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Exploring the depths of degenerate hyper-harmonic numbers in view of harmonic functions

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Abstract

This study aims to analyze several properties and relations of the degenerate hyper-harmonic numbers and the degenerate harmonic numbers. For this purpose, many identities including the Daehee numbers and derangement numbers, and degenerate Stirling numbers of the first kind are provided. Moreover, the first few values of the degenerate hyper-harmonic numbers are given and some graphical representations are shown.

Keywords: Hyper-harmonic numbers, degenerate hyper-harmonic numbers, degenerate Daehee numbers, differential equation.

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1. Introduction

Harmonic numbers and generalized harmonic numbers have been studied and analyzed in various branches of science such as theoretical physics, number theory, analysis of algorithms in computer science and elementary particle physics. In [5], finite series involving harmonic numbers and generalized harmonic numbers were derived by using the usual differential operator. The familiar hypergeometric summation theorems have been exploited to investigate diverse striking properties of harmonic numbers in [6]. Kargin et al. derived some summation formulae including harmonic numbers, whose coefficients involve r-Lah numbers. Recently hyperharmonic numbers have been examined and worked on by many mathematicians. For example, Benjamin et al. [3] obtained combinatorial interpretations of many interesting identities in terms of r-Stirling numbers for hyperharmonic numbers. Also, in [18], Kim and Kim investigated some identities, recurrence relations and properties covering degenerate harmonic numbers,

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hyperharmonic numbers and degenerate hyperharmonic numbers. Rim et al. [24] acquired some non-linear differential equations from the generating function of hyperharmonic numbers by which some identities including the hyperharmonic numbers and the Daehee numbers are derived.

Harmonic numbers H_ω given by the following sum (see [5–7, 13, 16, 24])

$$\sum_{k=1}^{\omega} \frac{1}{k} \text{ for } \omega \in \{1,2,3,\cdots\},$$

are a long-standing matter of survey and are momentous in diverse categories of elementary and analytic number theory. Several extensions of the mentioned numbers are extensively considered and investigated (see [3, 5–7, 9, 10, 14, 18, 19, 21, 24]. Inspired and motivated by these studies, here we work and analyze the hyper-harmonic numbers and the generalized harmonic numbers. The ω^{th} generalized harmonic number of order ξ is provided by

$$H_\omega^{(\xi)} := \sum_{k=1}^{\omega} \frac{1}{k^\xi},$$

where an integer ξ and a positive integer ω. It is obvious that $H_\omega^{(\xi)} = 0$ for $\omega \leqslant 0$. Also note that $H_\omega^{(\xi)}$ is the partial sum of the Riemann zeta function $\zeta(\xi)$ for $\xi > 1$, where the Riemann zeta function $\zeta(\xi)$ is provided by

$$\zeta(\xi) := \sum_{k=1}^{\infty} \frac{1}{k^\xi},$$

for its analytic continuation elsewhere and $\mathrm{Re}(\xi) > 1$. Another generalizations of the harmonic numbers are the hyper-harmonic numbers introduced by (see [3, 6, 7, 9, 10, 18, 24])

$$h_\omega^{(\rho)} = \sum_{k=1}^{\omega} h_k^{(\rho-1)}, \tag{1.1}$$

where $\rho \in \{1,2,3,\ldots\}$. Notice that $h_\omega^{(1)} := H_\omega, h_\omega^{(0)} := \frac{1}{\omega}(\omega \geqslant 1)$, and $h_0^{(\rho)} := 0, (\rho \geqslant 0)$. The aforesaid numbers possess many relations and applications in many branches of mathematics (see [2–5, 8, 11–14, 24]) and satisfy the following classical generating function (see [10])

$$\sum_{\omega=0}^{\infty} h_\omega^{(\rho)} z^\omega = -\frac{\log(1-z)}{(1-z)^\rho}, \tag{1.2}$$

which gives the following relation (see [10, 14, 23])

$$h_\omega^{(\rho)} = \binom{\omega+\rho-1}{\omega} \left(H_{\omega+\rho-1} - H_{\rho-1}\right). \tag{1.3}$$

The derangement numbers d_ω are introduced by the following exponential generating function (see [19])

$$\sum_{\omega=0}^{\infty} d_\omega \frac{z^\omega}{\omega!} = \frac{e^{-z}}{1-z}. \tag{1.4}$$

The degenerate form of the exponential function is provided by (see [1, 15, 17, 18, 20, 21, 25, 26])

$$e_\lambda^\zeta(z) := (1+\lambda z)^{\frac{\zeta}{\lambda}} \text{ with } e_\lambda^1(z) := e_\lambda(z), \tag{1.5}$$

which gives

$$e_{\lambda}^{\zeta}(z)=\sum_{\omega=0}^{\infty}(\zeta)_{\omega,\lambda}\frac{z^{\omega}}{\omega!}, \tag{1.6}$$

where $(\zeta)_{\omega,\lambda}:=\zeta(\zeta-\lambda)(\zeta-2\lambda)\cdots(\zeta-(\omega-1)\lambda)$ for $\omega\geqslant 1$ in conjunction with $(\zeta)_{0,\lambda}=1$. It is readily seen that $\lim_{\lambda\longrightarrow 0}e_{\lambda}^{\zeta}(z)=e^{\zeta z}$. The degenerate form of logarithm function $\log_{\lambda}(z)$, which is the compositional inverse of $e_{\lambda}^{\zeta}(z)$, is provided by (see [17, 18])

$$\log_{\lambda}(1+z)=\frac{(1+z)^{\lambda}-1}{\lambda}=\sum_{\omega=1}^{\infty}\lambda^{\omega-1}(1)_{\omega,1/\lambda}\frac{z^{\omega}}{\omega!}. \tag{1.7}$$

We notice that $e_{\lambda}(\log_{\lambda}(z))=\log_{\lambda}(e_{\lambda}(z))=z$. Note that

$$\lim_{\lambda\to 0}\log_{\lambda}(1+z)=\sum_{\omega=1}^{\infty}\frac{(-1)^{\omega-1}}{\omega}z^{\omega}=\log(1+z). \tag{1.8}$$

The degenerate form of derangement polynomials is provided by (see [21])

$$\sum_{\omega=0}^{\infty}d_{\omega,\lambda}(\zeta)\frac{z^{\omega}}{\omega!}=\frac{e_{\lambda}^{\zeta-1}(z)}{1-z}. \tag{1.9}$$

Upon setting $\zeta=0$, we attain $d_{\omega,\lambda}:=d_{\omega,\lambda}(0)$ termed the degenerate form of derangement numbers. The degenerate form of higher-order Cauchy numbers of the second kind is provided by (see [12])

$$\sum_{\omega=0}^{\infty}C_{\omega,\lambda}^{(\rho)}\frac{z^{\omega}}{\omega!}=z^{\rho}(\log_{\lambda}(1+z))^{-\rho}. \tag{1.10}$$

In [20], the degenerate form of higher-order Daehee numbers $D_{\omega,\lambda}$ is provided by

$$\sum_{\omega=0}^{\infty}D_{\omega,\lambda}^{(\rho)}\frac{z^{\omega}}{\omega!}=z^{-\rho}(\log_{\lambda}(1+z))^{\rho}. \tag{1.11}$$

The degenerate form of Stirling numbers of the first kind is provided by (see [18–20])

$$\sum_{\omega=k}^{\infty}S_{1,\lambda}(\omega,k)\frac{z^{\omega}}{\omega!}=\frac{(\log_{\lambda}(1+z))^{k}}{k!},\ \text{for } k\geqslant 0. \tag{1.12}$$

Note here that $\lim_{\lambda\to 0}S_{1,\lambda}(\omega,k)=S_{1}(\omega,k)$, where $S_{1}(\omega,k)$ are the Stirling numbers of the first kind given by (see [15, 17, 18, 21, 22])

$$\sum_{\omega=k}^{\infty}S_{1}(\omega,k)\frac{z^{\omega}}{\omega!}=\frac{(\log(1+z))^{k}}{k!}\ \text{for } k\geqslant 0. \tag{1.13}$$

The degenerate form of Stirling numbers of the second kind (see [15, 17, 18, 18, 19, 23]) is provided by

$$\sum_{\omega=\xi}^{\infty}S_{2,\lambda}(\omega,\xi)\frac{z^{\omega}}{\omega!}=\frac{(e_{\lambda}(z)-1)^{\xi}}{\xi!}\ \text{for } (\omega\geqslant 0), \tag{1.14}$$

which gives

$$(\zeta)_{\omega,\lambda}=\sum_{l=0}^{\omega}(\zeta)_{l}S_{2,\lambda}(\omega,l)\ \text{ for }\ (\omega\geqslant 0).$$

2. Main results

In this section, several relations including the degenerate harmonic numbers and the degenerate hyper-harmonic numbers are examined. Also, diverse relations including the degenerate harmonic numbers and the degenerate Daehee numbers are analyzed. Kim and Kim [16] introduced the degenerate form of harmonic numbers by

$$\sum_{\omega=1}^{\infty} H_{\omega,\lambda} z^{\omega} = -\frac{\log_{\lambda}(1-z)}{1-z}. \tag{2.1}$$

It is clear that $\lim_{\lambda\to 0} H_{\omega,\lambda} = H_{\omega}, (\omega \in \{1,2,3,\ldots\})$. In view of (1.7), we get

$$H_{\omega,\lambda} = \sum_{k=1}^{\omega} \frac{(-\lambda)^{k-1}(1)_{k,1/\lambda}}{k!}. \tag{2.2}$$

In view of (1.2) and (2.1), the degenerate hyper-harmonic numbers of the order $\rho \in \{1,2,3,\ldots\}$ are considered by (see [18])

$$\sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(\rho)} z^{\omega} = -\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho}}. \tag{2.3}$$

It is readily derived that

$$\begin{aligned} -\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho}}(1-z) &= \sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(\rho)} z^{\omega} - \sum_{\omega=1}^{\infty} H_{\omega}^{(\rho)} z^{\omega+1} \\ &= \sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(\rho)} z^{\omega} - \sum_{\omega=2}^{\infty} H_{\omega-1,\lambda}^{(\rho)} z^{\omega} = \sum_{\omega=1}^{\infty} \left(H_{\omega,\lambda}^{(\rho)} - H_{\omega-1,\lambda}^{(\rho)}\right) z^{\omega} \end{aligned}$$

and also

$$-\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho}}(1-z) = -\frac{\log_{\lambda}(1+z)}{(1-z)^{\rho-1}} = \sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(\rho-1)} z^{\omega},$$

which means the following relation:

$$H_{\omega,\lambda}^{(\rho)} = H_{\omega,\lambda}^{(\rho-1)} + H_{\omega-1,\lambda}^{(\rho)}, \quad (\omega \geqslant 1),$$

which is derived in [23] in another way. For $1 \leqslant s \leqslant \rho$, it is observed that

$$\begin{aligned} -\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho}} &= -\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho-s}} \frac{1}{(1-z)^{s}} \\ &= \sum_{l=1}^{\infty} H_{l,\lambda}^{(\rho-s)} z^{l} \sum_{\omega=0}^{\infty} \binom{-s}{\omega} (-1)^{\omega} z^{\omega} \\ &= \sum_{l=1}^{\infty} H_{l,\lambda}^{(\rho-s)} z^{l} \sum_{\omega=1}^{\infty} \binom{s+\omega-2}{s-1} z^{\omega-1} = \sum_{\omega=1}^{\infty} \sum_{l=1}^{\omega+1} H_{l,\lambda}^{(\rho-s)} \binom{s+\omega-l-1}{s-1} z^{\omega}. \end{aligned} \tag{2.4}$$

It is seen in (2.3) and (2.4) that

$$H_{\omega,\lambda}^{(\rho)} = \sum_{l=1}^{\omega+1} H_{l,\lambda}^{(\rho-s)} \binom{s+\omega-l-1}{s-1}.$$

It is obtained from (2.3) that

$$-\frac{\log_{\lambda}(1-z)}{(1-z)^{\rho}} = \frac{1}{(1-z)^{\rho-1}} \left(-\frac{\log_{\lambda}(1-z)}{1-z}\right) = \frac{1}{(1-z)^{\rho-1}} \left(\sum_{\omega=1}^{\infty} H_{\omega,\lambda} z^{\omega}\right)$$

$$= \frac{1}{(1-z)^{\rho-2}} \sum_{\omega=1}^{\infty} \left(\sum_{k=1}^{\omega} H_{k,\lambda} \right) z^{\omega} = \frac{1}{(1-z)^{\rho-2}} \sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(2)} z^{\omega} \tag{2.5}$$
$$= \frac{1}{(1-z)^{\rho-3}} \sum_{\omega=1}^{\infty} \left(\sum_{k=1}^{\omega} H_{k,\lambda}^{(2)} \right) z^{\omega} = \frac{1}{(1-z)^{\rho-3}} \sum_{\omega=1}^{\infty} H_{\omega,\lambda}^{(3)} z^{\omega}.$$

Theorem 2.1. *For $\omega, \rho \in \{1, 2, 3, \ldots\}$, we have*

$$\sum_{k=1}^{\omega} H_{k,\lambda}^{(\rho)} (-1)^k k! S_{2,\lambda}(\omega, k) = (-1)^{\omega} < \rho >_{\omega-1,\lambda} \omega,$$

where $< \rho >_{\omega,\lambda} := \rho(\rho+\lambda)(\rho+2\lambda)\cdots(\zeta+(\rho-1)\lambda)$ *for* $\omega \geqslant 1$ *in conjunction with* $< \rho >_{0,\lambda} = 1$.

Proof. Changing z by $1 - e_{\lambda}(z)$ in (2.3), we attain that

$$-ze_{\lambda}^{-\rho}(z) = \sum_{k=1}^{\infty} H_{k,\lambda}^{(\rho)} (-1)^k k! \frac{1}{k!} (e_{\lambda}(z) - 1)^k$$
$$= \sum_{k=1}^{\infty} H_{k,\lambda}^{(\rho)} (-1)^k k! \sum_{\omega=k}^{\infty} S_{2,\lambda}(\omega, k) \frac{z^{\omega}}{\omega!} = \sum_{\omega=1}^{\infty} \left(\sum_{k=1}^{\omega} H_{k,\lambda}^{(\rho)} (-1)^k k! S_{2,\lambda}(\omega, k) \right) \frac{z^{\omega}}{\omega!}$$

and also derive that

$$-ze_{\lambda}^{-\rho}(z) = -z \sum_{\omega=0}^{\infty} \frac{(-\rho)_{\omega,\lambda}}{\omega!} z^{\omega} = z \sum_{\omega=0}^{\infty} (-1)^{\omega-1} \frac{< \rho >_{\omega,\lambda}}{\omega!} z^{\omega}$$
$$= \sum_{\omega=1}^{\infty} (-1)^{\omega} < \rho >_{\omega-1,\lambda} \frac{z^{\omega}}{(\omega-1)!} = \sum_{\omega=1}^{\infty} (-1)^{\omega} < \rho >_{\omega-1,\lambda} \omega \frac{z^{\omega}}{\omega!},$$

which completes the proof of the theorem. □

Let $\omega \in \{1, 2, 3, \ldots\}$. It can be given that

$$\frac{1}{< 1 >_{\omega-1,\lambda}} \sum_{k=1}^{\omega} H_{k,\lambda} (-1)^{\omega-k} k! S_{2,\lambda}(\omega, k) = \omega.$$

Now, diverse relations including the degenerate harmonic numbers and the degenerate Daehee numbers are analyzed. It is observed from (1.11) that

$$\frac{\log_{\lambda}(1-z)}{z} = \frac{-\log_{\lambda}(1-z)}{(1+z)} \frac{1+z}{-z}$$
$$= \sum_{\omega=1}^{\infty} (-1)^{\omega+1} H_{\omega,\lambda} z^{\omega} \left(1 + \frac{1}{z}\right)$$
$$= \sum_{\omega=1}^{\infty} (-1)^{\omega+1} H_{\omega,\lambda} z^{\omega} + \sum_{\omega=0}^{\infty} (-1)^{\omega} H_{\omega+1,\lambda} z^{\omega}$$
$$= \sum_{\omega=1}^{\infty} \left((-1)^{\omega+1} H_{\omega,\lambda} + (-1)^{\omega} H_{\omega+1,\lambda} \right) z^{\omega} + H_{1,\lambda},$$

which means

$$D_{0,\lambda} = H_{1,\lambda}, \ D_{\omega,\lambda} = (-1)^{\omega} \omega! \left(H_{\omega+1,\lambda} - H_{\omega,\lambda} \right), \ (\omega \geqslant 1),$$

since $H_{\omega+1,\lambda} - H_{\omega,\lambda} = \frac{1}{\omega+1}$ and (1.6).

Theorem 2.2. *Let $\omega \in \{1,2,3,\dots\}$. The following formula holds*

$$D_{\omega,\lambda} = \omega! \sum_{i=0}^{\omega} H_{i+1,\lambda}^{(\rho)} \binom{\rho}{\omega-i} (-1)^i.$$

Proof. It is readily investigated from (1.11) and (2.3) that

$$\begin{aligned}\frac{\log_\lambda(1+z)}{z} = \frac{-\log_\lambda(1+z)}{(1+z)^\rho}\frac{(1+z)^\rho}{-z} &= \sum_{i=1}^{\infty}(-1)^{i+1}H_{i,\lambda}^{(\rho)}z^{i-1}\sum_{j=0}^{\infty}\binom{\rho}{j}z^j \\ &= \sum_{i=0}^{\infty}(-1)^{i}H_{i+1,\lambda}^{(\rho)}z^{i}\sum_{j=0}^{\infty}\binom{\rho}{j}z^j = \sum_{\omega=0}^{\infty}\left(\sum_{i=0}^{\omega}(-1)^i\binom{\rho}{\omega-i}H_{i+1,\lambda}^{(\rho)}\right)z^\omega,\end{aligned}$$

which implies the asserted result of the theorem. □

Theorem 2.3. *The following formula holds for any non-negative integer ω and $k \geqslant 1$:*

$$D_{\omega,\lambda}^{(\rho)} = \omega! \sum_{i=0}^{\omega}\sum_{j=0}^{\omega-i}(-1)^i\binom{\omega-i}{j}\frac{(k)_{\omega-i-j}D_{j,\lambda}^{(\rho-1)}H_{i+1,\lambda}^{(k)}}{(\omega-i)!}. \tag{2.6}$$

Proof. It is analyzed from (1.11) and (2.3) that

$$\begin{aligned}\left(\frac{\log_\lambda(1+z)}{z}\right)^\rho &= -\frac{\log_\lambda(1+z)}{(1+z)^k}\left(\frac{\log_\lambda(1+z)}{z}\right)^{\rho-1}\frac{(1+z)^k}{-z} \\ &= \left(\sum_{i=0}^{\infty}(-1)^iH_{i+1,\lambda}^{(\rho)}z^i\right)\left(\sum_{j=0}^{\infty}D_{j,\lambda}^{(\rho-1)}\frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty}(k)_l\frac{z^l}{l!}\right) \\ &= \left(\sum_{i=0}^{\infty}(-1)^iH_{i+1,\lambda}^{(\rho)}z^i\right)\left(\sum_{\xi=0}^{\infty}\sum_{j=0}^{\xi}\binom{\xi}{j}D_{j,\lambda}^{(\rho-1)}(k)_{\xi-j}\frac{z^\xi}{\xi!}\right) \\ &= \sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\sum_{j=0}^{\omega-i}(-1)^i\binom{\omega-i}{j}\frac{(k)_{\omega-i-j}D_{j,\lambda}^{(\rho-1)}H_{i+1,\lambda}^{(k)}}{(\omega-i)!}z^\omega,\end{aligned}$$

which proves (2.6). □

Theorem 2.4. *The following formula is valid for any non-negative integer ω:*

$$H_{\omega,\lambda} = \sum_{j=0}^{\omega-1}(-1)^j\frac{D_{j,\lambda}}{j!}. \tag{2.7}$$

Proof. It is noticeably acquired from (1.11) and (2.1) that

$$\sum_{\omega=1}^{\infty}H_{\omega,\lambda}z^\omega = -\frac{\log_\lambda(1+z)}{1-z} = \frac{\log_\lambda(1+z)}{-z}\frac{z}{1-z} = \sum_{j=0}^{\infty}(-1)^jD_{j,\lambda}\frac{z^j}{j!}\sum_{\omega=1}^{\infty}z^\omega = \sum_{\omega=1}^{\infty}\sum_{j=0}^{\omega-1}(-1)^j\frac{D_{j,\lambda}}{j!}z^\omega,$$

which provides the claimed result (2.7). □

Theorem 2.5. *The following formula is valid for any non-negative integer ω:*

$$\omega!H_{\omega,\lambda}^{(\rho)} = (\omega-k)\sum_{k=0}^{\omega-1}\binom{\omega}{k}(-1)^{\omega-1}D_{k,\lambda}(-\rho)_{k-1}. \tag{2.8}$$

Proof. It is clearly attained from (1.11) and (2.3) that

$$
\begin{aligned}
-\frac{\log_\lambda(1+z)}{(1-z)^\rho} &= \frac{\log_\lambda(1+z)}{-z}\frac{z}{(1-z)^\rho} \\
&= \sum_{k=0}^{\infty}(-1)^k D_{k,\lambda}\frac{z^k}{k!}\sum_{l=0}^{\infty}(-\rho)_l(-1)^l\frac{z^{l+1}}{l!} \\
&= \sum_{k=0}^{\infty}(-1)^k D_{k,\lambda}\frac{z^k}{k!}\sum_{l=1}^{\infty}(-\rho)_{l-1}(-1)^{l-1}l\frac{z^l}{l!} \\
&= \sum_{\omega=1}^{\infty}\sum_{k=0}^{\omega-1}\binom{\omega}{k}(-1)^{\omega-1}D_{k,\lambda}(-\rho)_{k-1}(\omega-k)\frac{z^\omega}{\omega!},
\end{aligned}
$$

which provides the asserted result (2.8). □

Theorem 2.6. *The following correlation is valid for any non-negative integer ω:*

$$
\omega! H^{(\rho)}_{\omega,\lambda} = \sum_{k=0}^{\omega-1}\sum_{l=0}^{\omega-1}\binom{\omega}{k}\binom{\omega}{l}D^{(\rho)}_{\omega-k,\lambda}(-1)^{k+l}C^{(\rho-1)}_{l,\lambda}(\omega-l)(-\rho)_{\omega-l-l}. \tag{2.9}
$$

Proof. It is obvious to compute from (1.10), (1.11), and (2.3) that

$$
\begin{aligned}
-\frac{\log_\lambda(1+z)}{(1-z)^\rho} &= \left(\frac{\log_\lambda(1+z)}{-z}\right)\left(\frac{-z}{\log_\lambda(1+z)}\right)^{\rho-1}\frac{z}{(1-z)^\rho} \\
&= \sum_{k=0}^{\infty}(-1)^k D^{(\rho)}_{k,\lambda}\frac{z^k}{k!}\sum_{l=0}^{\infty}(-1)^l C^{(\rho-1)}_{l,\lambda}\frac{z^l}{l!}\sum_{\xi=0}^{\infty}(-\rho)_\xi(-1)^\xi\frac{z^{\xi+1}}{\xi!} \\
&= \sum_{k=0}^{\infty}(-1)^k D^{(\rho)}_{k,\lambda}\frac{z^k}{k!}\sum_{l=0}^{\infty}(-1)^l C^{(\rho-1)}_{l,\lambda}\frac{z^l}{l!}\sum_{\xi=1}^{\infty}\xi(-\rho)_{\xi-1}(-1)^{\xi-1}\frac{z^\xi}{\xi!} \\
&= \sum_{k=0}^{\infty}(-1)^k D^{(\rho)}_{k,\lambda}\frac{z^k}{k!}\sum_{\omega=1}^{\infty}\sum_{l=0}^{\omega-1}\binom{\omega}{l}(-1)^{\omega-1}C^{(\rho-1)}_{l,\lambda}(\omega-l)(-\rho)_{\omega-l-l} \\
&= \sum_{\omega=1}^{\infty}\sum_{k=0}^{\omega-1}\sum_{l=0}^{\omega-1}\binom{\omega}{k}\binom{\omega}{l}D^{(\rho)}_{\omega-k,\lambda}(-1)^{k+l}C^{(\rho-1)}_{l,\lambda}(\omega-l)(-\rho)_{\omega-l-l}\frac{z^\omega}{\omega!},
\end{aligned}
$$

which means the claimed relation (2.9). □

Theorem 2.7. *The following formula holds for any non-negative integer ω:*

$$
\sum_{l=1}^{\omega+1}H^{(\rho)}_{l,\lambda}\frac{(-1)_{\omega-1,\lambda}(\omega-l+1)}{(\omega-l+1)!} = \sum_{l=1}^{\omega+1}H^{(\rho)}_{l,\lambda}\frac{d_{\omega-l,\lambda}(\omega-l+1)}{(\omega-l+1)!}. \tag{2.10}
$$

Proof. It is clearly derived from (1.9) and (2.3) that

$$
\begin{aligned}
-\frac{\log_\lambda(1+z)}{(1-z)^\rho}e_\lambda(-z) &= \sum_{l=1}^{\infty}H^{(\rho)}_{l,\lambda}z^l\sum_{\omega=1}^{\infty}(-1)_{\omega-1,\lambda}\frac{z^{\omega-1}}{(\omega-1)!} \\
&= \sum_{l=1}^{\infty}H^{(\rho)}_{l,\lambda}z^l\sum_{\omega=1}^{\infty}(-1)_{\omega-1,\lambda}\omega\frac{z^{\omega-1}}{\omega!}
\end{aligned}
$$

$$=\sum_{\omega=1}^{\infty}\sum_{l=1}^{\omega+1}H_{l,\lambda}^{(\rho)}(-1)_{\omega-1,\lambda}(\omega-l+1)\frac{z^{\omega}}{(\omega-l+1)!},$$

and also

$$-\frac{\log_{\lambda}(1+z)}{(1-z)^{\rho}}e_{\lambda}(-z)=\sum_{l=1}^{\infty}H_{l,\lambda}^{(\rho)}z^{l}\sum_{\omega=1}^{\infty}d_{\omega-1,\lambda}\frac{z^{\omega-1}}{(\omega-1)!}$$
$$=\sum_{l=1}^{\infty}H_{l,\lambda}^{(\rho)}z^{l}\sum_{\omega=1}^{\infty}d_{\omega-1,\lambda}\omega\frac{z^{\omega-1}}{\omega!}=\sum_{\omega=1}^{\infty}\sum_{l=1}^{\omega+1}H_{l,\lambda}^{(\rho)}d_{\omega-l,\lambda}(\omega-l+1)\frac{z^{\omega}}{(\omega-l+1)!},$$

which implies the asserted relation (2.10). □

Theorem 2.8. *The following formula holds for any non-negative integer ω:*

$$D_{\omega,\lambda}=\sum_{i=0}^{\omega}\sum_{\xi=0}^{\omega-i}(-1)^{i}H_{i+1,\lambda}^{(\rho)}(\rho)_{\xi,\lambda}S_{1,\lambda}(\omega-i,\xi)\frac{\omega!}{(\omega-i)!}. \tag{2.11}$$

Proof. It is obvious to investigate from (1.11), (1.12), and (2.3) that

$$\sum_{\omega=0}^{\infty}D_{\omega,\lambda}\frac{z^{\omega}}{\omega!}=\frac{\log_{\lambda}(1+z)}{z}=\frac{-\log_{\lambda}(1+z)}{(1+z)^{\rho}}\frac{(1+z)^{\rho}}{-z}\frac{-\log_{\lambda}(1+z)}{(1+z)^{\rho}}\frac{1}{-z}e_{\lambda}^{\rho}(\log_{\lambda}(1+z))$$
$$=\sum_{i=1}^{\infty}(-1)^{i+1}H_{i,\lambda}^{(\rho)}z^{i-1}\sum_{\xi=0}^{\infty}(\rho)_{\xi,\lambda}\frac{[\log_{\lambda}(1+z)]^{\xi}}{\xi!}$$
$$=\sum_{i=0}^{\infty}(-1)^{i}H_{i+1,\lambda}^{(\rho)}z^{i}\sum_{\omega=0}^{\infty}\sum_{\xi=0}^{\omega}(\rho)_{\xi,\lambda}S_{1,\lambda}(\omega,\xi)\frac{z^{\omega}}{\omega!}$$
$$=\sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\sum_{\xi=0}^{\omega-i}(-1)^{i}H_{i+1,\lambda}^{(\rho)}(\rho)_{\xi,\lambda}S_{1,\lambda}(\omega-i,\xi)\frac{z^{\omega}}{(\omega-i)!},$$

which means (2.13). □

Now, we introduce the generalized degenerate harmonic numbers and then investigate several properties and relations. For $\lambda\in\mathbb{C}$, we consider the generalized degenerate harmonic numbers $\hbar_{\omega,\lambda}^{(\rho)}$ given by

$$\sum_{\omega=0}^{\infty}\hbar_{\omega,\lambda}^{(\rho)}z^{\omega}=\frac{(-\log_{\lambda}(1-z))^{\rho+1}}{1-z}. \tag{2.12}$$

Note that $\hbar_{\omega,\lambda}^{(0)}=H_{\omega,\lambda}, (\omega\geqslant 0)$ in (2.1).

Now, we will provide some theorems.

Theorem 2.9. *The following formula holds for any non-negative integer ω:*

$$\hbar_{\omega,\lambda}^{(\rho)}=\sum_{i=0}^{\omega}\frac{(-1)^{\omega-i-\rho}S_{1,\lambda}(\omega-i,\rho)\rho!}{(\omega-i)!}H_{i,\lambda}. \tag{2.13}$$

Proof. It is obvious to investigate from (1.12), (2.1), and (2.12) that

$$\sum_{\omega=0}^{\infty}\hbar_{\omega,\lambda}^{(\rho)}z^{\omega}=\frac{(-\log_{\lambda}(1-z))^{\rho+1}}{1-z}$$

$$
\begin{aligned}
&= \frac{\left(-\log_{\lambda}(1-z)\right)}{1-z}\left(-\log_{\lambda}(1-z)\right)^{\rho} \\
&= \sum_{i=0}^{\infty} H_{i,\lambda} z^{i} \sum_{\omega=0}^{\infty} \frac{(-1)^{\omega-\rho} S_{1,\lambda}(\omega,\rho)\rho!}{\omega!} z^{\omega} \\
&= \sum_{\omega=0}^{\infty} \sum_{i=0}^{\omega} \frac{(-1)^{\omega-i-\rho} S_{1,\lambda}(\omega-i,\rho)\rho!}{(\omega-i)!} H_{i,\lambda} z^{\omega},
\end{aligned}
$$

which means the desired correlation (2.13). □

Theorem 2.10. *The following correlation is valid for any non-negative integer ω:*

$$
\hbar_{\omega,\lambda}^{(\rho)} = \sum_{i=0}^{\omega} \sum_{j=0}^{i} \binom{\xi-1}{\omega-i} (-1)^{\omega-j-\rho} \frac{H_{j,\lambda}^{(\xi)} S_{1,\lambda}(i-j,\rho)}{(i-j)!}. \tag{2.14}
$$

Proof. It is clearly attained from (1.12), (2.3), and (2.12) that

$$
\begin{aligned}
\sum_{\omega=0}^{\infty} \hbar_{\omega,\lambda}^{(\rho)} z^{\omega} &= \frac{\left(-\log_{\lambda}(1-z)\right)^{\rho+1}}{1-z} \\
&= \frac{\left(-\log_{\lambda}(1-z)\right)}{(1-z)^{\xi}} \left(-\log_{\lambda}(1-z)\right)^{\rho} (1-z)^{\xi-1} \\
&= \sum_{\omega=0}^{\infty} H_{\omega,\lambda}^{(\xi)} z^{\omega} \sum_{j=0}^{\infty} \frac{(-1)^{j-\rho} S_{1,\lambda}(j,\rho)\rho!}{j!} z^{j} \sum_{i=0}^{\infty} \binom{\xi-i}{i} (-1)^{i} z^{i} \\
&= \sum_{\omega=0}^{\infty} \sum_{i=0}^{\omega} \sum_{j=0}^{i} \binom{\xi-1}{\omega-i} (-1)^{\omega-j-\rho} \frac{H_{j,\lambda}^{(\xi)} S_{1,\lambda}(i-j,\rho)}{(i-j)!} z^{\omega},
\end{aligned}
$$

which implies the asserted relation (2.14). □

Theorem 2.11. *The following formula holds for any non-negative integer ω:*

$$
D_{\omega,\lambda}^{(\rho+1)} = \omega! \sum_{i=0}^{\omega} \sum_{j=0}^{i} \frac{(-1)^{j} k^{\omega-i}}{(\omega-i)!(i-j)!} H_{j+1,\lambda}^{(k)} D_{i-j,\lambda}^{(\rho)}.
$$

Proof. It is obvious to obtain from (1.11) and (2.3) that

$$
\begin{aligned}
\sum_{\omega=0}^{\infty} D_{\omega,\lambda}^{(\rho+1)} \frac{z^{\omega}}{\omega!} &= \left(\frac{\log_{\lambda}(1+z)}{z}\right)^{\rho+1} \\
&= \frac{-\log_{\lambda}(1+z)}{(-z)(1+z)^{k}} \left(\frac{\log_{\lambda}(1+z)}{z}\right)^{\rho} (1+z)^{k} \\
&= \sum_{\omega=0}^{\infty} (-1)^{\omega} H_{\omega+1,\lambda}^{(k)} z^{\omega} \sum_{j=0}^{\infty} D_{j,\lambda}^{(\rho-1)} \frac{z^{j}}{j!} \sum_{i=0}^{\infty} k^{i} \frac{z^{i}}{i!} \\
&= \sum_{\omega=0}^{\infty} \sum_{j=0}^{\omega} (-1)^{j} H_{j+1,\lambda}^{(k)} \frac{D_{\omega-j,\lambda}^{(\rho)}}{(\omega-j)!} z^{\omega} \sum_{i=0}^{\infty} k^{i} \frac{z^{i}}{i!} \\
&= \sum_{\omega=0}^{\infty} \sum_{i=0}^{\omega} \sum_{j=0}^{i} \frac{(-1)^{j} k^{\omega-i}}{(\omega-i)!(i-j)!} H_{j+1,\lambda}^{(k)} D_{i-j,\lambda}^{(\rho)} z^{\omega},
\end{aligned}
$$

which completes the proof of the theorem. □

Theorem 2.12. *The following correlation is valid for any non-negative integer ω:*

$$(-1)^{\omega+1-\rho}S_{1,\lambda}(\omega+1,\rho)\rho! = (\omega+1)!\left(\hbar^{(\rho-1)}_{\omega+1,\lambda} - \hbar^{(\rho-1)}_{\omega,\lambda}\right). \tag{2.15}$$

Proof. With the help of $S_{1,\lambda}(\omega,\rho) = 0$ for $0 \leqslant \omega < \rho$ and $\hbar^{(\rho)}_{0,\lambda} = 0$, it is clearly attained from (1.11) and (2.12) that

$$\begin{aligned}\sum_{\omega=\rho}^{\infty}\frac{(-1)^{\omega-\rho}S_{1,\lambda}(\omega,\rho)\rho!}{\omega!}z^{\omega-1} &= \frac{(-\log_\lambda(1-z))^\rho}{z}\\ &= \frac{(-\log_\lambda(1-z))^\rho}{1-z}\left(\frac{1}{z}-1\right)\\ &= \frac{(-\log_\lambda(1-z))^\rho}{z(1-z)} - \frac{(-\log_\lambda(1-z))^\rho}{1-z}\\ &= \sum_{\omega=0}^{\infty}\hbar^{(\rho-1)}_{\omega,\lambda}z^{\omega-1} - \sum_{\omega=0}^{\infty}\hbar^{(\rho-1)}_{\omega,\lambda}z^{\omega} = \sum_{\omega=0}^{\infty}\left(\hbar^{(\rho-1)}_{\omega+1,\lambda} - \hbar^{(\rho-1)}_{\omega,\lambda}\right)z^\omega,\end{aligned}$$

and also

$$\begin{aligned}\sum_{\omega=\rho}^{\infty}\frac{(-1)^{\omega-\rho}S_{1,\lambda}(\omega,\rho)\rho!}{\omega!}z^{\omega-1} &= \sum_{\omega=\rho-1}^{\infty}\frac{(-1)^{\omega+1-\rho}S_{1,\lambda}(\omega+1,\rho)\rho!}{(\omega+1)!}z^{\omega}\\ &= \sum_{\omega=0}^{\infty}\frac{(-1)^{\omega+1-\rho}S_{1,\lambda}(\omega+1,\rho)\rho!}{(\omega+1)!}z^{\omega}.\end{aligned}$$

Hence we arrive at the claimed relation (2.15). □

Theorem 2.13. *The following formula holds for any non-negative integer ω:*

$$\sum_{i=0}^{\omega}\binom{\omega-i+\rho}{\rho}\frac{(-1)^{\omega-\rho-1}S_{1,\lambda}(i,\rho+1)(\rho+1)!}{i!} = \sum_{i=0}^{\omega}\binom{\omega-i+\rho-1}{\rho-1}\hbar^{(\rho)}_{i,\lambda}. \tag{2.16}$$

Proof. Utilizing $\sum_{\omega=0}^{\infty}\binom{\omega}{\rho}z^\omega = \frac{z^\rho}{(1-z)^\rho}$ and then it is obvious to derive from (1.12) and (2.12) that

$$\begin{aligned}\left(\frac{-\log_\lambda(1-z)}{1-z}\right)^{\rho+1} &= \sum_{\omega=0}^{\infty}\frac{(-1)^{\omega-\rho-1}S_{1,\lambda}(\omega,\rho+1)(\rho+1)!}{\omega!}z^\omega\sum_{i=0}^{\infty}\binom{i+\rho}{\rho}z^i\\ &= \sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\binom{\omega-i+\rho}{\rho}\frac{(-1)^{\omega-\rho-1}S_{1,\lambda}(i,\rho+1)(\rho+1)!}{i!}z^\omega,\end{aligned}$$

and also

$$\begin{aligned}\left(\frac{-\log_\lambda(1-z)}{1-z}\right)^{\rho+1} &= \frac{\left(-\log_\lambda(1-z)\right)^{\rho+1}}{1-z}\frac{1}{(1-z)^\rho}\\ &= \sum_{\omega=0}^{\infty}\hbar^{(\rho)}_{\omega,\lambda}z^\omega\sum_{i=0}^{\infty}\binom{i+\rho-1}{\rho-1}z^i = \sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\binom{\omega-i+\rho-1}{\rho-1}\hbar^{(\rho)}_{i,\lambda}z^\omega,\end{aligned}$$

which means the claimed formula (2.16). □

Theorem 2.14. *The following correlation is valid for any non-negative integer ω:*

$$\sum_{i=0}^{\omega}\binom{\omega}{i}(-1)^{i-\rho-1}S_{1,\lambda}(i,\rho+1)(\rho+1)!d_{\omega-i,\lambda} = \omega!\sum_{i=0}^{\omega}\hbar^{(\rho)}_{i,\lambda}\frac{(-1)_{\omega-i,\lambda}}{(\omega-i)!}. \tag{2.17}$$

Proof. It is clearly attained from (1.9), (1.12), and (2.12) that

$$\frac{(-\log_\lambda(1-z))^{\rho+1}}{(1-z)}e_\lambda(-z)=\sum_{\omega=0}^{\infty}\hbar_{\omega,\lambda}^{(\rho)}z^\omega\sum_{i=0}^{\infty}\frac{(-1)_{i,\lambda}}{i!}z^i=\sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\hbar_{\omega,\lambda}^{(\rho)}\frac{(-1)_{\omega-i,\lambda}}{(\omega-i)!}z^\omega,$$

and also

$$\begin{aligned}\frac{(-\log_\lambda(1-z))^{\rho+1}}{1-z}e_\lambda(-z)&=(-\log_\lambda(1-z))^{\rho+1}\frac{e_\lambda(-z)}{1-z}\\&=\sum_{\omega=0}^{\infty}\frac{(-1)^{\omega-\rho-1}S_{1,\lambda}(\omega,\rho+1)(\rho+1)!}{\omega!}\sum_{i=0}^{\infty}d_{i,\lambda}\frac{z^i}{i!}\\&=\sum_{\omega=0}^{\infty}\sum_{i=0}^{\omega}\binom{\omega}{i}(-1)^{i-\rho-1}S_{1,\lambda}(i,\rho+1)(\rho+1)!d_{\omega-i,\lambda}\frac{z^\omega}{\omega!},\end{aligned}$$

which implies the asserted formula (2.17). □

3. Further remarks

The degenerate hyper-harmonic numbers of the order $\rho\in\{1,2,3,\dots\}$ are given by

$$-\frac{\log_\lambda(1-z)}{(1-z)^\rho}=\sum_{\omega=1}^{\infty}H_{\omega,\lambda}^{(\rho)}z^\omega.$$

In this section, computational values, and graphical representations of degenerate hyper-harmonic numbers are shown. A few of them are

$$\begin{aligned}H_{1,\lambda}^{(\rho)}&=1,\\H_{2,\lambda}^{(\rho)}&=\frac{1}{2}-\frac{\lambda}{2}+\rho,\\H_{3,\lambda}^{(\rho)}&=\frac{1}{3}+\rho+\frac{\rho^2}{2}-\frac{\lambda}{2}-\frac{\rho\lambda}{2}+\frac{\lambda^2}{6},\\H_{4,\lambda}^{(\rho)}&=\frac{1}{4}+\frac{11\rho}{12}+\frac{3\rho^2}{4}+\frac{\rho^3}{6}-\frac{11\lambda}{24}-\frac{3\rho\lambda}{4}-\frac{\rho^2\lambda}{4}+\frac{\lambda^2}{4}+\frac{\rho\lambda^2}{6}-\frac{\lambda^3}{24},\\H_{5,\lambda}^{(\rho)}&=\frac{1}{5}+\frac{5\rho}{6}+\frac{7\rho^2}{8}+\frac{\rho^3}{3}+\frac{\rho^4}{24}-\frac{5\lambda}{12}-\frac{7\rho\lambda}{8}-\frac{\rho^2\lambda}{2}-\frac{\rho^3\lambda}{12}+\frac{7\lambda^2}{24}+\frac{\rho\lambda^2}{3}+\frac{\rho^2\lambda^2}{12}-\frac{\lambda^3}{12}-\frac{\rho\lambda^3}{24}+\frac{\lambda^4}{120},\\H_{6,\lambda}^{(\rho)}&=\frac{1}{6}+\frac{137\rho}{180}+\frac{15\rho^2}{16}+\frac{17\rho^3}{36}+\frac{5\rho^4}{48}+\frac{\rho^5}{120}-\frac{137\lambda}{360}-\frac{15\rho\lambda}{16}-\frac{17\rho^2\lambda}{24}-\frac{5\rho^3\lambda}{24}\\&\quad-\frac{\rho^4\lambda}{48}+\frac{5\lambda^2}{16}+\frac{17\rho\lambda^2}{36}+\frac{5\rho^2\lambda^2}{24}+\frac{\rho^3\lambda^2}{36}-\frac{17\lambda^3}{144}-\frac{5\rho\lambda^3}{48}-\frac{\rho^2\lambda^3}{48}+\frac{\lambda^4}{48}+\frac{\rho\lambda^4}{120}-\frac{\lambda^5}{720}.\end{aligned}$$

We investigate the graphical representations of degenerate hyper-harmonic numbers $H_{\omega,\lambda}^{(\rho)}$ (Figure 1). This shows the ten plots combined into one.

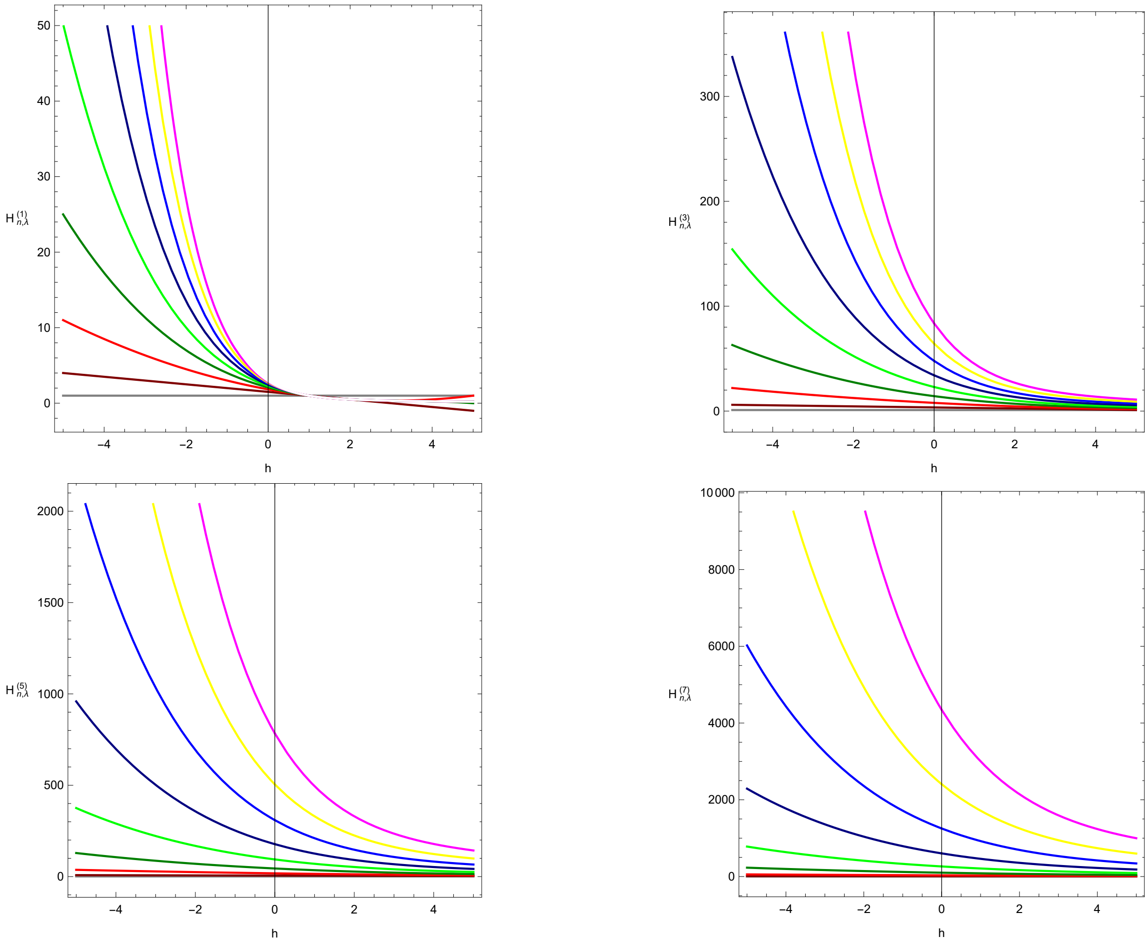

Figure 1: Degenerate hyper-harmonic numbers $H^{(r)}_{n,\lambda}$.

In Figure 1 (top-left), we choose $\rho = 1$ and $-5 \leqslant \lambda \leqslant 5$. In Figure 1 (top-right), we choose $\rho = 3$ and $-5 \leqslant \lambda \leqslant 5$. In Figure 1 (bottom-left), we choose $\rho = 5$ and $-5 \leqslant \lambda \leqslant 5$. In Figure 1 (bottom-right), we choose $\rho = 7$ and $-5 \leqslant \lambda \leqslant 5$.

Table 1: Degenerate hyper-harmonic numbers $H^{(3)}_{n,\lambda}$.

ω	$\lambda \to 0$	$\lambda = \frac{1}{2}$
1	1	1
2	7/2	13/4
3	47/6	55/8
4	57/4	765/64
5	459/20	2373/128
6	341/10	13657/512
7	3349/70	37263/1024
8	3601/56	781725/16384
9	42131/504	1987865/32768
10	44441/420	9865075/131072
11	605453/4620	23993489/262144
12	631193/3960	229572889/2097152

4. Conclusion

In this paper, we have analyzed diverse properties and relations of the degenerate hyper-harmonic numbers, and the degenerate harmonic numbers. We have also derived many identities and formulas including the Daehee numbers and derangement numbers, and degenerate Stirling numbers of the first kind. Furthermore, the first few values of the degenerate hyper-harmonic numbers are given and some graphical representations are shown, as applications.

It is possible that this paper's idea can be applied to polynomials that are similar and these polynomials have potential applications in other fields of science in addition to the applications at the end of the article. We will continue to explore this opinion in various directions in our next scientific works to advance the purpose of this article. For future directions, we will consider that the polynomials introduced in this paper can be examined within the context of the monomiality principle and umbral calculus to have alternative ways of deriving our results.

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