\Lambda\times\Lambda$.

Definition 2.5 ([41, 58, 62])**.** Let $u\in U$ and $(\mathcal{O},\Lambda)$ be a $\bowtie$-set. Then we say that

(i) $u_\lambda\in(\mathcal{O},\Lambda)$ whenever $u\in\mathcal{O}(\lambda)$;
(ii) $u\in(\mathcal{O},\Lambda)$ (resp., $u\Subset(\mathcal{O},\Lambda)$) if $u\in\mathcal{O}(\lambda)$ for every (resp., some) $\lambda\in\Lambda$.

The negation of the above-mentioned relations is given as

(i) $u_\lambda\notin(\mathcal{O},\Lambda)$ whenever $u\notin\mathcal{O}(\lambda)$.
(ii) $u\notin(\mathcal{O},\Lambda)$ (resp., $u\not\Subset(\mathcal{O},\Lambda)$) if $u\notin\mathcal{O}(\lambda)$ for some (resp., every) $\lambda\in\Lambda$.

Definition 2.6 ([16])**.** A soft mapping $h_g:\mathfrak{F}(U,\Lambda)\to\mathfrak{F}(V,\Xi)$, where $h:U\to V$ and $g:\Lambda\to\Xi$ are crisp mappings, is a relation associating each $u_\lambda\in\mathfrak{F}(U,\Lambda)$ with one and only one $v_\xi\in\mathfrak{F}(V,\Xi)$ such that $h_g(u_\lambda)=h(u)_{g(\lambda)}$ for every $u\in U$ and $\lambda\in\Lambda$.

In addition, $h_g^{-1}(v_\xi)=\bigsqcup_{\substack{u\in h^{-1}(v)\\ \lambda\in g^{-1}(\xi)}}u_\lambda$ for every $v\in V$ and $\xi\in\Xi$.

2.2. *Supra soft topology*

Definition 2.7 ([40])**.** A family $\mathbb{T}$ of $\bowtie$-sets over U with Λ forms a supra soft-topology over U with Λ providing that the absolute and null $\bowtie$-sets are elements of it, and it is closed under arbitrary soft-unions.

In this case, the triplet $(U,\mathbb{T},\Lambda)$ is called a supra soft-topological space (briefly, SST-space). Every element of $\mathbb{T}$ is called supra soft-open (in short, supra $\bowtie$-open) and its complement is called supra soft-closed (in short, supra $\bowtie$-closed).

We term an SST-space $(U,\mathbb{T},\Lambda)$ supra soft locally indiscrete if $(\mathcal{O},\Lambda)$ is supra $\bowtie$-closed whenever it is supra $\bowtie$-open.

Definition 2.8 ([58])**.** For a $\bowtie$-subset $(\mathcal{O},\Lambda)$ of an SST-space $(U,\mathbb{T},\Lambda)$, the soft-union of all supra $\bowtie$-open subsets of $(\mathcal{O},\Lambda)$ is named a supra soft-interior of $(\mathcal{O},\Lambda)$, and the soft-intersection of all supra $\bowtie$-closed supersets of $(\mathcal{O},\Lambda)$ is named a supra soft-closure of $(\mathcal{O},\Lambda)$. These operators are respectively denoted by $\square(\mathcal{O},\Lambda)$ and $\blacksquare(\mathcal{O},\Lambda)$. The soft difference between the operators of supra soft-closure and supra soft-interior of a soft subset $(\mathcal{O},\Lambda)$ is termed supra soft-boundary and denoted by $\mathbf{B}(\mathcal{O},\Lambda)$. That is, $\mathbf{B}(\mathcal{O},\Lambda) = \blacksquare(\mathcal{O},\Lambda)\Delta\square(\mathcal{O},\Lambda)$.

Definition 2.9 ([55, 61])**.**

(i) if the only supra $\bowtie$-clopen subsets of an SST-space $(U,\mathbb{T},\Lambda)$ are null and absolute $\bowtie$-sets, then $(U,\mathbb{T},\Lambda)$ is named supra soft connected.

(ii) if every two non-null supra $\bowtie$-open subsets of an SST-space $(U,\mathbb{T},\Lambda)$ has a non-null soft-intersection, then $(U,\mathbb{T},\Lambda)$ is named supra soft hyperconnected.

Definition 2.10 ([4, 11, 37])**.** A $\bowtie$-subset $(\mathcal{O},\Lambda)$ of $(U,\mathbb{T},\Lambda)$ is called supra $\bowtie$ α-open, supra $\bowtie$-pre open, supra $\bowtie$-semi open, supra $\bowtie$ b-open, supra $\bowtie$ β-open, and supra $\bowtie$-somewhere dense subsets if the following conditions are respectively satisfied

$(\mathcal{O},\Lambda) \sqsubseteq \square(\blacksquare(\square(\mathcal{O},\Lambda)))$, $(\mathcal{O},\Lambda) \sqsubseteq \square(\blacksquare(\mathcal{O},\Lambda))$, $(\mathcal{O},\Lambda) \sqsubseteq \blacksquare(\square(\mathcal{O},\Lambda))$,
$(\mathcal{O},\Lambda) \sqsubseteq \square(\blacksquare(\mathcal{O},\Lambda)) \sqcup \blacksquare(\square(\mathcal{O},\Lambda))$, $(\mathcal{O},\Lambda) \sqsubseteq \blacksquare(\square(\blacksquare(\mathcal{O},\Lambda)))$, and $(\mathcal{O},\Lambda) = \Phi$ or $\square(\blacksquare(\mathcal{O},\Lambda)) \neq \Phi$.

It was exhibited the next formula that explains how to inherit crisp supra topologies from a supra soft-topology.

Proposition 2.11 ([58])**.** *Let $(U,\mathbb{T},\Lambda)$ be an SST-space. Then for each $\lambda \in \Lambda$ the class*

$$\mathbb{T}_\lambda = \{\mathcal{O}(\lambda) : (\mathcal{O},\Lambda) \in \mathbb{T}\}, \tag{2.1}$$

represents a crisp supra topology on U. Since this class is produced for each parameters, we shall name a parametric supra topology (in short, p-supra topology).

Definition 2.12 ([24])**.** Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of an SST-space $(U,\mathbb{T},\Lambda)$. Then $(\square(\mathcal{O}),\Lambda)$ and $(\blacksquare(\mathcal{O}),\Lambda)$ are given by $\square(\mathcal{O})(\lambda) = \square(\mathcal{O}(\lambda))$ and $\blacksquare(\mathcal{O})(\lambda) = \blacksquare(\mathcal{O}(\lambda))$, respectively, where $\square(\mathcal{O}(\lambda))$ and $\blacksquare(\mathcal{O}(\lambda))$ are operators of supra interior and supra closure of $\mathcal{O}(\lambda)$ in a p-supra topological space $(U,\mathbb{T}_\lambda)$, respectively.

Theorem 2.13 ([40])**.** *A mapping $h : (U,\mathbb{T}) \to (V,\mathbb{Y})$ is supra continuous if and only if the inverse image of each supra open set is supra open.*

Theorem 2.14 ([24])**.** *If $h_g : (U,\mathbb{T},\Lambda) \to (V,\mathbb{Y},\Lambda)$ is supra soft-continuous, then $h : (U,\mathbb{T}_\lambda) \to (V,\mathbb{Y}_{g(\lambda)})$ is supra continuous for all $\lambda \in \Lambda$.*

3. Finite supra $\bowtie$-open sets

In the present section, we will study the concept of finite $\bowtie$-open sets [29] in the habitat of supra soft-topology. We clarify that the family of supra finite $\bowtie$-open sets fails to be supra soft-topology or infra soft-topology. We also delve into studying its essential properties and figure out the equivalent classes of it under some conditions such as supra soft locally indiscrete and supra soft submaximal. Furthermore, we manifest that each component of a supra finite $\bowtie$-open set is supra finite open, but having this property by each component of a soft set does not imply that it is supra finite $\bowtie$-open.

Definition 3.1. A supra $\bowtie$-open subset $(\mathcal{O},\Lambda)$ of an SST-space $(U,\mathbb{T},\Lambda)$ satisfying $\blacksquare(\mathcal{O},\Lambda) \triangle (\mathcal{O},\Lambda)$ is finite is called a finite supra $\bowtie$-open. That is, $(\mathcal{O},\Lambda)$ is stated to be a finite supra $\bowtie$-open set if it is supra $\bowtie$-open and its soft boundary is a finite $\bowtie$-set. The complement of a finite supra $\bowtie$-open set is a finite supra $\bowtie$-closed set.

The analogous characteristic of a finite supra $\bowtie$-closed set is provided in the following.

Proposition 3.2. *Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of an SST-space $(\mathrm{U},\mathbb{T},\Lambda)$. Then, $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-closed and only if it is supra $\bowtie$-closed and $(\mathcal{O},\Lambda)\bigtriangleup\square(\mathcal{O},\Lambda)$ is finite.*

Proof.

$\Rightarrow$ Assume that $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-closed. Directly, we obtain $(\mathcal{O},\Lambda)$ is supra $\bowtie$-closed and $\blacksquare(\mathcal{O}^c,\Lambda)\bigtriangleup(\mathcal{O}^c,\Lambda)$ is finite. The next equivalence is obvious:

$$\blacksquare(\mathcal{O}^c,\Lambda)\bigtriangleup(\mathcal{O}^c,\Lambda)=[\square(\mathcal{O},\Lambda)]^c\cap(\mathcal{O},\Lambda)=(\mathcal{O},\Lambda)\bigtriangleup\square(\mathcal{O},\Lambda).$$

Subsequently, $(\mathcal{O},\Lambda)\bigtriangleup\square(\mathcal{O},\Lambda)$ is finite.

$\Leftarrow$ Let the sufficient conditions be hold. Then, $(\mathcal{O}^c,\Lambda)$ is supra $\bowtie$-open. By the above equivalence, we get $\blacksquare(\mathcal{O}^c,\Lambda)\bigtriangleup(\mathcal{O}^c,\Lambda)$ is finite. So that, $(\mathcal{O}^c,\Lambda)$ is finite supra $\bowtie$-open, which proves the required result. □

Before we discover the main properties of these types of $\bowtie$-sets, we shall display the next proposition which we need to understand the sequels of this manuscript.

Proposition 3.3. *The next properties hold for every $\bowtie$-subset $(\mathcal{O},\Lambda)$ of an SST-space $(\mathrm{U},\mathbb{T},\Lambda)$.*

(i) $\mathbf{B}(\mathcal{O},\Lambda)=\blacksquare(\mathcal{O},\Lambda)\sqcap\blacksquare(\mathcal{O}^c,\Lambda)$.
(ii) $\mathbf{B}(\mathcal{O},\Lambda)$ *is supra $\bowtie$-closed.*
(iii) $(\mathcal{O},\Lambda)$ *is supra $\bowtie$-closed (resp., supra $\bowtie$-open) if and only if* $\mathbf{B}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O},\Lambda)$ *(resp.,* $\mathbf{B}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O}^c,\Lambda)$*).*
(iv) $(\mathcal{O},\Lambda)$ *is supra $\bowtie$-clopen if and only if* $\mathbf{B}(\mathcal{O},\Lambda)=\Phi$.

Proof. Analogous to the proof technique given in classical topology. □

Remark 3.4.

(i) A supra $\bowtie$-open set $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-open providing that $\mathbf{B}(\mathcal{O},\Lambda)$ is finite.
(ii) A finite $\bowtie$-set is finite supra $\bowtie$-open providing that $\mathbf{B}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O}^c,\Lambda)$.

Proposition 3.5. *The inverse image of finite supra $\bowtie$-open set is preserved under injective soft continuity.*

Proof. Let $(\mathcal{O},\Lambda)$ be a finite supra $\bowtie$-open subset of $(\mathrm{W},\mathbb{Y},\Lambda)$ and $h_g:(\mathrm{U},\mathbb{T},\Lambda)\to(\mathrm{W},\mathbb{Y},\Lambda)$ be a soft continuous mapping. We directly obtain $h_g^{-1}(\mathcal{O},\Lambda)$ is soft open and

$$\blacksquare[h_g^{-1}(\mathcal{O},\Lambda)]\bigtriangleup h_g^{-1}(\mathcal{O},\Lambda)\sqsubseteq h_g^{-1}[\blacksquare(\mathcal{O},\Lambda)]\bigtriangleup h_g^{-1}(\mathcal{O},\Lambda)=h_g^{-1}[\blacksquare(\mathcal{O},\Lambda)\bigtriangleup(\mathcal{O},\Lambda)].$$

Since $\blacksquare(\mathcal{O},\Lambda)\bigtriangleup(\mathcal{O},\Lambda)$ is finite, then by injectiveness of h_g we find $h_g^{-1}[\blacksquare(\mathcal{O},\Lambda)\bigtriangleup(\mathcal{O},\Lambda)]$ is finite. Hence, $\blacksquare[h_g^{-1}(\mathcal{O},\Lambda)]\bigtriangleup h_g^{-1}(\mathcal{O},\Lambda)$ is finite. This completes the proof. □

Corollary 3.6. *Finite supra $\bowtie$-open set is an invariant property.*

The condition of injective provided the aforementioned proposition is dispensable as the below example points out.

Example 3.7. Pretend we have SST-spaces $(\mathcal{R},\mathcal{S},\Lambda)$ and $(\mathrm{U},\mathbb{T},\Lambda)$ such that $\mathcal{R}$ is the set of real numbers, $\mathrm{U}=\{u,v\}$, $\Lambda=\{\lambda_1,\lambda_2,\lambda_3\}$, and the soft topologies $\mathcal{S}$ over $\mathcal{R}$ with Λ and $\mathbb{T}$ over U with Λ are given by

$$\mathcal{S}=\{\widetilde{\mathcal{R}},(\mathcal{O},\Lambda)\sqsubseteq\widetilde{\mathcal{R}}:4\in(\mathcal{O},\Lambda)\}\cup\{\Phi\}\ \text{ and }\ \mathbb{T}=\{\Phi,\widetilde{\mathrm{U}},(\mathcal{Z},\Lambda)=\{(\lambda_1,\{u\}),(\lambda_2,\{u\}),(\lambda_3,\{u\})\}\}.$$

where a soft mapping $h_g:(\mathcal{R},\mathcal{S},\Lambda)\to(\mathrm{U},\mathbb{T},\Lambda)$ is defined by the following:

$$g \text{ is the identity mapping and } h(4)=u \text{ and for each } r\neq 4 \text{ we have } h(r)=v.$$

It is obvious that h_g is not injective. Now, $(\mathcal{Z},\Lambda)$ is a finite supra $\bowtie$-open subset of $(\mathrm{U},\mathbb{T},\Lambda)$ but $h_g^{-1}(\mathcal{Z},\Lambda)=\{(\lambda_1,\{4\}),(\lambda_2,\{4\}),(\lambda_3,\{4\})\}$ is not finite soft open in $(\mathcal{R},,\Lambda)$ in spite of h_g is supra soft-continuous.

Proposition 3.8. *Finite supra ⋈-open subsets is closed under finite cartesian product.*

Proof. Let $(\mathcal{O},\Lambda)$ and $(\mathcal{Z},\Lambda)$ be finite supra ⋈-open sets. As we know the product of finite numbers of supra ⋈-open sets is supra ⋈-open, also, we know that $\blacksquare[(\mathcal{O},\Lambda)\times(\mathcal{Z},\Lambda)] = \blacksquare(\mathcal{O},\Lambda)\times\blacksquare(\mathcal{Z},\Lambda)$. By hypothesis, the following is finite ⋈-set

$$\begin{aligned}[\blacksquare(\mathcal{O},\Lambda)\bigtriangleup(\mathcal{O},\Lambda)]\times[\blacksquare(\mathcal{Z},\Lambda)\bigtriangleup(\mathcal{Z},\Lambda)] &= [\blacksquare(\mathcal{O},\Lambda)\times\blacksquare(\mathcal{Z},\Lambda)]\bigtriangleup[(\mathcal{O},\Lambda)\times(\mathcal{Z},\Lambda)]\\ &= \blacksquare[(\mathcal{O},\Lambda)\times(\mathcal{Z},\Lambda)]\bigtriangleup[(\mathcal{O},\Lambda)\times(\mathcal{Z},\Lambda)],\end{aligned}$$

which ends the proof of that $(\mathcal{O},\Lambda)\times(\mathcal{Z},\Lambda)$ is a finite supra ⋈-open set. □

Proposition 3.9. *Let* $(U,\mathcal{S}_1,\Lambda)$ *and* $(U,\mathcal{S}_2,\Lambda)$ *be* SST*-spaces such that* $\mathcal{S}_1\subseteq\mathcal{S}_2$. *If* $(\mathcal{O},\Lambda)$ *is a finite supra ⋈-open subset of* $\mathcal{S}_1$, *then it is also a finite supra ⋈-open subset of* $\mathcal{S}_2$.

Proof. Let $(\mathcal{O},\Lambda)$ be a finite supra ⋈-open subset of $(U,\mathcal{S}_1,\Lambda)$. Since $\mathcal{S}_1\subseteq\mathcal{S}_2$, $(\mathcal{O},\Lambda)\in\mathcal{S}_2$ and $\blacksquare_{\mathcal{S}_2}(\mathcal{O},\Lambda)\sqsubseteq\blacksquare_{\mathcal{S}_1}(\mathcal{O},\Lambda)$, this implies that $\mathbf{B}_{\mathcal{S}_2}(\mathcal{O},\Lambda)$ is finite. Hence, we finish the proof. □

One can see that the converse of Proposition 3.9 is false by taking a member of $\mathcal{S}_2$ which is not a member of $\mathcal{S}_1$.

Proposition 3.10. *Let* $(U,\mathbb{T},\Lambda)$ *be an* SST*-space and* $(\mathcal{O}_j,\Lambda)$ *be ⋈-subsets of it for each* $j\in J$. *Then,* $\mathbf{B}[\sqcup_{j\in J}(\mathcal{O}_j,\Lambda)]\sqsubseteq\sqcup_{j\in J}\mathbf{B}(\mathcal{O}_j,\Lambda)$.

Proof. We provide the proof for two ⋈-subsets. To do this, let $u_\lambda\in\mathbf{B}[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]$. Then

$$\begin{aligned}u_\lambda\in\blacksquare[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]\text{ and }u_\lambda\notin\square[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)] &\Longrightarrow u_\lambda\in\blacksquare(\mathcal{O},\Lambda)\text{ or }\blacksquare(\mathcal{Z},\Lambda),\text{ and}\\ u_\lambda\notin\square(\mathcal{O},\Lambda)\text{ and }u_\lambda\notin\square(\mathcal{Z},\Lambda) &\Longrightarrow u_\lambda\in\blacksquare(\mathcal{O},\Lambda)\triangle\square(\mathcal{O},\Lambda)\text{ or }u_\lambda\in\blacksquare(\mathcal{Z},\Lambda)\triangle\square(\mathcal{Z},\Lambda)\\ &\Longrightarrow u_\lambda\in\mathbf{B}(\mathcal{O},\Lambda)\sqcup\mathbf{B}(\mathcal{Z},\Lambda).\end{aligned}$$

Hence, we obtain $\mathbf{B}[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]\sqsubseteq\mathbf{B}(\mathcal{O},\Lambda)\sqcup\mathbf{B}(\mathcal{Z},\Lambda)$, as required. One can obtain the proof for arbitrary number of ⋈-subsets using mathematical induction proof. □

Corollary 3.11. *The finite soft-union (resp., soft-intersection) of finite supra ⋈-open (resp., finite supra ⋈-closed) subsets of an* SST*-space* $(U,\mathbb{T},\Lambda)$ *is finite supra ⋈-open (resp., finite supra ⋈-closed).*

Corollary 3.12. *The family of finite supra ⋈-open sets forms a minimal soft structure.*

Proof. It is obvious that the absolute and null ⋈-sets are finite supra ⋈-open; therefore, the required finding is obtained. □

In the next example, we elaborate on that the converse of Proposition 3.10 is not always true as well as we reveal that both classes of finite supra ⋈-open and finite supra ⋈-closed subsets are not always closed under arbitrary numbers of soft-union and intersection.

Example 3.13. Pretend we have an SST-space $(\mathcal{R},\mathbb{T},\Lambda)$ such that $\mathcal{R}$ is real numbers set, $\Lambda=\{\lambda_1,\lambda_2\}$ and $\mathbb{T}$ is the usual soft-topology; that is, $\mathbb{T}$ is generated by the soft basis $\{(\mathcal{O},\Lambda):\mathcal{O}(\lambda_1)$ and $\mathcal{O}(\lambda_2)$ are supra open intervals in the forms of $(a,b)\}$. By taking the following ⋈-sets

$$(\mathcal{O},\Lambda)=\{(\lambda_1,(6,7)),(\lambda_2,(6,7))\}\ \text{ and }\ (\mathcal{Z},\Lambda)=\{(\lambda_1,[7,8)),(\lambda_2,[7,8))\},$$

we find that

$$\mathbf{B}[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]=\{(\lambda_1,\{6,8\}),(\lambda_2,\{6,8\})\},\ \text{ whereas }\ \mathbf{B}(\mathcal{O},\Lambda)\sqcup\mathbf{B}(\mathcal{O},\Lambda)=\{(\lambda_1,\{6,7,8\}),(\lambda_2,\{6,7,8\})\}.$$

Therefore, $\mathbf{B}[\sqcup_{j\in J}(\mathcal{O}_j,\Lambda)]$ is a proper $\bowtie$-subset of $\sqcup_{j\in J}\mathbf{B}(\mathcal{O}_j,\Lambda)$. Also, by taking the $\bowtie$-sets of the form $(\mathcal{O}_r,\Lambda)=\{(\lambda_1,(r,r+1)),(\lambda_2,(r,r+1))\}$, we get $(\mathcal{O}_r,\Lambda)$ is finite supra $\bowtie$-open for each r in the set of natural numbers $\mathbb{N}$. In contrast, the $\bowtie$-set $\sqcup_{r\in\mathbb{N}}(\mathcal{O}_r,\Lambda)$ is not finite supra $\bowtie$-open because

$$\begin{aligned}\blacksquare[\sqcup_{r\in\mathbb{N}}(\mathcal{O}_r,\Lambda)]\triangle\sqcup_{r\in\mathbb{N}}(\mathcal{O}_r,\Lambda)&=\blacksquare[\{(\lambda_1,[1,\infty)\backslash\mathbb{N}),(\lambda_2,[1,\infty)\backslash\mathbb{N})\}]\triangle\{(\lambda_1,[1,\infty)\backslash\mathbb{N}),(\lambda_2,[1,\infty)\backslash\mathbb{N})\}\\&=\{(\lambda_1,[1,\infty)),(\lambda_2,[1,\infty))\}]\triangle\{(\lambda_1,[1,\infty)\backslash\mathbb{N}),(\lambda_2,[1,\infty)\backslash\mathbb{N})\}\\&=\{(\lambda_1,\mathbb{N}),(\lambda_2,\mathbb{N})\},\end{aligned}$$

which is infinite $\bowtie$-set. So, the class of finite supra $\bowtie$-open sets is not closed under arbitrary numbers of soft-union. Moreover, by taking the $\bowtie$-sets of the form $(\mathcal{O}_i,\Lambda)=\{(\lambda_1,(\frac{-1}{r},\frac{1}{r})),(\lambda_2,(\frac{-1}{r},\frac{1}{r}))\}$, we get $(\mathcal{O}_r,\Lambda)$ is finite supra $\bowtie$-open for each r in the set of natural numbers $\mathbb{N}$. But the $\bowtie$-set $\sqcap_{r\in\mathbb{N}}(\mathcal{O}_r,\Lambda)=\{(\lambda_1,\{0\}),(\lambda_2,\{0\})\}$ is not supra $\bowtie$-open, thereby it is not finite supra $\bowtie$-open. Hence, the class of finite supra $\bowtie$-open sets is not closed under arbitrary numbers of soft-intersection. By taking the complement of the second and third computations we also infer that the class of finite supra $\bowtie$-closed sets is not closed under arbitrary numbers of soft-union and intersection.

By the next example, we clarify that the families of finite supra $\bowtie$-open and finite supra $\bowtie$-closed subsets are not always closed under finite numbers of soft-intersection and union.

Example 3.14. Pretend we have an SST-space $(U,\mathbb{T},\Lambda)$ such that $U=\{u,v,w\}$, $\Lambda=\{\lambda_1,\lambda_2\}$, and $\mathbb{T}=\{\widetilde{\Phi},\widetilde{U},(\mathcal{O}_1,\Lambda),(\mathcal{O}_2,\Lambda),(\mathcal{O}_3,\Lambda)\}$ is a supra soft-topology, where

$$(\mathcal{O}_1,\Lambda)=\{(\lambda_1,\{u\}),(\lambda_2,\{u\})\},\quad(\mathcal{O}_2,\Lambda)=\{(\lambda_1,\{u,w\}),(\lambda_2,\{u,w\})\},\quad\text{and}\quad(\mathcal{O}_3,\Lambda)=\{(\lambda_1,\{v,w\}),(\lambda_2,\{v,w\})\}.$$

It is clear that $(\mathcal{O}_2,\Lambda)$ and $(\mathcal{O}_3,\Lambda)$ are supra finite $\bowtie$-open sets but their soft intersection $\{(\lambda_1,\{w\}),(\lambda_2,\{w\})\}$ is not supra finite $\bowtie$-open. Also, $(\mathcal{O}_2^c,\Lambda)$ and $(\mathcal{O}_3^c,\Lambda)$ are supra finite $\bowtie$-closed sets but their soft union $\{(\lambda_1,\{u,v\}),(\lambda_2,\{u,v\})\}$ is not supra finite $\bowtie$-closed.

In the following, we will debate the connections of this kind of $\bowtie$-sets with supra $\bowtie$-open sets and their known generalizations.

Proposition 3.15.

(i) *Every supra $\bowtie$-clopen set is finite supra $\bowtie$-open.*

(ii) *Every finite supra $\bowtie$-open set is supra $\bowtie$-open (supra $\bowtie$ α-open, supra $\bowtie$ pre-open, supra $\bowtie$ semi-open, supra $\bowtie$ b-open, supra $\bowtie$ β-open).*

Proof. Since $\mathbf{B}(\mathcal{O},\Lambda)$ is the null $\bowtie$-set for every supra $\bowtie$-clopen subset $(\mathcal{O},\Lambda)$, the proof of (i) follows. The proof of (ii) is obvious by Definition 3.1 for the case of supra $\bowtie$-open and by the previous relationships that are well-known in the published articles for the case of between brackets. □

By Example 3.13, it can be seen that the $\bowtie$-set $(\mathcal{O},\Lambda)=\{(\lambda_1,(4,5]),(\lambda_2,[2,3))\}$ is supra $\bowtie$ semi-open, supra $\bowtie$ b-open, supra $\bowtie$ β-open but not finite supra $\bowtie$-open. Also, $\{(\lambda_1,[1,\infty)\backslash\mathbb{N}),(\lambda_2,[1,\infty)\backslash\mathbb{N})\}$ is a supra $\bowtie$-open set that is not finite supra $\bowtie$-open. On the other hand, in a soft topology consists of three elements over the finite universal set we get a finite supra $\bowtie$-open set that is not supra $\bowtie$-clopen. So that, Proposition 3.15 is not invertible.

Proposition 3.16. *If all proper supra $\bowtie$-closed subsets of an infinite SST-space are finite, then a $\bowtie$-set is supra $\bowtie$-open if and only if it is finite supra $\bowtie$-open.*

Proof. To prove the necessary side, let $(\mathcal{O},\Lambda)$ be supra $\bowtie$-open. Suppose, to the contrary, that $\mathbf{B}(\mathcal{O},\Lambda)$ is infinite. Then, $\blacksquare(\mathcal{O},\Lambda)$ is infinite. By hypothesis, $\blacksquare(\mathcal{O},\Lambda)$ must be equal the absolute $\bowtie$-set. Therefore, $\mathbf{B}(\mathcal{O},\Lambda)$ is the null $\bowtie$-set. But this contradicts that $\mathbf{B}(\mathcal{O},\Lambda)$ is infinite, so $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-open. The sufficient side is obvious by (ii) of Proposition 3.15. □

Example 3.13 elucidates the necessity of the condition "all proper supra $\bowtie$-closed subsets are finite" to satisfy the equality in Proposition 3.16. That is, $\{(\lambda_1, [1,\infty)\backslash\mathbb{N}), (\lambda_2, [1,\infty)\backslash\mathbb{N})\}$ is a supra $\bowtie$-open subset of an SST-space $(\mathcal{R}, \mathbb{T}, \Lambda)$ defined in Example 3.13 that is not finite supra $\bowtie$-open.

The following lemma will be useful to demonstrate some of upcoming results.

Proposition 3.17. *The families of supra $\bowtie$-open and finite supra $\bowtie$-open subsets of an* SST*-space $(U, \mathbb{T}, \Lambda)$ are identical providing that one of the following conditions holds:*

(i) *the absolute $\bowtie$-set is finite;*
(ii) *the* SST *is supra $\bowtie$-clopen;*
(iii) *the* SST *is locally indiscrete.*

Proof.

(i) Obvious.

(ii) Follows from (iv) of Proposition 3.3.

(iii) It is sufficient to show that $\mathbf{B}(\mathcal{O}, \Lambda)$ is null for any supra $\bowtie$-open subset of a soft locally indiscrete topological space. To do so, let $(\mathcal{O}, \Lambda)$ be supra $\bowtie$-open. By hypothesis, $(\mathcal{O}, \Lambda)$ is expressed as a soft-union of supra $\bowtie$-clopen subsets; say,

$$(\mathcal{O}, \Lambda) = \sqcup_{j\in J}(\mathcal{Z}_j, \Lambda), \tag{3.1}$$

where $(\mathcal{Z}_j, \Lambda)$ is an element of soft basis for each j. By Proposition 3.10, we have the following inclusion

$$\mathbf{B}[\sqcup_{j\in J}(\mathcal{Z}_j, \Lambda)] \sqsubseteq \sqcup_{j\in J}\mathbf{B}(\mathcal{Z}_j, \Lambda).$$

Since every $(\mathcal{Z}_j, \Lambda)$ is supra $\bowtie$-clopen, $\mathbf{B}(\mathcal{Z}_j, \Lambda) = \Phi$. This implies that $\mathbf{B}[\sqcup_{j\in J}(\mathcal{Z}_j, \Lambda)] = \Phi$. From (3.1) we prove that $(\mathcal{O}, \Lambda)$ is finite supra $\bowtie$-open. □

Corollary 3.18. *The closeness of finite supra $\bowtie$-open sets with respect to arbitrary soft-union is satisfied providing that one of the following conditions holds:*

(i) *the absolute $\bowtie$-set is finite;*
(ii) *the* SST *is supra $\bowtie$-clopen;*
(iii) *the* SST *is locally indiscrete.*

Proposition 3.19. *Let $(\mathcal{O}, \Lambda)$ be a finite supra $\bowtie$-open subset of a supra soft extremally disconnected space $(U, \mathbb{T}, \Lambda)$. Then $\blacksquare(\mathcal{O}, \Lambda)$ is finite supra $\bowtie$-open.*

Proof. Since $\blacksquare(\mathcal{O}, \Lambda)$ is supra $\bowtie$-open and $\mathbf{B}[\blacksquare(\mathcal{O}, \Lambda)] \sqsubseteq \blacksquare(\mathcal{O}, \Lambda)$, the proof follows. □

We close this part by looking at the transition of the feature of being a finite supra $\bowtie$-open set between soft and classical realms.

The next proposition can be proved readily, so the proof is omitted.

Proposition 3.20. *Let $(\mathcal{O}, \Lambda)$ be a $\bowtie$-subset of an* SST*-space $(U, \mathbb{T}, \Lambda)$. Then $(\blacksquare(\mathcal{O}), \Lambda) \sqsubseteq (\blacksquare(\mathcal{O}), \Lambda)$.*

Proposition 3.21. *If $(\mathcal{O}, \Lambda)$ is finite supra $\bowtie$-open subset of $(U, \mathbb{T}, \Lambda)$, then $\mathcal{O}(\lambda)$ is finite supra open subset of $(U, \mathbb{T}_a)$ for every $\lambda \in \Lambda$.*

Proof. It is obvious that $\mathcal{O}(\lambda)$ is a supra open subset of $(U, \mathbb{T}_a)$ for every $\lambda \in \Lambda$ when $(\mathcal{O}, \Lambda)$ is a finite supra $\bowtie$-open subset of $(U, \mathbb{T}, \Lambda)$. It remains to demonstrate that $\mathbf{B}(\mathcal{O}(\lambda))$ is finite. From Proposition 3.20, $(\blacksquare(\mathcal{O}), \Lambda) \sqsubseteq \blacksquare(\mathcal{O}, \Lambda)$, which automatically means that $(\mathbf{B}(\mathcal{O}), \Lambda)$ is finite. Hence, $\mathbf{B}(\mathcal{O}(\lambda))$ is finite, as required. □

The following counterexample is built to clarify that Proposition 3.21 is not invertible, in general.

Example 3.22. Let the real numbers and natural numbers sets (respectively denoted by $\mathcal{R}$ and $\mathbb{N}$) be the universal and parameters sets, respectively. Define an SST $\mathbb{T}$ over $\mathcal{R}$ with $\mathbb{N}$ as following:

$$\mathbb{T} = \{\emptyset, \widetilde{\mathcal{R}}, (\mathcal{O}_i, \mathbb{N}) : \mathcal{O}_i(n) = \mathcal{R} \text{ for all but finitely many } n \in \mathbb{N}\}$$

Now, one can remark that a $\bowtie$-set $(\mathcal{Z}, \mathbb{N})$ defined by

$$\mathcal{Z}(5) = \{5\} \text{ and for } n \neq 5 : \mathcal{Z}(n) = \mathcal{R}$$

is not finite supra $\bowtie$-open in spite of all components are finite open sets.

Lemma 3.23. *An* SST*-space* $(U, \mathbb{T}, \Lambda)$ *is supra* $\bowtie$*-submaximal iff* $\blacksquare(\mathcal{O}, \Lambda)\Delta\square(\mathcal{O}, \Lambda)$ *is a supra discrete* $\bowtie$*-set.*

Proof. Following similar argument given in the proof of Theorem 3.3 of [39]. □

Theorem 3.24. *Let* $(\mathcal{O}, \Lambda)$ *be a* $\bowtie$*-subset of an* SST*-space* $(U, \mathbb{T}, \Lambda)$ *which is supra* $\bowtie$*-compact and supra* $\bowtie$*-submaximal. Then,* $\blacksquare(\mathcal{O}, \Lambda)\Delta(\mathcal{O}, \Lambda)$ *is a finite soft set.*

Proof. Since $\blacksquare(\mathcal{O}, \Lambda)\Delta(\mathcal{O}, \Lambda)$ is a supra $\bowtie$-closed set, it follows from Lemma 3.23 that it $\bowtie$-discrete relative SST. By hypothesis of supra $\bowtie$-compactness, we obtain $\blacksquare(\mathcal{O}, \Lambda)\Delta(\mathcal{O}, \Lambda)$ is a supra $\bowtie$-compact. Hence, $\blacksquare(\mathcal{O}, \Lambda)\Delta(\mathcal{O}, \Lambda)$ is finite. □

4. Interior and closure points over the finite supra $\bowtie$-open sets

This segment applies the notions of finite supra $\bowtie$-open and finite supra $\bowtie$-closed sets to present some topological operators. We investigate major features and describe the relationships between them with some counterexamples that show the false directions.

Definition 4.1. Let $(\mathcal{O}, \Lambda)$ be a $\bowtie$-set in $(U, \mathbb{T}, \Lambda)$, the union of all finite supra $\bowtie$-open subsets which are contained in $(\mathcal{O}, \Lambda)$ is defined as the fsso-interior of $(\mathcal{O}, \Lambda)$, it will be symbolized by $\square_{\text{fsso}}(\mathcal{O}, \Lambda)$.

The next result follows directly from the above definition. The validity of its converse is false in general as revealed by example given after this proposition.

Proposition 4.2. *If* $(\mathcal{O}, \Lambda)$ *is a finite supra* $\bowtie$*-open subset of* $(U, \mathbb{T}, \Lambda)$*, then* $(\mathcal{O}, \Lambda) = \square_{\text{fsso}}(\mathcal{O}, \Lambda)$*.*

Example 4.3. In Example 3.13, we illuminate that $\{(\lambda_1, [1, \infty)\backslash\mathbb{N}), (\lambda_2, [1, \infty)\backslash\mathbb{N})\}$ is not finite supra $\bowtie$-open. On the other hand, notice that $\{(\lambda_1, [1, \infty)\backslash\mathbb{N}), (\lambda_2, [1, \infty)\backslash\mathbb{N})\} = \sqcup_{r\in\mathbb{N}}\{(\lambda, (r, r + 1)), (\lambda_2, (r, r + 1))\}$, where $\{(\lambda_1, (r, r + 1)), (\lambda_2, (r, r + 1))\}$ is finite supra $\bowtie$-open for each $r \in \mathbb{N}$. Hence,

$$\{(\lambda_1, [1, \infty)\backslash\mathbb{N}), (\lambda_2, [1, \infty)\backslash\mathbb{N})\} = \square_{\text{fsso}}(\{(\lambda_1, [1, \infty)\backslash\mathbb{N}), (\lambda_2, [1, \infty)\backslash\mathbb{N})\}).$$

Proposition 4.2 says that fsso-interior does not fulfill a main feature of soft interior operator; that is, if for every a soft-point u_λ belongs to $(\mathcal{O}, \Lambda)$ there exists a finite supra $\bowtie$-open set $(\mathcal{Z}, \Lambda)$ s.t. $u_\lambda \in (\mathcal{Z}, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda)$, then $(\mathcal{O}, \Lambda)$ need not be finite supra $\bowtie$-open.

The following properties of fsso-interior operators can be proved easily, so we omit their proofs.

Proposition 4.4. *For every* $\bowtie$*-subsets* $(\mathcal{O}, \Lambda)$, $(\mathcal{Z}, \Lambda)$ *of* $(U, \mathbb{T}, \Lambda)$*, we have*

$$\square_{\text{fsso}}(\mathcal{O}, \Lambda) \sqsubseteq \square_{\text{fsso}}(\mathcal{Z}, \Lambda), \text{ when } (\mathcal{O}, \Lambda) \sqsubseteq (\mathcal{Z}, \Lambda).$$

Corollary 4.5. *Let* $(\mathcal{O}, \Lambda)$ *and* $(\mathcal{Z}, \Lambda)$ *be* $\bowtie$*-subsets of* $(U, \mathbb{T}, \Lambda)$*. Then*

(i) $\square_{\text{fsso}}(\mathcal{O}, \Lambda) \sqcup \square_{\text{fsso}}(\mathcal{Z}, \Lambda) \sqsubseteq \square_{\text{fsso}}[(\mathcal{O}, \Lambda) \sqcup (\mathcal{Z}, \Lambda)]$;
(ii) $\square_{\text{fsso}}[(\mathcal{O}, \Lambda) \sqcap (\mathcal{Z}, \Lambda)] \sqsubseteq \square_{\text{fsso}}(\mathcal{O}, \Lambda) \sqcap \square_{\text{fsso}}(\mathcal{Z}, \Lambda)$.

The relations of inclusion deduced in the aforementioned results (Proposition 4.4 and Corollary 4.5) can not be replaced by equalities as clarified by the following counterexamples.

Example 4.6. Let $(U, \mathbb{T}, \Lambda)$ be an SST-space, where $U = \{u, v, w\}$, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and the members of $\mathbb{T}$ are the null and absolute $\bowtie$-sets and the following:

$$(\mathcal{O}, \Lambda) = \{(\lambda_1, U), (\lambda_2, \emptyset), (\lambda_3, \emptyset)\} \text{ and } (\mathcal{Z}, \Lambda) = \{(\lambda_1, \emptyset), (\lambda_2, U), (\lambda_3, U)\}.$$

Take the $\bowtie$-sets as following $(\mathcal{H}, \Lambda) = \{(\lambda_1, \emptyset), (\lambda_2, U), (\lambda_3, \emptyset)\}$, and $(\mathcal{P}, \Lambda) = \{(\lambda_1, \emptyset), (\lambda_2, \emptyset), (\lambda_3, U)\}$. Then, $\square_{fsso}(\mathcal{H}, \Lambda) = \square_{fsso}(\mathcal{P}, \Lambda) = \Phi$ in spite of $(\mathcal{H}, \Lambda)$ and $(\mathcal{P}, \Lambda)$ are independent of each other with respect to the relation of inclusion. Moreover,

$$\square_{fsso}(\mathcal{H}, \Lambda) \sqcup \square_{fsso}(\mathcal{P}, \Lambda) = \Phi,$$

whereas

$$\square_{fsso}[(\mathcal{H}, \Lambda) \sqcup (\mathcal{P}, \Lambda)] = (\mathcal{Z}, \Lambda).$$

Now, we go into fsso-closure, which is considered as the dual of fsso-interior.

Definition 4.7. Let $(\mathcal{O}, \Lambda)$ be a $\bowtie$-set in $(U, \mathbb{T}, \Lambda)$, the intersection of all finite supra $\bowtie$-closed subsets containing $(\mathcal{O}, \Lambda)$ is defined as the fsso-closure of $(\mathcal{O}, \Lambda)$, it will be symbolized by $\blacksquare_{fsso}(\mathcal{O}, \Lambda)$.

Proposition 4.8. *Pretend a $\bowtie$-subset $(\mathcal{O}, \Lambda)$ of $(U, \mathbb{T}, \Lambda)$ and $u_\lambda \in \widetilde{U}$. Then $u_\lambda \in \blacksquare_{fsso}(\mathcal{O}, \Lambda)$ if and only if $(\mathcal{Z}, \Lambda) \sqcap (\mathcal{O}, \Lambda) \neq \Phi$ for each finite supra $\bowtie$-open set $(\mathcal{Z}, \Lambda)$ containing u_λ.*

Proof.

Necessity: Let $u_\lambda \in \blacksquare_{fsso}(\mathcal{O}, \Lambda)$ and let $(\mathcal{Z}, \Lambda)$ be a finite supra $\bowtie$-open set such that $u_\lambda \in (\mathcal{Z}, \Lambda)$. Suppose that $(\mathcal{Z}, \Lambda) \sqcap (\mathcal{O}, \Lambda) = \Phi$. Now, we have $(\mathcal{O}, \Lambda) \sqsubseteq (\mathcal{Z}^c, \Lambda)$. This leads to that $\blacksquare_{fsso}(\mathcal{O}, \Lambda) \sqsubseteq (\mathcal{Z}^c, \Lambda)$. But this contradicts that $u_\lambda \in \blacksquare_{fsso}(\mathcal{O}, \Lambda)$. Hence, the intersection of $(\mathcal{Z}, \Lambda)$ and $(\mathcal{O}, \Lambda)$ must be the non-null $\bowtie$-set.

Sufficiency: Let us consider the sufficient part holds. Suppose, to the contrary, that $u_\lambda \notin \blacksquare_{fsso}(\mathcal{O}, \Lambda)$. This means we can find a finite supra $\bowtie$-closed set $(\mathcal{H}, \Lambda)$ containing $(\mathcal{Z}, \Lambda)$ s.t. $u_\lambda \notin (\mathcal{H}, \Lambda)$. Accordingly, we have $(\mathcal{H}^c, \Lambda)$ as a finite supra $\bowtie$-open set containing u_λ and its intersection with $(\mathcal{O}, \Lambda)$ is the null $\bowtie$-set. This is a contradiction. Hence, we demonstrate that $u_\lambda \in \blacksquare_{fsso}(\mathcal{O}, \Lambda)$. □

Corollary 4.9. *Let $(\mathcal{O}, \Lambda)$ be a finite supra $\bowtie$-open set and $(\mathcal{Z}, \Lambda)$ a $\bowtie$-set in $(U, \mathbb{T}, \Lambda)$. Then, $(\mathcal{O}, \Lambda) \sqcap (\mathcal{Z}, \Lambda) = \Phi$ implies that $(\mathcal{O}, \Lambda) \sqcap \blacksquare_{fsso}(\mathcal{Z}, \Lambda) = \Phi$. Moreover, if $(\mathcal{O}, \Lambda)$ and $(\mathcal{Z}, \Lambda)$ are finite supra $\bowtie$-open sets, then $\blacksquare_{fsso}(\mathcal{O}, \Lambda) \sqcap \blacksquare_{fsso}(\mathcal{Z}, \Lambda) = \Phi$.*

Proposition 4.10. *If $(\mathcal{O}, \Lambda)$ is a $\bowtie$-subset of $(U, \mathbb{T}, \Lambda)$, then*

(i) *if $(\mathcal{O}, \Lambda)$ is a finite supra $\bowtie$-closed set, then $(\mathcal{O}, \Lambda) = \blacksquare_{fsso}(\mathcal{O}, \Lambda)$;*
(ii) *$[\square_{fsso}(\mathcal{O}, \Lambda)]^c = \blacksquare_{fsso}(\mathcal{O}^c, \Lambda)$ and $[\blacksquare_{fsso}(\mathcal{O}, \Lambda)]^c = \square_{fsso}(\mathcal{O}^c, \Lambda)$.*

Proof.

(i) It is straightforward.

(ii)

$$\begin{aligned}[\square_{fsso}(\mathcal{O}, \Lambda)]^c &= [\sqcup_{j \in J}(\mathcal{H}_j, \Lambda) : (\mathcal{H}_j, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda), \text{ where } (\mathcal{H}_j, \Lambda) \text{ is a finite supra } \bowtie\text{-open set}]^c \\ &= [\sqcap_{j \in J}(\mathcal{H}_j^c, \Lambda) : (\mathcal{O}^c, \Lambda) \sqsubseteq (\mathcal{H}_j^c, \Lambda), \text{ where } (\mathcal{H}_j^c, \Lambda) \text{ is a finite supra } \bowtie\text{-closed set}] \\ &= \blacksquare_{fsso}(\mathcal{O}^c, \Lambda).\end{aligned}$$

□

Proposition 4.11. *Let $(\mathcal{O}, \Lambda)$ be a $\bowtie$-subset of $(U, \mathbb{T}, \Lambda)$. Then, $\blacksquare_{fsso}(\mathcal{O}, \Lambda) \sqsubseteq \blacksquare_{fsso}(\mathcal{Z}, \Lambda)$, when $(\mathcal{O}, \Lambda) \sqsubseteq (\mathcal{Z}, \Lambda)$.*

Proof. Owing to the fact reporting that every finite supra $\bowtie$-closed set contains $(\mathcal{Z},\Lambda)$ containing $(\mathcal{O},\Lambda)$ as well, the proof is gotten. □

Corollary 4.12. *Let* $(\mathcal{O},\Lambda)$ *and* $(\mathcal{Z},\Lambda)$ *be* $\bowtie$*-subsets of* $(U,\mathbb{T},\Lambda)$*. Then,*

(i) $\blacksquare_{fsso}[(\mathcal{O},\Lambda)\sqcap(\mathcal{Z},\Lambda)]\sqsubseteq\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcap\blacksquare_{fsso}(\mathcal{Z},\Lambda)$;

(ii) $\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcup\blacksquare_{fsso}(\mathcal{Z},\Lambda)\sqsubseteq\blacksquare_{fsso}[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]$.

By Example 4.6, it can be confirmed that the inclusion relations deduced in Proposition 4.11 and Corollary 4.12 can not be replaced by equalities.

Definition 4.13. Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of $(U,\mathbb{T},\Lambda)$. The complement of $\square_{fsso}(\mathcal{O},\Lambda)\sqcup\square_{fsso}(\mathcal{O}^c,\Lambda)$ is said to be a fsso-boundary of $(\mathcal{O},\Lambda)$, it will be symbolized by $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)$.

Proposition 4.14. *Let* $(\mathcal{O},\Lambda)$ *be a* $\bowtie$*-subset of* $(U,\mathbb{T},\Lambda)$*. Then*

$$\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcap\blacksquare_{fsso}(\mathcal{O}^c,\Lambda).$$

Proof.

$$\begin{aligned}\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=[\square_{fsso}(\mathcal{O},\Lambda)\sqcup\square_{fsso}(\mathcal{O}^c,\Lambda)]^c&=[\square_{fsso}(\mathcal{O},\Lambda)]^c\sqcap[\square_{fsso}(\mathcal{O}^c,\Lambda)]^c \quad \text{(De Morgan's law)}\\&=\blacksquare_{fsso}(\mathcal{O}^c,\Lambda)\sqcap\blacksquare_{fsso}(\mathcal{O},\Lambda) \quad \text{(Proposition 4.10 (ii))}.\end{aligned}$$

□

Corollary 4.15. *Let* $(\mathcal{O},\Lambda)$ *be a* $\bowtie$*-subset of* $(U,\mathbb{T},\Lambda)$*. Then*

(i) $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\mathbf{B}_{fsso}(\mathcal{O}^c,\Lambda)$;

(ii) $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\blacksquare_{fsso}(\mathcal{O},\Lambda)\Delta\,\square_{fsso}(\mathcal{O},\Lambda)$;

(iii) $\blacksquare_{fsso}(\mathcal{O},\Lambda)=\square_{fsso}(\mathcal{O},\Lambda)\sqcup\,\mathbf{B}_{fsso}(\mathcal{O},\Lambda)$;

(iv) $\square_{fsso}(\mathcal{O},\Lambda)=(\mathcal{O},\Lambda)\Delta\,\mathbf{B}_{fsso}(\mathcal{O},\Lambda)$.

Proof.

(i) It is straightforward.

(ii) $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcap\blacksquare_{fsso}(\mathcal{O}^c,\Lambda)=\blacksquare_{fsso}(\mathcal{O},\Lambda)\Delta\,[\blacksquare_{fsso}(\mathcal{O}^c,\Lambda)]^c$. By (ii) of Proposition 4.10 the required relation is obtained.

(iii) $\square_{fsso}(\mathcal{O},\Lambda)\sqcup\,\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\square_{fsso}(\mathcal{O},\Lambda)\sqcup\,[\blacksquare_{fsso}(\mathcal{O},\Lambda)\Delta\,\square_{fsso}(\mathcal{O},\Lambda)]=\blacksquare_{fsso}(\mathcal{O},\Lambda)$.

(iv)

$$\begin{aligned}(\mathcal{O},\Lambda)\Delta\,\mathbf{B}_{fsso}(\mathcal{O},\Lambda)&=(\mathcal{O},\Lambda)\Delta\,[\blacksquare_{fsso}(\mathcal{O},\Lambda)\Delta\square_{fsso}(\mathcal{O},\Lambda)]\\&=(\mathcal{O},\Lambda)\sqcap[\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcap(\square_{fsso}(\mathcal{O},\Lambda))^c]^c\\&=(\mathcal{O},\Lambda)\sqcap[(\blacksquare_{fsso}(\mathcal{O},\Lambda))^c\sqcup\,\square_{fsso}(\mathcal{O},\Lambda)]\\&=[(\mathcal{O},\Lambda)\sqcap(\blacksquare_{fsso}(\mathcal{O},\Lambda))^c]\sqcup[(\mathcal{O},\Lambda)\sqcap\,\square_{fsso}(\mathcal{O},\Lambda)]=\square_{fsso}(\mathcal{O},\Lambda).\end{aligned}$$

□

Note that $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)$ need not be a finite supra $\bowtie$-closed set.

From (ii) of Corollary 4.15, the proof of the next proposition follows.

Proposition 4.16. *The next formula holds for any* $\bowtie$*-subset* $(\mathcal{O},\Lambda)$ *of* $(U,\mathbb{T},\Lambda)$*:*

$$\mathbf{B}_{fsso}(\square_{fsso}(\mathcal{O},\Lambda))\sqcup\mathbf{B}_{fsso}(\blacksquare_{fsso}(\mathcal{O},\Lambda))\sqsubseteq\mathbf{B}_{fsso}(\mathcal{O},\Lambda).$$

The inclusion relation of Proposition 4.16 cannot be replaced by equality as exhibited by the following.

Example 4.17. Let $(\mathcal{O},\Lambda)=\{(\lambda_1,\mathbb{Q}),(\lambda_2,\mathbb{Q})\}$ be a $\bowtie$-subset an SST-space $(U,\mathbb{T},\Lambda)$ provided in Example 3.13. Then $\mathbf{B}_{fsso}(\square_{fsso}(\mathcal{O},\Lambda))\sqcup\mathbf{B}_{fsso}(\blacksquare_{fsso}(\mathcal{O},\Lambda))=\Phi$ whereas $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\{(\lambda_1,\mathcal{R}),(\lambda_2,\mathcal{R})\}$.

Proposition 4.18. *Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of $(U,\mathbb{T},\Lambda)$.*

(i) *If $(\mathcal{O},\Lambda)$ is a finite supra $\bowtie$-open set, then $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)\sqcap(\mathcal{O},\Lambda)=\Phi$;*
(ii) *If $(\mathcal{O},\Lambda)$ of $(U,\mathbb{T},\Lambda)$ is a finite supra $\bowtie$-closed set, then $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O},\Lambda)$.*

Proof.

(i) Since $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-open, then by (iv) of Corollary 4.15, we get $\square_{fsso}(\mathcal{O},\Lambda)=(\mathcal{O},\Lambda)=(\mathcal{O},\Lambda)\Delta\mathbf{B}_{fsso}(\mathcal{O},\Lambda)$. So that, $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)\sqcap(\mathcal{O},\Lambda)=\Phi$.

(ii) Obviously, $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqcap\blacksquare_{fsso}(\mathcal{O}^c,\Lambda)\sqsubseteq\blacksquare_{fsso}(\mathcal{O},\Lambda)$. Since $(\mathcal{O},\Lambda)$ is finite supra $\bowtie$-closed, we obtain the proof. □

Corollary 4.19. *If $(\mathcal{O},\Lambda)$ is a finite supra $\bowtie$-clopen subset of an SST-space, then $\mathbf{B}_{fsso}(\mathcal{O},\Lambda)=\Phi$.*

Definition 4.20. Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of an SST-space $(U,\mathbb{T},\Lambda)$, the union of soft points u_λ satisfying that

$$[(\mathcal{Z},\Lambda)\backslash u_\lambda]\sqcup(\mathcal{O},\Lambda)\neq\Phi \text{ for each finite supra }\bowtie\text{-open set containing } u_\lambda$$

is said to be an fsso-derived of $(\mathcal{O},\Lambda)$, it will be symbolized by $\mathbf{L}_{fsso}(\mathcal{O},\Lambda)$.

The following properties of fsso-derived operator can be proved easily, so we omit their proofs.

Proposition 4.21. *Let $(\mathcal{O},\Lambda)$ and $(\mathcal{Z},\Lambda)$ be $\bowtie$-subsets of an SST-space $(U,\mathbb{T},\Lambda)$. Then $(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{Z},\Lambda)$ implies that $\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqsubseteq\mathbf{L}_{fsso}(\mathcal{Z},\Lambda)$.*

Corollary 4.22. *Let $(\mathcal{O},\Lambda)$ and $(\mathcal{Z},\Lambda)$ be $\bowtie$-subsets of an SST-space $(U,\mathbb{T},\Lambda)$. Then*

(i) $\mathbf{L}_{fsso}[(\mathcal{O},\Lambda)\sqcap(\mathcal{Z},\Lambda)]\sqsubseteq\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqcap\mathbf{L}_{fsso}(\mathcal{Z},\Lambda)$;
(ii) $\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqcup\mathbf{L}_{fsso}(\mathcal{Z},\Lambda)\sqsubseteq\mathbf{L}_{fsso}[(\mathcal{O},\Lambda)\sqcup(\mathcal{Z},\Lambda)]$.

Theorem 4.23. *Let $(\mathcal{O},\Lambda)$ be a $\bowtie$-subset of an SST-space $(U,\mathbb{T},\Lambda)$.*

(i) *if $(\mathcal{O},\Lambda)$ is a finite supra $\bowtie$-closed set, then $\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O},\Lambda)$;*
(ii) $\blacksquare_{fsso}(\mathcal{O},\Lambda)=(\mathcal{O},\Lambda)\sqcup\mathbf{L}_{fsso}(\mathcal{O},\Lambda)$.

Proof.

(i) Assume that $(\mathcal{O},\Lambda)$ is a finite supra $\bowtie$-closed set set and $u_\lambda\notin(\mathcal{O},\Lambda)$. Then $u_\lambda\in(\mathcal{O},\Lambda)^c$, which is a finite supra $\bowtie$-open set. From the fact that $(\mathcal{O},\Lambda)^c\sqcap(\mathcal{O},\Lambda)=\Phi$, we obtain $u_\lambda\notin\mathbf{L}_{fsso}(\mathcal{O},\Lambda)$. Thus $\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O},\Lambda)$.

(ii) Let $u_\lambda\notin[(\mathcal{O},\Lambda)\sqcup\mathbf{L}_{fsso}(\mathcal{O},\Lambda)]$. Then $u_\lambda\notin(\mathcal{O},\Lambda)$ and $u_\lambda\notin\mathbf{L}_{fsso}(\mathcal{O},\Lambda)$. Therefore, there is a finite supra $\bowtie$-open set $(\mathcal{Z},\Lambda)$ containing u_λ with $(\mathcal{Z},\Lambda)\sqcap(\mathcal{O},\Lambda)=\Phi$. Thus, $\blacksquare_{fsso}(\mathcal{O},\Lambda)\sqsubseteq(\mathcal{O},\Lambda)\sqcup\mathbf{L}_{fsso}(\mathcal{O},\Lambda)$. On the other hand, it is well-known that $(\mathcal{O},\Lambda)\sqcup\mathbf{L}_{fsso}(\mathcal{O},\Lambda)\sqsubseteq\blacksquare_{fsso}(\mathcal{O},\Lambda)$. □

5. Some types of supra soft continuous mappings

We close this work by introducing fresh types soft continuity between supra soft-topological spaces and illustrating some characterizations. We amply discuss how these sorts of soft continuity behave between the spaces of soft and classical supra topologies. Illustrative examples are constructed to ensure the obtained findings.

Definition 5.1. A soft mapping $h_g:(U,\mathbb{T},\Lambda)\to(V,\mathbb{S},\Lambda)$ is said to be supra soft ff-continuous (resp., supra soft of-continuous, supra soft fo-continuous) if the inverse image of each finite supra $\bowtie$-open (resp., finite supra $\bowtie$-open, supra $\bowtie$-open) subset is finite supra $\bowtie$-open (resp., supra $\bowtie$-open, finite supra $\bowtie$-open).

Proposition 5.2.

(i) *A supra soft* $\mathtt{fo}$*-continuous mapping is supra soft* $\mathtt{ff}$*-continuous.*
(ii) *A supra soft* $\mathtt{ff}$*-continuous mapping is supra soft* $\mathtt{of}$*-continuous.*

It is not necessary for the above proposition's converse to be true. The example which follows illustrates it.

Example 5.3. Pretend we have SST-spaces $(\mathcal{R}, \mathbb{R}, \Lambda)$, $(\mathcal{R}, \mathbb{S}, \Lambda)$ and $(U, \mathbb{T}, \Lambda)$ such that $\mathcal{R}$ is the set of real numbers, $U = \{u, v\}$, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and the soft topologies $\mathbb{R}$, $\mathbb{S}$ over $\mathcal{R}$ with Λ and $\mathbb{T}$ over U with Λ are given by

$$\mathbb{R} = \{\widetilde{\mathcal{R}}, (\mathcal{O}, \Lambda) \sqsubseteq \widetilde{\mathcal{R}} : 1 \in (\mathcal{O}, \Lambda)\} \cup \{\Phi\},$$
$$\mathbb{S} = \{\Phi, \widetilde{\mathcal{R}}, (\mathcal{Z}, \Lambda) = \{(\lambda_1, \{4\}), (\lambda_2, \{4\}), (\lambda_3, \{4\})\}\}, \text{ and}$$
$$\mathbb{T} = \{\Phi, \widetilde{U}, (\mathcal{Z}, \Lambda) = \{(\lambda_1, \{v\}), (\lambda_2, \{v\}), (\lambda_3, \{v\})\}\}.$$

Take soft mappings $l_g : (\mathcal{R}, \mathbb{R}, \Lambda) \to (\mathcal{R}, \mathbb{S}, \Lambda)$ and $h_g : (\mathcal{R}, \mathbb{S}, \Lambda) \to (U, \mathbb{T}, \Lambda)$ such that the mappings $l : \mathcal{R} \to \mathcal{R}$ and $g : \Lambda \to \Lambda$ are identity, and $h : \mathcal{R} \to U$ is given by

$$h(4) = v \text{ and for each } r \neq 4 \text{ we have } h(r) = u.$$

Then, l_g and h_g are soft $\mathtt{ff}$-continuous and soft $\mathtt{of}$-continuous, respectively. In contrast, neither l_g is soft $\mathtt{fo}$-continuous nor h_g is soft $\mathtt{ff}$-continuous.

Proposition 5.4. *The concepts of supra soft* $\mathtt{ff}$*-continuity and supra soft* $\mathtt{of}$*-continuity are identical under an injective condition.*

Proof. By Proposition 3.5, we get the corresponding between them. □

Some enumerates for these sorts of soft continuity are displayed in the following.

Proposition 5.5. *Let* $h_g : (U, \mathbb{T}, \Lambda) \to (V, \mathbb{S}, \Lambda)$ *be a soft mapping. Then*

(i) *if* h_g *is supra soft* $\mathtt{ff}$*-continuous, then for every* $u_\lambda \in U$ *and every finite supra* $\bowtie$*-open subset* $(\mathcal{O}, \Lambda)$ *containing* $h_g(u_\lambda)$*, there is a finite supra* $\bowtie$*-open set* $(\mathcal{Z}, \Lambda)$ *containing* u_λ *with* $h_g(\mathcal{Z}, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda)$*;*
(ii) *if* h_g *is supra soft* $\mathtt{of}$*-continuous, then for every* $u_\lambda \in U$ *and every finite supra* $\bowtie$*-open set* $(\mathcal{O}, \Lambda)$ *containing* $h_g(u_\lambda)$*, there is a supra* $\bowtie$*-open set* $(\mathcal{Z}, \Lambda)$ *containing* u_λ *with* $h_g(\mathcal{Z}, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda)$*;*
(iii) *if* h_g *is supra soft* $\mathtt{fsso}$*-continuous, then for every* $u_\lambda \in U$ *and every supra* $\bowtie$*-open set* $(\mathcal{O}, \Lambda)$ *containing* $h_g(u_\lambda)$*, there is a finite supra* $\bowtie$*-open set* $(\mathcal{Z}, \Lambda)$ *containing* u_λ *with* $h_g(\mathcal{Z}, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda)$*.*

Proof. We prove (i) and one can prove the other cases following similar arguments. Let $u_\lambda \in U$ and $(\mathcal{O}, \Lambda)$ be a finite supra $\bowtie$-open set containing $h_g(u_\lambda)$. Since $h_g^{-1}(\mathcal{O}, \Lambda)$ is a finite supra $\bowtie$-open set containing u_λ, there is a finite supra $\bowtie$-open set $(\mathcal{Z}, \Lambda)$ containing u_λ with $h_g(\mathcal{Z}, \Lambda) \sqsubseteq (\mathcal{O}, \Lambda)$. □

Decompositions theorem for these supra soft continuity types are exhibited in the following result.

Proposition 5.6. *Let* $l_\Phi : (V, \mathbb{Q}, \Lambda) \to (Y, \mathbb{R}, \Lambda)$ *and* $h_g : (U, \mathbb{T}, \Lambda) \to (V, \mathbb{S}, \Lambda)$ *be soft mappings. Then,*

(i) *if* h_g *and* l_Φ *are supra soft* $\mathtt{ff}$*-continuous (resp., supra soft* $\mathtt{fo}$*-continuous), then* $l_\Phi \circ h_g$ *is soft* $\mathtt{ff}$*-continuous (resp., supra soft* $\mathtt{fo}$*-continuous) too;*
(ii) *if* h_g *is supra soft* $\mathtt{ff}$*-continuous and* l_Φ *is supra soft* $\mathtt{fo}$*-continuous, then* $l_\Phi \circ h_g$ *is supra soft* $\mathtt{fo}$*-continuous;*
(iii) *if* h_g *is supra soft* $\mathtt{fo}$*-continuous (resp., supra soft* $\mathtt{of}$*-continuous) and* l_Φ *is soft* $\mathtt{of}$*-continuous (resp., supra soft* $\mathtt{fo}$*-continuous), then* $l_\Phi \circ h_g$ *is supra soft* $\mathtt{ff}$*-continuous (resp., supra soft continuous).*

Proof. It is straightforward. □

Note that the decomposition of soft of-continuous mappings need not be a soft of-continuous mapping.

Now, we will derive some characterizations of supra soft ff-continuity, supra soft fo-continuity, and supra soft of-continuity.

Proposition 5.7. *Let* $h_g : (U, \mathbb{T}, \Lambda) \to (V, \mathbb{S}, \Lambda)$ *be a soft mapping. Then,* h_g *is supra soft* ff*-continuous (resp., supra soft* fo*-continuous, supra soft* of*-continuous) if and only if the pre-image of every finite supra* $\bowtie$*-closed (resp., supra* $\bowtie$*-closed, finite supra* $\bowtie$*-closed) subset is finite supra* $\bowtie$*-closed (resp., finite supra* $\bowtie$*-closed, supra* $\bowtie$*-closed).*

The next theorems and counterexamples will describe the transmissions of these soft continuity in realms of soft and crisp topologies.

Theorem 5.8. *If* $h_g : (U, \mathbb{T}, \Lambda) \to (V, \mathbb{S}, \Lambda)$ *is a supra soft* fo*-continuous mapping, then* $h : (U, \mathbb{T}_\lambda) \to (V, \mathbb{S}_{g(\lambda)})$ *is a supra* fo*-continuous mapping.*

Proof. Let $\mathcal{U}$ be a supra open subset of $(V, \mathbb{S}_{g(\lambda)})$. Then there is a supra $\bowtie$-open subset $(\mathcal{O}, \Lambda)$ of $(V, \mathbb{S}, \Lambda)$ such that $\mathcal{O}(g(\lambda)) = \mathcal{U}$. By hypothesis, h_g is supra soft fo-continuous, so $h_g^{-1}(\mathcal{O}, \Lambda)$ is a finite supra $\bowtie$-open set. It follows from Proposition 3.21 that each component of $h_g^{-1}(\mathcal{O}, \Lambda)$ is a finite supra open set. Remark that a $\bowtie$-subset $h_g^{-1}(\mathcal{O}, \Lambda) = (\mathcal{P}, \Lambda)$ of $(U, \mathbb{T}, \Lambda)$ is calculated by $\mathcal{P}(\lambda) = h^{-1}(\mathcal{O}(g(\lambda)))$ for each $\lambda \in \Lambda$. Accordingly, we obtain $h^{-1}(\mathcal{O}(g(\lambda))) = h^{-1}(\mathcal{U})$ is finite supra open. Hence, we show fo-continuity of $h : (U, \mathbb{T}_\lambda) \to (V, \mathbb{S}_{g(\lambda)})$. □

In general, the converse of Theorem 5.8 fails as the next counterexample shows.

Example 5.9. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and let $\mathbb{T} = \{\widetilde{\Phi}, \widetilde{U}, (\mathcal{O}, \Lambda)\}$, $\mathbb{S} = \{\widetilde{\Phi}, \widetilde{U}, (\mathcal{Z}, \Lambda)\}$ be soft topologies on $U = \{u, v\}$, where

$$(\mathcal{O}, \Lambda) = \{(\lambda_1, \{v\}), (\lambda_2, U), (\lambda_3, \{v\})\} \text{ and } (\mathcal{Z}, \Lambda) = \{(\lambda_1, \emptyset), (\lambda_2, U), (\lambda_3, \{v\})\}.$$

Consider the following identity mappings $g : \Lambda \to \Lambda$ and $h : U \to U$. Then $h : (U, \mathbb{T}_{\lambda_1}) \to (U, \mathbb{S}_{g(\lambda_1)=\lambda_1})$, $h : (U, \mathbb{T}_{\lambda_2}) \to (V, \mathbb{S}_{g(\lambda_2)=\lambda_2})$, and $h : (U, \mathbb{T}_{\lambda_3}) \to (U, \mathbb{S}_{g(\lambda_3)=\lambda_3})$ are fo-continuous. On the other hand, $h_g : (U, \mathbb{T}, \Lambda) \to (U, \mathbb{S}, \Lambda)$ is not a soft fo-continuous mapping because $h_g^{-1}(\mathcal{Z}, \Lambda) = (\mathcal{Z}, \Lambda) \notin \mathbb{T}$.

6. Conclusion and future work

Generalizations of topological structures and their uncertain counterparts like supra soft-topology are crucial instruments to treat many practical problems existing in different situations of our life [14, 15, 18, 20, 28, 57, 60]. The unit build of supra soft-topology is supra $\bowtie$-open sets, so expanding or restricting this unit helps researchers to define diverse types of topological concepts via supra soft-topologies such as continuity, compactness, separation axioms, etc as well as to handle some real-life situations.

It has been established, in this manuscript, the idea of "finite supra $\bowtie$-open sets" as a novel subclass of supra soft-topology includes the class of supra $\bowtie$-clopen. Some properties of this class have been presented and the relationships between this class and other celebrated generalizations of supra soft-open sets have been elucidated. Also, it has been derived the conditions they guarantee some equivalences between this class and others. Moreover, it has been exploited this class to introduce some types of operators and continuity and founded their main characterizations. Some results that describe how these concepts behave via the spaces of supra soft-topologies and their parametric supra topologies have been scrutinized with the aid of some counterexamples.

In the end, we refer to some possible directions one can take into account as future works. First, one can describe the main topological concepts in relation to the class of finite supra $\bowtie$-open sets such as soft connectedness and compactness, soft axioms of separation, etc. This line can be also conducted by discovering the behaviors of the current concepts in the realms of infra soft-topologies and weak soft structures. Second, one can research the application of this class to information systems aiming to make an accurate decision and select the optimal choice since topological operators represent alternative tools to approximate subsets and measure their accuracy.

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