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C-erasure reconstruction error of GC-frame of subspaces

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Abstract

In [M. H. Faroughi, R. Ahmadi, Math. Nachr., **284** (2010), 681–693], we generalized the concept of fusion frames, namely, c-fusion integral, which is a continuous version of the fusion frames and in [M. H. Faroughi, A. Rahimi, R. Ahmadi, Methods Funct. Anal. Topology, **16** (2010), 112–119] we extended it for generalized frames. In this article we give some important properties about it namely erasures of subspaces, the bound of gc-erasure reconstruction error for Parseval gc-frame of subspaces.

Keywords: Operator theory, frame, Hilbert space, fusion frame, generalized frame.
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1. Introduction

Throughout this paper, H will be a Hilbert space and $\mathbb{H}$ the collection of all closed subspace of H. Also, (X, μ) will be a measure space, and $v : X \to [0, +\infty)$ will be a measurable mapping such that $v \neq 0$ a.e.. We shall denote the unit closed ball of H by H_1.

Frames was first introduced in the context of non-harmonic Fourier series [8]. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work [7]. Since then the theory of frames began to be more widely studied. During the last 20 years, the theory of frames has grown up rapidly, several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence

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to improve the robustness of data transmission on [4], and to design high-rate constellation with full diversity in multiple-antenna code design [16]. In [1–3], some applications have been developed.

The fusion frames were considered by Casazza, Kutyniok and Li in connection with distributed processing and are related to the construction of global frames [5, 6]. The fusion frame theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights.

In [9], we extended the fusion frames to their continuous versions in measure spaces and in [10], we extended the C-fusion frames to GC-fusion frames. In this paper, we shall investigate some properties about it and interested to C-erasure reconstruction error of gc-frame of subspaces.

Definition 1.1. Let $\{f_i\}_{i\in I}$ be a sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H, if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \sum_{i\in I} | < f_i, h > |^2 \leq B\|h\|^2.$$

The constants A and B are called the frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$, it is called a Parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$$T : l^2(I) \to H, \quad T(c_i) = \sum_{i\in I} c_i f_i,$$

$$T^* : H \to l^2(I), \quad T^*(f) = \{< f, f_i >\}_{i\in I},$$

$$Sf = TT^*f = \sum_{i\in I} < f, f_i > f_i.$$

These operators are called synthesis operator, analysis operator and frame operator, respectively.

Definition 1.2. For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspace in H, and let $\{v_i\}_{i\in I}$ be a family of real numbers, called weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a frame of subspaces for H, if there exist $0 < C \leq D < \infty$ such that for all $h \in H$

$$C\|h\|^2 \leq \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 \leq D\|h\|^2, \tag{1.1}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

We call C and D the frame of subspaces bounds. The family $\{(W_i, v_i)\}_{i\in I}$ is called a c-tight frame of subspaces, if in (1.1) the constants C and D can be chosen so that $C = D$, a Parseval frame of subspaces provided $C = D = 1$ and an orthonormal frame of subspaces basis, if $H = \bigoplus_{i\in I} W_i$. If $\{(W_i, v_i)\}_{i\in I}$ possesses an upper frame of subspaces bound, but not necessarily a lower bound, we call it a Bessel frame of subspaces sequence with Bessel frame of subspaces bound D. The representation space employed in this setting is

$$(\sum_{i\in I} \oplus W_i)_{l_2} = \{\{f_i\}_{i\in I} | f_i \in W_i \quad \text{and} \quad \{||f_i||\}_{i\in I} \in l^2(I)\}.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a frame of subspaces for H. The synthesis operator, analysis operator and frame operator are defined, respectively, by

$$T_W : (\sum_{i\in I} \oplus W_i)_{l_2} \to H \quad \text{with} \quad T_W(f) = \sum_{i\in I} v_i f_i,$$

$$T_W^*: H \to (\sum_{i\in I} \oplus W_i)_{l_2} \quad \text{with} \quad T_W^*(f) = \{v_i \pi_{W_i}(f)\}_{i\in I},$$

$$S_W(f) = T_W T_W^* = \sum_{i\in I} v_i^2 \pi_{W_i}(f).$$

By Proposition 3.7 in [6], if $\{(W_i, v_i)\}_{i\in I}$ is a frame of subspaces for H with bounds C and D, then S_W is a positive and invertible operator on H with $CId \leq S_W \leq DId$. The theory of frames has a continuous version as follows.

Definition 1.3. Let (X, μ) be a measure space. Let $f : X \to H$ be weakly measurable, i.e., for all $h \in H$, the mapping $x \to< f(x), h >$ is measurable. Then f is called a continuous frame or c-frame for H, if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X | < f(x), h > |^2 d\mu \leq B\|h\|^2.$$

The representation space employed in this setting is

$$L^2(X, \mu) = \{\varphi : X \to H | \, \varphi \quad \text{is measurable and} \quad \|\varphi\|_2 < \infty\},$$

which $\|\varphi\|_2 = (\int_X ||\varphi(x)||^2 d\mu)^{\frac{1}{2}}$. The synthesis operator, analysis operator and frame operator are defined, respectively, by

$$T_f : L^2(X, \mu) \to H,$$

$$< T_f\varphi, h >= \int_X \varphi(x) < f(x), h > d\mu(x),$$

$$T_f^* : H \to L^2(X, \mu),$$

$$(T_f^* h)(x) =< h, f(x) >, \quad x \in X,$$

$$S_f = T_f T_f^*.$$

Also by Theorem 2.5. in [12], S_f is positive, self-adjoint and invertible.

We need the following theorems and the proofs can be found in [12].

Theorem 1.4. *Let f be a continuous frame for H with the frame operator S_f and let $V : H \to K$ be a bounded and invertible operator. Then $V \circ f$ is a continuous frame for K with the frame operator VS_fV^*.*

Theorem 1.5. *Let K be a closed subspace of H and let $P : H \to K$ be an orthogonal projection. Then the following hold:*

(i) *If f is a continuous frame for H with bounds A and B, then Pf is a continuous frame for K with the bounds A and B.*

(ii) *If f is a continuous frame for K with the frame operator S_f, then for each $h, k \in H$,*

$$< Ph, k >= \int_X < h, S_f^{-1} f(x) >< f(x), k > d\mu(x).$$

The following lemmas and theorems can be found in operator theory text books [11, 13–15] which we shall use them in the paper.

Lemma 1.6. *Let $u : H \to K$ be a bounded operator. Then*

(i) $\|u\| = \|u^*\|$ *and* $\|uu^*\| = \|u\|^2$.

(ii) R_u *is closed, if and only if* R_{u^*} *is closed.*

(iii) u *is surjective, if and only if there exists* $c > 0$ *such that for each* $h \in H$

$$c\|h\| \leq \|u^*(h)\|.$$

Lemma 1.7. *Let u be a self-adjoint bounded operator on H. Let*

$$m_u = \inf_{h \in H} < uh, h >, \quad M_u = \sup_{h \in H} < uh, h > .$$

Then, $m_u, M_u \in \sigma(u)$.

Theorem 1.8. *Let $u : K \to H$ be a bounded operator with closed range R_u. Then there exists a bounded operator $u^\dagger : H \to K$ for which $uu^\dagger f = f$, $f \in R_u$. Also, $u^* : H \to K$ has closed range and $(u^*)^\dagger = (u^\dagger)^*$. The operator $u^\dagger$ is called the pseudo-inverse of u.*

Theorem 1.9. *Let $u : K \to H$ be a bounded surjective operator. Given $y \in H$, the equation $ux = y$ has a unique solution of minimal norm, namely, $x = u^\dagger y$.*

2. *C*-Frame of subspaces

In this section we introduce the continuous version of frame of subspaces and we obtain some useful properties of it.

Definition 2.1. Let $F : X \to \mathbb{H}$ be such that for each $h \in H$, the mapping $x \mapsto \pi_{F(x)}(h)$ is measurable (i.e., is weakly measurable), and let $v : X \to [0, +\infty)$ be a measurable mapping such that $v \neq 0$ a.e.. We say that (F, v) is a c-frame of subspaces for H, if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu \leq B\|h\|^2. \tag{2.1}$$

(F, v) is called a tight c-frame of subspaces for H, if $A = B$, and Parseval if $A = B = 1$. If we only have the upper bound, we call (F, v) is a Bessel c-frame of subspaces mapping for H.

Definition 2.2. Let $F : X \to \mathbb{H}$. Let $L^2(X, H, F)$ be the class of all measurable mappings $f : X \to H$ such that for each $x \in X$, $f(x) \in F(x)$ and

$$\int_X \|f(x)\|^2 d\mu < \infty.$$

It can be verified that $L^2(X, H, F)$ is a Hilbert space with inner product defined by

$$< f, g >= \int_X < f(x), g(x) > d\mu,$$

for $f, g \in L^2(X, H, F)$.

Remark 2.3. For brevity, we denote $L^2(X,H,F)$ by $L^2(X,F)$. Let (F,v) be a Bessel c-frame of subspaces mapping, $f \in L^2(X,F)$ and $h \in H$. Then

$$\begin{aligned}
|\int_X v(x) < f(x), h > d\mu| &= |\int_X v(x) < \pi_{F(x)}(f(x)), h > d\mu| \\
&= |\int_X v(x) < f(x), \pi_{F(x)}(h) > d\mu| \leq \int_X v(x)\|f(x)\|.\|\pi_{F(x)}(h)\|d\mu \\
&\leq (\int_X \|f(x)\|^2 d\mu)^{1/2}(\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu)^{1/2} \\
&\leq B^{1/2}\|h\|(\int_X \|f(x)\|^2 d\mu)^{1/2}.
\end{aligned}$$

So we may define:

Definition 2.4. Let (F,v) be a Bessel c-frame of subspaces mapping for H. We define the c-frame of subspaces pre-frame operator (synthesis operator) $T_F : L^2(X,F) \to H$, by

$$< T_F(f), h >= \int_X v(x) < f(x), h > d\mu,$$

where $f \in L^2(X,F)$ and $h \in H$.

By Remark 2.3, $T_F : L^2(X,F) \to H$ is a bounded linear mapping. Its adjoint

$$T_F^* : H \to L^2(X,F),$$

will be called c-frame of subspaces analysis operator, and $S_F = T_F \circ T_F^*$ will be called c-frame of subspaces operator. The representation space in this setting is $L^2(X,F)$.

Remark 2.5. Let (F,v) be a Bessel c-frame of subspaces mapping for H. Then $T_F : L^2(X,F) \to H$ is indeed a vector-valued integral, which for $f \in L^2(X,F)$ we shall put

$$T_F(f) = \int_X v f d\mu,$$

where

$$< \int_X v f d\mu, h >=< \int_X v(x) < f(x), h > d\mu, h \in H.$$

For each $h \in H$ and $f \in L^2(X,F)$, we have

$$\begin{aligned}
< T_F^*(h), f > &=< h, T_F(f) > \\
&= \int_X v(x) < h, f(x) > d\mu \\
&= \int_X v(x) < \pi_{F(x)}(h), f(x) > d\mu \\
&=< v\pi_F(h), f > .
\end{aligned}$$

Hence for all $h \in H$,

$$T_F^*(h) = v\pi_F(h). \tag{2.2}$$

So $T_F^* = v\pi_F$.

Remark 2.6. A c-frame of subspaces is indeed a generalization of frame of subspaces. In Definition 2.1, if we put $X = I$ and μ be the counting measure, then F is a frame of subspaces according to Definition 1.2. Also with this hypothesis, $L^2(X, F)$ changes to $(\sum_{i\in I} \oplus W_i)_{l_2}$, the representation space of frame of subspaces.

Definition 2.7. Let (F, v) and (G, v) be Bessel c-frame of subspaces mappings for H. We say (F, v) and (G, v) are weakly equal, if $T_F^* = T_G^*$, which is equivalent to $v\pi_F(h) = v\pi_G(h)$, a.e. for all $h \in H$. Since, $v \neq 0$ a.e., (F, v) and (G, v) are weakly equal, if $\pi_F(h) = \pi_G(h)$, a.e. for all $h \in H$.

Remark 2.8. Let $T_F = 0$. Now, let $O : X \to \mathbb{H}$ be defined by $O(x) = \{0\}$, for almost all $x \in X$. Then (O, v) is a Bessel c-frame of subspaces mapping and $T_O = 0$. Let $h \in H$. Since $v\pi_F(h) \in L^2(X, F)$, so

$$\int_X v^2(x) < \pi_{F(x)}(h), \pi_{F(x)}(h) > d\mu = \int_X v(x) < v(x)\pi_{F(x)}(h), h > d\mu =< T_F(v\pi_F(h)), h >= 0.$$

Thus, $\pi_{F(x)}(h) = 0$, a.e. Therefore, $\pi_F(h) = \pi_O(h)$, a.e. Hence (F, v) and (O, v) are weakly equal.

Definition 2.9. For any Bessel c-frame of subspaces mapping (F, v) for H, we shall denote

$$A_{F,v} = \inf_{h\in H_1} \|v\pi_F(h)\|^2,$$

$$B_{F,v} = \sup_{h\in H_1} \|v\pi_F(h)\|^2 = \|v\pi_F\|^2.$$

Remark 2.10. Let (F, v) be a Bessel c-frame of subspaces mapping for H. Since, for each $h \in H$

$$< T_F T_F^*(h), h >= \|v\pi_F(h)\|^2 = \int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu,$$

$A_{F,v}$ and $B_{F,v}$ are optimal scalars which satisfies

$$A_{F,v} \leq T_F T_F^* \leq B_{F,v}.$$

In other words $A_{F,v}$ is the supremum of all positive numbers A, and $B_{F,v}$ is the infimum of all positive numbers B which satisfies in (2.1). So (F, v) is a c-frame of subspaces for H, if and only if $A_{F,v} > 0$.

Lemma 2.11. *Let (F, v) be a Parseval c-frame of subspaces for H. Then $T_F^* T_F$ is the orthogonal projection of $L^2(X, F)$ onto $T_F^*(H)$.*

Proof. By Remark 2.5 we have $T_F^*(h) = v\pi_F(h)$. Thus

$$||T_F^*(h)||^2 = ||v\pi_F(h)||^2 = \int_X ||v(x)\pi_{F(x)}(h)||^2 d\mu = \int_X v^2(x)||\pi_{F(x)}(h)||^2 d\mu = ||h||^2.$$

Thus T_F^* is an isometry. So we can embed H into $L^2(X, F)$ by identifying H with $T_F^*(H)$. Let $P : L^2(X, F) \to T_F^*(H)$ be the orthogonal projection. For each $f \in L^2(X, F)$ and $h \in H$ we have

$$< Pf, T_F^*(h) >=< f, PT_F^*(h) >=< f, T_F^*(h) >=< T_F(f), h >=< T_F^* T_F(f), T_F^*(h) > .$$

Thus

$$Pf - T_F^* T_F(f) \perp T_F^*(H).$$

But $ran(P) = T_F^*(H)$, hence $P = T_F^* T_F$. $\square$

Example 2.12. Let $X = [-1, 1]$ with the Lebesgue measure μ and let H_n be a n-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$. For each $x \in X$, let

$$F(x) = \Big\{ \lambda \sum_{i=0}^{n-1} x^i e_{i+1} : \lambda \in \mathbb{C} \Big\}, \quad \text{and} \quad \upsilon(x) = \sqrt{\sum_{i=1}^{n} x^{2i-2}}.$$

Then $F : X \to \mathbb{H}$ and (F, υ) is a c-frame of subspaces for H. For the proof of the example we refer the reader to [12].

3. Main result

Theorem 3.1. *Let (F, υ) be a c-frame of subspaces for H with bounds C and D, and let $Y \subseteq X$ be measurable. Then the following assertions are satisfied:*

(i) *If $\int_Y \upsilon^2(x)d\mu > D$, then $\cap_{x\in Y} F(x) = \{0\}$.*

(ii) *If $\int_Y \upsilon^2(x)d\mu = D$, then $\cap_{x\in Y} F(x) \perp span\{F(x)\}_{x\in X-Y}$ a.e.*

(iii) *If $c = \int_Y \upsilon^2(x)d\mu < C$, then $F : X - Y \to \mathbb{H}$ is a c-frame of subspaces with bounds $C - c$ and D.*

Proof. (i) Suppose $h \in \cap_{x\in Y} F(x)$, then $\pi_{F(x)}(h) = h$ for all $x \in Y$. We have

$$\begin{aligned} D\|h\|^2 < \|h\|^2 (\int_Y \upsilon^2(x)d\mu) &= \int_Y \|h\|^2 \upsilon^2(x)d\mu \\ &= \int_Y \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \\ &\le \int_Y \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \\ &= \int_X \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \le D\|h\|^2, \end{aligned}$$

hence $h = 0$.

(ii) If $\int_Y \upsilon^2(x)d\mu = D$ and $h \in \cap_{x\in Y} F(x)$, then

$$\begin{aligned} D\|h\|^2 &= \int_Y \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \\ &\le \int_Y \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \\ &\le D\|h\|^2, \end{aligned}$$

thus

$$D\|h\|^2 + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \le D\|h\|^2.$$

Hence

$$\int_{X-Y} \|\pi_{F(x)}(h)\|^2 \upsilon^2(x)d\mu \le 0.$$

Therefore we have $\pi_{F(x)}(h) = 0$, for all $x \in X - Y$ (a.e.) and we conclude that $h \perp F(x)$ for all $x \in X - Y$ (a.e.). Thus $h \perp span\{F(x)\}_{x\in X-Y}$ (a.e.) and we get

$$\cap_{x\in Y} F(x) \perp span\{F(x)\}_{x\in X-Y}.$$

(iii) For all $h \in H$ we have

$$\int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu = \int_X \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu - \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$
$$\geq C\|h\|^2 - \|h\|^2 \int_Y v^2(x) d\mu = (C-c)\|h\|^2.$$

The upper bound is obvious. □

The following corollary immediately follows from Theorem 3.1.

Corollary 3.2. *Let (F, v) be a c-frame of subspaces for H with bounds C and D, and let $Y \subseteq X$ be measurable. Then the following statements are equivalent:*

(i) $c = \int_Y v^2(x) d\mu < C$.

(ii) $F : (X - Y) \to \mathbb{H}$ *is a c-frame of subspaces with bounds $C - c$ and D.*

Definition 3.3. Let $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. for each $x \in X$, suppose that $\Lambda_x \in B(F(x), K_x)$ and put

$$\Lambda = \{\Lambda_x \in B(F(x), K_x) : x \in X\}.$$

Then (Λ, F, v) is a gc-frame of subspaces for H, if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X v^2(x) \|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \leq B\|h\|^2,$$

where $\pi_{F(x)}$ is the orthogonal projection onto the subspace $F(x)$.

(Λ, F, v) is called a tight gc-frame of subspaces for H, if A, B can be chosen so that $A = B$, and parseval, if $A = B = 1$. If we only have the upper bound, we call (Λ, F, v) is a Bessel gc-frame of subspaces mapping for H.

Let $K = \oplus_{x \in X} K_x$ and $L^2(X, K)$ be a collection of all measurable functions $\varphi : X \longrightarrow K$ such that for each $x \in X$, $\varphi(x) \in K_x$ and

$$\int_X ||\varphi(x)||^2 d\mu < \infty.$$

It can be verified that $L^2(X, K)$ is a Hilbert space with inner product defined by

$$\langle \varphi, \gamma \rangle = \int_X \langle \varphi(x), \gamma(x) \rangle d\mu,$$

for $\varphi, \gamma \in L^2(X, K)$ and the representation space in this setting is $L^2(X, K)$.

Remark 3.4. Let (Λ, F, v) be a Bessel gc-frame of subspaces mapping with Bessel bound B, $\varphi \in L^2(X, K)$ and $h \in H$. Then

$$|\int_X v(x) \langle \Lambda_x^*(\varphi(x)), h \rangle d\mu| = |\int_X v(x) \langle \Lambda_x^*(\varphi(x)), \pi_{F(x)}(h) \rangle d\mu|$$
$$= |\int_X v(x) \langle \varphi(x), \Lambda_x(\pi_{F(x)}(h)) \rangle d\mu|$$

$$\begin{aligned}
&\leq \int_X v(x)\|\varphi(x)\|.\|\Lambda_x(\pi_{F(x)}(h))\|d\mu \\
&\leq (\int_X \|\varphi(x)\|^2 d\mu)^{1/2}(\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu)^{1/2} \\
&\leq B^{1/2}\|h\|(\int_X \|\varphi(x)\|^2 d\mu)^{1/2}.
\end{aligned}$$

So we may define the following:

Definition 3.5. Let (Λ, F, v) be a Bessel gc-frame of subspaces mapping for H. We define the gc-pre-frame of subspaces operator (synthesis operator) $T_{gf} : L^2(X,K) \to H$, by

$$\langle T_{gf}(\varphi), h\rangle = \int_X v(x)\langle \Lambda_x^*(\varphi(x)), h\rangle d\mu,$$

where $\varphi \in L^2(X,K)$ and $h \in H$. It is obvious that T_{gf} is linear and by Remark 3.4 T_{gf} is a bounded linear mapping. Its adjoint

$$T_{gf}^* : H \to L^2(X,K),$$

will be called gc-frame of subspaces analysis operator, and $S_{gf} = T_{gf} \circ T_{gf}^*$ will be called gc-frame of subspaces operator. For each $h \in H$ and $\varphi \in L^2(X,K)$, we have

$$\begin{aligned}
\langle T_{gf}^*(h), \varphi\rangle &= \langle h, T_{gf}(f)\rangle \\
&= \int_X v(x)\langle h, \Lambda_x^*(\varphi(x))\rangle d\mu \\
&= \int_X v(x)\langle \pi_{F(x)}(h), \Lambda_x^*(\varphi(x))\rangle d\mu \\
&= \int_X v(x)\langle \Lambda_x(\pi_{F(x)}(h)), \varphi(x)\rangle d\mu \\
&= \langle v\Lambda_.\pi_{F_.}(h), f\rangle.
\end{aligned}$$

Hence for each $h \in H$,

$$T_{gf}^* = v\Lambda_.\pi_{F_.}.$$

Definition 3.6. For each Bessel gc-frame of subspaces mapping (F, v) for H, we denote

$$A_{\Lambda,v} = \inf_{h\in H_1} \|v\Lambda_.\pi_{F_.}(h)\|^2,$$

$$B_{\Lambda,v} = \sup_{h\in H_1} \|v\Lambda_.\pi_{F_.}(h)\|^2 = \|v\Lambda_.\pi_{F_.}\|^2.$$

Remark 3.7. Let (Λ, F, v) be a Bessel gc-frame of subspaces mapping for H. Since, for each $h \in H$

$$\langle T_{gf}T_{gf}^*(h), h\rangle = \|v\Lambda_.\pi_{F_.}(h)\|^2,$$

$A_{\Lambda,v}$ and $B_{\Lambda,v}$ are optimal scalars which satisfies

$$A_{\Lambda,v} \leq T_{gf}T_{gf}^* \leq B_{\Lambda,v}.$$

So (Λ, F, v) is a gc-frame of subspaces for H, if and only if $A_{\Lambda,v} > 0$.

Proposition 3.8. *The following conditions are equivalent.*

(i) (Λ, F, v) *is a gc-frame of subspaces for* H *with bounds* C *and* D.

(ii) $CId \leq S_{gf} \leq DId$.

Moreover, the optimal bounds are $\|S_{gf}\|$ *and* $\|S_{gf}^{-1}\|^{-1}$.

Proof. (i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (i), let T_{gf}^* denote the analysis operator of (Λ, F, v). Since $S_{gf} = T_{gf}T_{gf}^*$ and then $\|T_{gf}\|^2 = \|S_{gf}\|$, for each $h \in H$, we have

$$\int_X v^2 \|\Lambda_x(\pi_{F(x)}(h))\|^2 \, d\mu = \|T_{gf}^*(h)\|^2 \leq \|T_{gf}^*\|^2 \|h\|^2 = \|S_{gf}\| \|h\|^2 \leq D\|h\|^2.$$

Also for all $h \in H$,

$$\|T_{gf}^*(h)\|^2 = \left\langle T_{gf}T_{gf}^*(h), h \right\rangle = \left\langle S_{gf}h, h \right\rangle = \left\langle S_{gf}^{\frac{1}{2}}h, S_{gf}^{\frac{1}{2}}h \right\rangle = \|S_{gf}^{\frac{1}{2}}h\|^2 \geq C\|h\|^2.$$

Also

$$\|S_{gf}\| = \sup_{h \in H_1} \left\langle S_{gf}(h), h \right\rangle = \sup_{h \in H_1} \|v\Lambda_{.}\pi_{F_{.}}(h)\|^2 = B_{\Lambda,v}.$$

So the optimal upper bound is $\|S_{gf}\|$. For the optimal lower bound, if C be the lower bound, we have

$$C\|h\|^2 \leq \left\langle S_{gf}^{1/2}(h), S_{gf}^{1/2}(h) \right\rangle \leq D\|h\|^2.$$

Now put $h = S_{gf}^{-1/2}(h)$. We have

$$C\|S_{gf}^{-1/2}(h)\|^2 \leq \left\langle h, h \right\rangle \leq D\|S_{gf}^{-1/2}(h)\|^2.$$

Thus

$$\|S_{gf}^{-1}\| = \sup_{h \in H_1} \|S_{gf}^{-1/2}(h)\|^2 \leq C^{-1}.$$

We conclude that $A_{\Lambda,v} \leq \|S_{gf}^{-1}\|^{-1}$. In other implication we have

$$\|h\| \leq \|S_{gf}^{-1/2}\| \|S_{gf}^{1/2}(h)\|.$$

Hence

$$\inf_{h \in H_1} \|S_{gf}^{1/2}(h)\|^2 \geq \inf_{h \in H_1} \|h\|^2 \|S_{gf}^{-1/2}\|^{-2} = \|S_{gf}^{-1}\|^{-1}$$

We conclude that $A_{\Lambda,v} \geq \|S_{gf}^{-1}\|^{-1}$. Finally $A_{\Lambda,v} = \|S_{gf}^{-1}\|^{-1}$. □

Corollary 3.9. S_{gf} *is a positive and invertible operator from* H *into* H.

Proof. From Proposition 3.8 it is obvious. □

Theorem 3.10. *Let* (Λ, F, v) *be a gc-frame of subspaces for* H *with bounds* C *and* D, *and let* $Y \subseteq X$ *be measurable. Then the following assertions are satisfied:*

(i) *If* $\int_Y v^2(x) d\mu > D$, *then* $\cap_{x \in Y} K_x = \{0\}$.

(ii) *If* $\int_Y v^2(x) d\mu = D$, *then* $\cap_{x \in Y} K_x \perp span\{K_x\}_{x \in X-Y}$ *a.e.* .

(iii) *If* $c = \int_Y v^2(x)d\mu < C$*, then* (Λ, F, v) *with* $F : X - Y \to \mathbb{H}$ *is a gc-frame of subspaces with bounds* $C - c$ *and* D*.*

Proof. (i) Suppose $h \in \cap_{x\in Y} K_x$, then $\Lambda_x(\pi_{F(x)}(h)) = h$ for all $x \in Y$. We have

$$\begin{aligned} D\|h\|^2 &< \|h\|^2 (\int_Y v^2(x)d\mu) \\ &= \int_Y \|h\|^2 v^2(x)d\mu \\ &= \int_Y \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \\ &\leq \int_Y \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu + \int_{X-Y} \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \\ &= \int_X \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \leq D\|h\|^2, \end{aligned}$$

hence $h = 0$.

(ii) If $\int_Y v^2(x)d\mu = D$ and $h \in \cap_{x\in Y} K_x$, then

$$\begin{aligned} D\|h\|^2 &= \int_Y \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \\ &\leq \int_Y \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu + \int_{X-Y} \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \\ &\leq D\|h\|^2. \end{aligned}$$

Thus

$$D\|h\|^2 + \int_{X-Y} \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \leq D\|h\|^2.$$

Hence

$$\int_{X-Y} \|\Lambda_x(\pi_{F(x)}(h))\|^2 v^2(x)d\mu \leq 0.$$

Therefore, we have $\Lambda_x(\pi_{F(x)}(h)) = 0$, for all $x \in X - Y$ (a.e.) and we conclude that $h \perp F(x)$ for all $x \in X - Y$ (a.e.) Thus $h \perp span\{F(x)\}_{x\in X-Y}$ (a.e.) and we get

$$\cap_{x\in Y} F(x) \perp span\{F(x)\}_{x\in X-Y}.$$

(iii) For all $h \in H$ we have

$$\begin{aligned} \int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x)d\mu &= \int_X \|\pi_{F(x)}(h)\|^2 v^2(x)d\mu - \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x)d\mu \\ &\geq C\|h\|^2 - \|h\|^2 \int_Y v^2(x)d\mu = (C - c)\|h\|^2. \end{aligned}$$

The upper bound is obvious. □

Definition 3.11. For any $X_\circ \subseteq X$ measurable, we define

$$D_{X_\circ} : L^2(X, K) \to L^2(X, K),$$

$$D_{X_\circ}(f)(x) = \begin{cases} f(x) & \text{if } x \in X_\circ, \\ 0 & \text{if } x \in X - X_\circ, \end{cases}$$

for all $f \in L^2(X, K)$.

Definition 3.12. Let (Λ, F, v) be a gc-frame of subspaces with the pre-frame operator T_{gf}. We define the c-erasure reconstruction error $\xi_1(gf)$ to be

$$\xi_1(gf) = \sup\{(\|T_{gf}D_{X_\circ}T^*_{gf}\| : X_\circ \subseteq X\},$$

which $X_\circ$ is measurable.

Theorem 3.13. *Let $v \in L^2(X)$ and let (Λ, F, v) be a Parseval gc-frame of subspaces with c-erasure reconstruction error $\xi_1(gf)$. Then $D_{X_\circ}(f) \in L^2(X, K)$ and*

$$\xi_1(F) \leq \|v\|_2.$$

Proof. Since

$$\|D_{X_\circ}\| = \sup\{\|D_{X_\circ}(f)\| : \|f\| = 1 \quad \text{and} \quad f \in L^2(X, F)\},$$
$$\|D_{X_\circ}(f)\|^2 = \int_X |D_{X_\circ}(f)(x)|^2 d\mu = \int_{X_\circ} |f(x)|^2 d\mu \leq \|f\|^2,$$

so

$$\|D_{X_\circ}\| \leq 1.$$

Choose $X_\circ \subseteq X$ measurable, and fix it. By Remark 2.5 we have

$$\begin{aligned}
\|T_{gf}D_{X_\circ}T^*_{gf}\| &= \sup_{h\in H_1} \|T_{gf}D_{X_\circ}T^*_{gf}(h)\| \\
&= \sup_{h\in H_1} \|T_{gf}D_{X_\circ}(v\Lambda_.\pi_{F_.}(h))\| \\
&= \sup_{h\in H_1} \sup_{k\in H_1} | < T_{gf}D_{X_\circ}(v\Lambda_.\pi_{F_.}(h)), k > | \\
&= \sup_{h\in H_1} \sup_{k\in H_1} |\int_{X_\circ} < v^2(x)\Lambda_x(\pi_{F(x)}(h)), k > d\mu| \\
&\leq \sup_{h\in H_1} (\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu)^{1/2} (\int_X v^2(x) d\mu)^{1/2} \\
&= \sup_{h\in H_1} \|h\| (\int_X v^2(x) d\mu)^{1/2} = \|v\|_2.
\end{aligned}$$

Since $X_\circ \subseteq X$ is arbitrary

$$\xi_1(gf) \leq \|v\|_2.$$

□

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