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On the solutions set of non-local Hilfer fractional orders of an Itô stochastic differential equation

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Abstract

In this paper, an issue of Hilfer fractional order of an Itô stochastic differential equation with two non-local conditions is considered, studying case is split into two problems, one of them gives its solution as a second-order stochastic process and the other gives its solution as a non-standard Brownian motion in the same space of continuous second order processes. The existence of the solutions of both problems will be studied, the maximal and minimal solution will be defined, the sufficient conditions for uniqueness and some continuous dependencies will be shown. For some examples of non-standard Brownian motion as a Brownian motion with drift, geometric Brownian motion, Brownian bridge, and integrated Brownian bridge see [M.-Y. Chen, Department of Finance National Chung Hsing University, (2013), 1–19], [B. Kafash, R. Lalehzari, A. Delavarkhalafi, E. Mahmoudi, MATCH Commum. Math. Comput. Chem., **71** (2014), 265–277], [B. Øksendal, Springer-Verlag, Heidelberg New York, (2000)] and [O. Posch, University of Hamburg, (2010)].

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1. Introduction

Stochastic problems have got attention of many researchers, in particular Ito's problems that are related to Brownian motion, and also simple stochastic problems in which fractional operators appear, or more complicated problems, see [1, 3–5, 9, 12, 15, 18–20, 26]. Let (Ω, G, P) be a probability space where Ω is a sample space, G is a σ-algebra of subsets of Ω and P is the probability measure (see [7, 15, 25, 27]). Let $I = [a, T]$, $a \geqslant 0$, and $\vartheta(t; w) = \vartheta(t)$, $t \in I$, $w \in \Omega$, considering the space of all mean square stochastic processes $L_2(\Omega)$, which is identified with the norm

$$\| \vartheta(.) \|_{L_2} = \sqrt{E(\vartheta(.; w))^2},$$

where the expectation E is recognized with $E(h_1) = \int_\Omega h_1(w) dP$. Let $C(I, L_2(\Omega))$ be the Banach space of

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all continuous maps from I into $L_2(\Omega)$ that is satisfying the condition $\sup_{t\in I} E\|\vartheta\|^2_{L_2} < \infty$, $t \in I$, where

$$\|\vartheta\|_C = \sup_{t\in I} E\|\vartheta\|^2_{L_2}.$$

Consider the fractional problem of Itô stochastic differential equation, whereas the Hilfer fractional operator is appeared

$$dß(t) = \mathcal{F}(t, D^{\alpha,\beta}_{a^+}\vartheta(t))dt + \mathcal{G}(t, ß(t))dW(t), \quad t \in (a, T], \tag{1.1}$$

with non-local random initial conditions

$$ß(a) + \sum_{k=1}^{m} \xi_k\, ß(\tau_k) = ß_a \tag{1.2}$$

and

$$\vartheta(a) + \sum_{k=1}^{m} \eta_k\, \vartheta(\tau_k) = I^{\gamma-1}_a \vartheta_a, \tag{1.3}$$

where $\alpha, \beta \in (0, 1], \beta \leqslant \alpha, \gamma = \alpha + \beta - \alpha\beta, D^{\alpha,\beta}_{a^+}$ is the generalized Riemann-Lioville fractional derivative operator of order α and type β provided by Hilfer in [14], $ß(t)$, $t \in [a, T]$ is non-standard Brownian motion, η_k's, ξ_k's, $k = 1, 2, \dots, m$ are some positive real numbers, and ϑ_a and $ß_a$ are second-order random variable.

2. Preliminaries

Here, let's present some basic definitions.

Definition 2.1 ([2, 23])**.** Let $\vartheta \in C(I, L_2(\Omega))$ and $\alpha, \beta \in (0, 1]$. The stochastic integral operator of order β is defined by

$$I^{\beta}_{a^+}\vartheta(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\vartheta(s)ds$$

and the stochastic fractional order derivative is defined by

$$D^{\alpha}_{a^+}\vartheta(t) = I^{1-\alpha}_{a^+}\frac{d\vartheta}{dt}.$$

Definition 2.2 ([14, 23])**.** The Hilfer fractional derivative operator of order $0 \leqslant \nu \leqslant 1$ and $0 < \mu < 1$ for a function ϑ can be defined as

$$D^{\nu,\mu}_{a^+}\vartheta(t) = I^{\nu(1-\mu)}_{a^+} D I^{(1-\nu)(1-\mu)}_{a^+}\vartheta(t).$$

For some features of stochastic fractional calculus, see [10, 13].

3. Integral representation

During the research, it was assumed that the following assumptions are fulfilled.

(i) The functions $\mathcal{F}(t, \vartheta(t)) : I \times L_2(\Omega) \to L_2(\Omega)$ and $\mathcal{G}(t, ß(t)) : I \times L_2(\Omega) \to L_2(\Omega)$ are measurable in $t \in I$ and continuous in the second argument *w*.p.1.

(ii) For all $t \in I$, $\exists\ \mu_i > 0$, $i = 1, 2$, and two bounded measurable functions $\nu_i(t) : I \to R$ such that $\nu_i = \sup_{t\in I} \|\nu_i(t)\|_{L_2}$ and

$$\| \mathcal{F}(t, \vartheta(t)) \|_{L_2} \leqslant \nu_1 + \mu_1 \| \vartheta(t) \|_{L_2} \quad \text{and} \quad \| \mathcal{G}(t, ß(t)) \|_{L_2} \leqslant \nu_2 + \mu_2 \| ß(t) \|_{L_2},$$

where

$$\nu = \max\{\nu_1, \nu_2\}, \quad \mu = \max\{\mu_1, \mu_2\}.$$

Now, consider the following lemma.

Lemma 3.1. *Consider*

$$D_{a^+}^{\alpha,\beta}\vartheta(t) = U(t). \tag{3.1}$$

Then, the solution of the fractional stochastic problem (3.1) *with the initial elementary condition* (1.3) *can be given through*

$$\vartheta(t) = \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds, \tag{3.2}$$

where $\eta = \left(1+\sum_{k=1}^{m}\eta_k\right)^{-1}$.

Proof. Operating with $I_{a^+}^{\alpha}$ in equation (3.1), we obtain

$$I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}\vartheta(t) = I_{a^+}^{\alpha}U(t).$$

Using the definition of $D_{a^+}^{\alpha,\beta}$,

$$I_{a^+}^{\alpha}I_{a^+}^{\beta(1-\alpha)}DI_{a^+}^{(1-\beta)(1-\alpha)}\vartheta(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds.$$

So

$$\vartheta(t) = \vartheta(a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds$$

and

$$\begin{aligned}\vartheta(a) &= I_{a^+}^{\gamma}\vartheta_a - \sum_{k=1}^{m}\eta_k\vartheta(a) - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\\ &= \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right),\end{aligned}$$

where $\eta = \left(1+\sum_{k=1}^{m}\eta_k\right)^{-1}$, then equation (3.2) is given. Conversely, let

$$\begin{aligned}\vartheta(t) &= \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds\\ &= \vartheta(a) + I_{a^+}^{\alpha}U(s).\end{aligned}$$

Hence

$$\begin{aligned}\vartheta(t)-\vartheta(a) &= I_{a^+}^{\alpha}U(t),\\ I_{a^+}D\vartheta(t) &= I_{a^+}^{\alpha}U(t),\\ \vartheta(t) &= I_{a^+}^{\alpha-1}U(t),\\ DI_{a^+}^{(1-\beta)(1-\alpha)}\vartheta(t) &= DI_{a^+}^{(1-\beta)(1-\alpha)}I_{a^+}^{\alpha-1}U(t),\\ DI_{a^+}^{(1-\beta)(1-\alpha)}\vartheta(t) &= I_{a^+}^{-\beta(1-\alpha)}U(t),\\ I_{a^+}^{\beta(1-\alpha)}DI_{a^+}^{(1-\beta)(1-\alpha)}\vartheta(t) &= U(t),\\ D_{a^+}^{\alpha,\beta}\vartheta(t) &= U(t).\end{aligned}$$

□

Now, consider the following lemma.

Lemma 3.2. *The solution of the stochastic fractional problem* (1.1)-(1.3) *can be given by*

$$\begin{aligned}\beta(t) = \xi\left(\beta_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s)\right)\\ + \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\beta(s))dW(s),\end{aligned} \tag{3.3}$$

where $U(t)$ *is defined in equation* (3.1) *and* $\xi = \left(1 + \sum_{k=1}^{m}\xi_k\right)^{-1}$.

Proof. Integrating equation (1.1), we obtain

$$\beta(t) = \beta(a) + \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\beta(s))dW(s).$$

Then

$$\begin{aligned}\sum_{k=1}^{m}\xi_k\beta(\tau_k) &= \sum_{k=1}^{m}\xi_k\beta(a) + \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds + \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s),\\ \beta_a - \beta(a) &= \sum_{k=1}^{m}\xi_k\beta(a) + \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds + \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s).\end{aligned}$$

So,

$$\beta(a) = \left(1 + \sum_{k=1}^{m}\xi_k\right)^{-1}\left(\beta_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s)\right).$$

Hence, equation (3.3) is given. Conversely, let

$$\begin{aligned}\beta(t) &= \xi\left(\beta_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s)\right) + \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\beta(s))dW(s)\\ &= \beta(a) + \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\beta(s))dW(s).\end{aligned}$$

Hence

$$\begin{aligned}\beta(t) - \beta(a) &= \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\beta(s))dW(s),\\ \int_a^t d\beta(t) &= I_{a^+}[\mathcal{F}(t,U(t))dt + \mathcal{G}(t,\beta(t))dW(t)],\\ d\beta(t) &= \mathcal{F}(t,U(t))dt + \mathcal{G}(t,\beta(t))dW(t).\end{aligned}$$

Thus, equations (3.2) and (3.3) are proved. Now, let's discuss the existence of their solutions in the defined space $C(I, L_2(\Omega))$. □

4. Solutions of the problem

Theorem 4.1. *Let the assumptions* (i)-(ii) *be satisfied and* $4(T-a)^{\alpha} < \Gamma(\alpha+1)$, *then the stochastic integral equation* (3.2) *has at least one solution* $\vartheta(t) \in C(I, L_2(\Omega))$.

Proof. Consider the set Q_{r_1}, such that $Q_{r_1} = \{\vartheta(t) \in C \text{ w.p.1. } : \|\vartheta\|_C \leqslant r_1\} \subset C$, where

$$r_1 \leqslant \frac{\sqrt{2}\|\vartheta_a\|_C A_1}{1-4A_2}, \quad A_1 = \frac{(T-a)^{\gamma-1}}{\Gamma(\gamma)}, \quad A_2 = \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)},$$

and $\vartheta(t)$ is any continuous second-order stochastic process. Now, define the mapping $F\vartheta(t)$, where

$$F\vartheta(t) = \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds.$$

Noting that $(a_1+a_2)^2 \leqslant 2a_1^2+2a_2^2$, we get $(a_1+a_2+a_3)^2 \leqslant 2a_1^2+4a_2^2+4a_3^2$ and so on. Let $X \in Q_{r_1}$, then

$$\begin{aligned} E(F\vartheta(t))^2 \leqslant{}& 2E\left[\frac{\eta\,\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right]_H^2 + 4E\left[\frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right]^2 \\ &+ 4E\left[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds\right]^2. \end{aligned}$$

Thus

$$\begin{aligned} \|F\vartheta(t)\|_{L_2} \leqslant{}& \sqrt{2}\left\|\frac{\eta\,\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right\|_{L_2} + 2\left\|\frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds\right\|_{L_2} \\ &+ 2\left\|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U(s)ds\right\|_{L_2}. \end{aligned}$$

So

$$\begin{aligned} \|F\vartheta\|_C &\leqslant \frac{\sqrt{2}\eta\|\vartheta_a\|_C(T-a)^{\gamma-1}}{\Gamma(\gamma)} + \frac{2\eta\sum_{k=1}^{m}\eta_k\|U\|_C(T-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{2\|U\|_C(T-a)^{\alpha}}{\Gamma(\alpha+1)} \\ &\leqslant \sqrt{2}\|\vartheta_a\|_C A_1 + 4\|U\|_C A_2 = r_1, \end{aligned}$$

where $\left(1+\eta\sum_{k=1}^{m}\eta_k\right) < 2$, and $\eta < 1$, thus

$$r_1 = \sqrt{2}\|\vartheta_a\|_C A_1 + 4\|U\|_C A_2 \leqslant \sqrt{2}\|\vartheta_a\|_C A_1 + 4r_1 A_2,$$

which implies that

$$r_1 \leqslant \frac{\sqrt{2}\|\vartheta_a\|_C A_1}{1-4A_2}.$$

That proves $F : Q_{r_1} \to Q_{r_1}$ also, the class $\{FQ_{r_1}\}$ is uniformly bounded on Q_{r_1}. Now, considering $t_1, t_2 \in [0,T]$ such that $|t_2-t_1| < \delta$, then

$$F\vartheta(t_2) - F\vartheta(t_1) = \eta\vartheta_a\left[\frac{(t_2-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t_1-a)^{\gamma-1}}{\Gamma(\gamma)}\right] + \frac{1}{\Gamma(\alpha)}\left[\int_a^{t_2}(t_2-s)^{\alpha-1}U(s)ds - \int_a^{t_1}(t_1-s)^{\alpha-1}U(s)ds\right]$$

$$= \eta\vartheta_a\left[\frac{(t_2-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t_1-a)^{\gamma-1}}{\Gamma(\gamma)}\right] + \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}U(s)ds + \int_a^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]U(s)ds\right].$$

So

$$||F\vartheta(t_2)-F\vartheta(t_1)||_{L_2} \leqslant \sqrt{2}\eta||\vartheta_a||_{L_2}\left|\frac{(t_2-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t_1-a)^{\gamma-1}}{\Gamma(\gamma)}\right| + \frac{\sqrt{2}||U(t)||_{L_2}}{\Gamma(\alpha)}\left|\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds + \int_a^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds\right|\right|_{L_2}.$$

Thus

$$\begin{aligned}||F\vartheta(t_2)-F\vartheta(t_1)||_C &\leqslant \sqrt{2}\eta||\vartheta_a||_C\left|\frac{(t_2-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t_1-a)^{\gamma-1}}{\Gamma(\gamma)}\right| \\ &\quad + \frac{\sqrt{2}||U||_C}{\Gamma(\alpha+1)}|(t_2-t_1)^{\alpha}-(t_2-t_1)^{\alpha}+(t_2-a)^{\alpha}-(t_1-a)^{\alpha}| \\ &\leqslant \sqrt{2}\eta||\vartheta_a||_C\left|\frac{(t_2-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t_1-a)^{\gamma-1}}{\Gamma(\gamma)}\right| + \frac{\sqrt{2}||U||_C}{\Gamma(\alpha+1)}|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}|.\end{aligned}$$

This proves the equi-continuity of the class $\{FQ_{r_1}\}$ on Q_{r_1}. Now, let $\vartheta_n \in Q_{r_1}$, $\vartheta_n \to \vartheta$ *w*.p.1 (see [7]).

$$\begin{aligned}\lim_{n\to\infty} F\vartheta_n(t) &= \lim_{n\to\infty}\left[\frac{\eta\,\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U_n(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}U_n(s)ds\right] \\ &= \frac{\eta\,\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}\lim_{n\to\infty}U_n(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\lim_{n\to\infty}U_n(s)ds \\ &= F\vartheta(t).\end{aligned}$$

This proves that $\{FQ_{r_1}\}$ is continuous. Therefore, the closure of $\{FQ_{r_1}\}$ is compact (see [7]). Thus, equation (3.2) has a solution $\vartheta \in C$ and the Hilfer stochastic fractional differential equation (3.1) with the nonlocal random condition (1.3) consequently has a solution $\vartheta \in C$. □

Now, consider the following theorem.

Theorem 4.2. *Let the assumptions* (i)-(ii) *be satisfied and* $\mu < \frac{1}{4\sqrt{2}\sqrt{T-a}}$, *then the problem* (3.3) *has at least one solution* $\beta(t) \in C$.

Proof. Consider a set Q_{r_2} such that $Q_{r_2} = \{\varpi(t) \in C$ *w*.p.1 $: ||\varpi||_C \leqslant r_2\} \subset C$, where

$$r_2 \leqslant \frac{\sqrt{2}||\beta_a||_C + 4[\nu+\mu r_1](T-a) + 4\sqrt{2}\nu\sqrt{T-a}}{1-4\sqrt{2}\mu\sqrt{T-a}}$$

and ϖ is any Brownian motion in the space of continuous second-order stochastic processes, whatever it is standard or nonstandard. Now, define the mapping $\aleph B(t)$, where

$$\aleph\beta(t) = \xi\left(\beta_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\beta(s))dW(s)\right)$$

$$+\int_a^t \mathcal{F}(s,U(s))ds+\int_a^t \mathcal{G}(s,ß(s))dW(s).$$

Let $B \in Q_{r_2}$, then

$$\|\aleph ß(t)\|_{L_2} \leqslant \xi\left(\sqrt{2}\|ß_a\|_C+2[\nu+\mu\|U(t)\|_{L_2}]\sum_{k=1}^{m}\xi_k(\tau_k-a)+2\sqrt{2}\sum_{k=1}^{m}\xi_k\sqrt{\int_a^{\tau_k}\|\mathcal{G}(s,ß(s))\|_{L_2}^2 ds}\right)$$
$$+4[\nu+\mu\|U(t)\|_{L_2}](t-a)+4\sqrt{\int_a^t\|\mathcal{G}(s,ß(s))\|_{L_2}^2 ds}).$$

So

$$\begin{aligned}\|\aleph ß\|_C &\leqslant \xi\left(\sqrt{2}\|ß_a\|_C+2[\nu+\mu\|U\|_C]\sum_{k=1}^{m}\xi_k(\tau_k-a)+2\sqrt{2}\sum_{k=1}^{m}\xi_k[\nu+\mu\|B\|_C]\sqrt{\tau_k-a}\right)\\ &\quad+4[\nu+\mu\|U\|_C](t-a)+4[\nu+\mu\|ß\|_C]\sqrt{t-a}\\ &\leqslant \sqrt{2}\|ß_a\|_C+4[\nu+\mu\|U\|_C](T-a)+4\sqrt{2}[\nu+\mu\|ß\|_C]\sqrt{T-a}=r_2,\end{aligned}$$

where

$$\begin{aligned}r_2&=\sqrt{2}\|ß_a\|_C+4[\nu+\mu\|U\|_C](T-a)+4\sqrt{2}[\nu+\mu\|ß\|_C]\sqrt{T-a}\\ &\leqslant \sqrt{2}\|ß_a\|_C+4[\nu+\mu r_1](T-a)+4\sqrt{2}[\nu+\mu r_2]\sqrt{T-a}.\end{aligned}$$

Then

$$r_2 \leqslant \frac{\sqrt{2}\|ß_a\|_C+4[\nu+\mu r_1](T-a)+4\sqrt{2}\nu\sqrt{T-a}}{1-4\sqrt{2}\mu\sqrt{T-a}},$$

which proves $\aleph : Q_{r_2} \to Q_{r_2}$, also, the class $\{\aleph Q_{r_2}\}$ is uniformly bounded on Q_{r_2}. Now, considering $t_1, t_2 \in [0,T]$ such that $|t_2-t_1|<\delta$, then

$$\aleph ß(t_2)-\aleph ß(t_1)=\int_{t_1}^{t_2}\mathcal{F}(s,U(s))ds+\int_{t_1}^{t_2}\mathcal{G}(s,ß(s))dW(s).$$

So

$$\begin{aligned}\|\aleph ß(t_2)-\aleph ß(t_1)\|_C &\leqslant \sqrt{2}[[\nu+\mu\|ß\|_C][t_2-t_1]+\sqrt{2}\sqrt{\int_{t_1}^{t_2}[\nu+\mu\|ß\|_C^2]ds}\\ &\leqslant \sqrt{2}[[\nu+\mu\|ß\|_C][t_2-t_1]+\sqrt{2}[\nu+\mu\|ß\|_C]\sqrt{(t_2-t_1)}\\ &\leqslant \sqrt{2}[\nu+\mu\|ß\|_C][t_2-t_1+\sqrt{(t_2-t_1)}].\end{aligned}$$

This leads to the equi-continuity of the class $\{\aleph Q_{r_2}\}$ on Q_{r_2}. Now, let $B_n \in Q_{r_2}$, $ß_n \to ß$ *w.p.1*,

$$\lim_{n\to\infty}\aleph ß_n(t)=\lim_{n\to\infty}\left[\xi\left(ß_a-\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds-\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß_n(s))dW(s)\right)\right.$$
$$\left.+\int_a^t\mathcal{F}(s,U(s))ds+\int_a^t\mathcal{G}(s,ß_n(s))dW(s)\right]$$

$$= \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,\lim_{n\to\infty}ß_n(s))dW(s)\right)$$
$$+\int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,\lim_{n\to\infty}ß_n(s))dW(s) = \aleph ß(t).$$

That is, $\{\aleph Q_{r_2}\}$ is continuous. Accordingly, the closure of $\{\aleph Q_{r_2}\}$ is compact (see [7]). Thus, equation (3.3) has a solution $ß(t) \in C$. Therefore, the stochastic fractional problem (1.1)-(1.3) has a solution $ß(t) \in C$. □

5. Maximal and minimal solution

In this section, consider the next definitions (see [11]).

Definition 5.1. Let $Y(t)$ be a solution of the stochastic integral equation (3.2), then $\vartheta(t)$ is said to be a maximal solution of (3.2) if every solution $Y(t)$ of (3.2) satisfies the inequality

$$E(Y^2(t)) < E(\vartheta^2(t)) \quad \text{or} \quad \|Y(t)\|_{L_2} < \|\vartheta(t)\|_{L_2}.$$

A minimal solution $s(t)$ can be defined by a similar way by reversing the above inequality, i.e.,

$$E(\vartheta^2(t)) > E(s^2(t)) \quad \text{or} \quad \|\vartheta(t)\|_{L_2} > \|s(t)\|_{L_2}.$$

Definition 5.2. Let $\varpi(t)$ be a solution of the stochastic integral equation (3.3), then $\omega(t)$ is said to be a maximal solution of (3.3) if every solution $\varpi(t)$ of (3.2) satisfies the inequality

$$E(\varpi^2(t)) < E(\omega^2(t)) \quad \text{or} \quad \|\varpi(t)\|_{L_2} < \|\omega(t)\|_{L_2}.$$

A minimal solution $\psi(t)$ can be defined by a similar way by reversing the above inequality, i.e.,

$$E(\omega^2(t)) > E(\psi^2(t)) \quad \text{or} \quad \|\omega(t)\|_{L_2} > \|\psi(t)\|_{L_2}.$$

Definition 5.3. The function $f(t,\varphi(t)) : [0,T]\times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically decreasing if for any $\varphi_1(t), \varphi_2(t) \in L_2(\Omega)$ satisfying $\| \varphi_1(t) \|_{L_2} > \| \varphi_2(t) \|_{L_2}$, implies that

$$\| f(t,\varphi_1(t)) \|_{L_2} < \| f(t,\varphi_2(t)) \|_{L_2}.$$

Definition 5.4. The function $f(t,\varphi(t)) : [0,T]\times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically increasing if for any $\varphi_1(t), \varphi_2(t) \in L_2(\Omega)$ satisfying $\| \varphi_1(t) \|_{L_2} > \| \varphi_2(t) \|_{L_2}$ implies that

$$\| f(t,\varphi_1(t)) \|_{L_2} > \| f(t,\varphi_2(t)) \|_{L_2}.$$

Now, the following theorems will be proved.

Theorem 5.5. *Let the assumptions of Theorem 4.1 be satisfied. If* $U(t)(= U(\vartheta(t))) = D^{\alpha,\beta}_{a^+}\vartheta(t)$ *satisfies the Definition 5.3, then there exists a maximal solution of the stochastic integral equation* (3.2).

Proof. Firstly, we prove the existence of the maximal solution of the stochastic integral equation (3.2). Let $\epsilon > 0$ be given. Now consider the integral equation

$$\vartheta_\epsilon(t) = \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}U_\epsilon(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U_\epsilon(s)ds, \tag{5.1}$$

and for $\epsilon > \epsilon_1 > \epsilon_2$,

$$U_{\epsilon_i}(t) = U_\epsilon(t) + \epsilon_i, \ i = 1,2, \quad 0 < \epsilon_2 < \epsilon_1.$$

Thus $U_\epsilon(t) < U_{\epsilon_1}(t) < U_{\epsilon_2}(t)$. Also, it is clear that the function $U_{\epsilon_i}(t)$, $i = 1,2$ satisfies the conditions (i)-(ii). Then equation (5.1) is a solution of problem (3.2) according to Theorem 4.1. Now

$$\vartheta_{\epsilon_1}(t) = \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}[U_\epsilon(s)+\epsilon_1]ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}[U_\epsilon(s)+\epsilon_1]ds,$$

so

$$\vartheta_{\epsilon_1}(t) + \epsilon_1\frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds - \epsilon_1\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds$$
$$= \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}U_\epsilon(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U_\epsilon(s)ds$$

and hence

$$\vartheta_{\epsilon_1}(t) + \epsilon_1\frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds + \epsilon_1\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds$$
$$\geqslant \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}U_\epsilon(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U_\epsilon(s)ds.$$

In the same way, but for the opposite inequality,

$$\vartheta_{\epsilon_2}(t) - \epsilon_2\frac{1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds - \epsilon_2\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds$$
$$\leqslant \eta\left(\frac{\vartheta_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}U_\epsilon(s)ds\right) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}U_\epsilon(s)ds.$$

This implies that, for $\epsilon_1 > \epsilon_2$,

$$\vartheta_{\epsilon_1}(t) + \frac{\epsilon_1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds + \frac{\epsilon_1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds$$
$$\geqslant \vartheta_{\epsilon_2}(t) - \frac{\epsilon_2}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds - \frac{\epsilon_2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds$$
$$\geqslant \vartheta_{\epsilon_2}(t) - \frac{\epsilon_1}{\Gamma(\alpha)}\eta\sum_{k=1}^{m}\eta_k\int_a^{\tau_k}(\tau_k - s)^{\alpha-1}ds - \frac{\epsilon_1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds, \quad \epsilon_1 > \epsilon_2.$$

Thus

$$\vartheta_{\epsilon_2}(t) \leqslant \vartheta_{\epsilon_1}(t) + \frac{2\epsilon_1}{\Gamma(\alpha)}\eta \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} ds + \frac{2\epsilon_1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds.$$

Thus, as $\epsilon_1 \to 0$, $\|\vartheta_{\epsilon_2}(t)\|_{L_2} \leqslant \|\vartheta_{\epsilon_1}(t)\|_{L_2}$ for $\epsilon_2 < \epsilon_1$. By a same way for $\epsilon > \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$,

$$\|\vartheta_{\epsilon_n}(t)\|_{L_2} \leqslant \|\vartheta_{\epsilon_{n-1}}(t)\|_{L_2} \leqslant \cdots \leqslant \|\vartheta_{\epsilon_1}(t)\|_{L_2} \leqslant \|\vartheta_{\epsilon}(t)\|_{L_2}.$$

Finally, Theorem 4.1 shows that the family of solutions $\vartheta_\epsilon(t)$ defined by equation (5.1) is uniformly bounded and equi-continuous functions. Hence, by Arzela-Ascoli theorem [21], there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \vartheta_{\epsilon_n}(t)$ exists uniformly in C and denote this limit by $\vartheta(t)$ from the continuity of the solutions ϑ_{ϵ_n} and by applying Lebesgue dominated convergence theorem,

$$\vartheta(t) = \lim_{n\to\infty} \vartheta_{\epsilon_n}(t),$$

which proves that $\vartheta(t)$ is a solution of the problem (3.2). Finally, we show that $\vartheta(t)$ is the maximal solution of the problem (3.2). Let $\tilde{\vartheta}(t)$ be any solution of problem (3.2) such that

$$\tilde{\vartheta}(t) = \eta \left(\frac{\vartheta_a (t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} V(s) ds \right) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} V(s) ds,$$

where $D^{\alpha,\beta}\tilde{\vartheta}(t) = V(t)$ and $V(t) = U_\epsilon(t) + \Delta$, such that $\Delta > 0$ is a real positive number. We can get

$$\| \vartheta_\epsilon(t) \|_{L_2} > \| \tilde{\vartheta}(t) \|_{L_2} .$$

From the uniqueness of the maximal solution (see [8]), it is clear that $\vartheta_\epsilon(t)$ tends to $\vartheta(t)$ uniformly as $\epsilon \to 0$. This finishes the proof. □

The minimal solution of the problem (3.2) can be defined in the same fashion as done above, it means that for $\epsilon > \epsilon_1 > \epsilon_2$,

$$U_{\epsilon_i}(t) = U_\epsilon(t) - \epsilon_i, \ i = 1,2, \quad 0 < \epsilon_2 < \epsilon_1.$$

Thus $U_\epsilon(t) > U_{\epsilon_1}(t) > U_{\epsilon_2}(t)$. Also, it is clear that the function $U_{\epsilon_i}(t)$, $i = 1,2$ satisfies the conditions (i)-(ii), i.e., the function $U(t)$ is assumed to satisfy the Definition 5.4. Now, for the problem (3.3), the non-standard Brownian motion can be shown as a maximal solution.

Theorem 5.6. *Let the assumptions of Theorem 4.2 be satisfied. If $\mathcal{G}(t, ß(t))$ satisfies the Definition 5.3, then there exists a maximal solution of problem (3.3).*

Proof. Firstly, for proving the existence of the maximal solution of the problem, let $\epsilon > 0$ be given, consider the integral equation

$$\begin{aligned} ß_\epsilon(t) = \xi \left(ß_a - \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{F}(s, U(s)) ds - \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{G}(s, ß_\epsilon(s)) dW(s) \right) \\ + \int_a^t \mathcal{F}(s, U(s)) ds + \int_a^t \mathcal{G}(s, ß_\epsilon(s)) dW(s) \end{aligned} \tag{5.2}$$

and for $\epsilon > \epsilon_1 > \epsilon_2$,

$$\mathcal{G}(t, ß_{\epsilon_i}(t)) = \mathcal{G}(t, ß_\epsilon(t)) + \epsilon_i, \ i = 1,2, \quad 0 < \epsilon_2 < \epsilon_1.$$

Clearly, the functions $\mathcal{G}(t, ß_{\epsilon_i}(t))$, $i = 1, 2$, satisfy the conditions (i)-(ii). Then equation (5.2) is a solution of problem (3.3) according to Theorem 4.2. Now,

$$ß_{\epsilon_1}(t) = \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s, U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}[\mathcal{G}(s, ß_\epsilon(s)) - \epsilon_1]dW(s)\right) + \int_a^t\mathcal{F}(s, U(s))ds + \int_a^t[\mathcal{G}(s, ß_\epsilon(s)) - \epsilon_1]dW(s),$$

so

$$ß_{\epsilon_1}(t) + \epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] - \epsilon_1[W(t) - W(a)] = \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s, U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s, ß_\epsilon(s))dW(s)\right) + \int_a^t\mathcal{F}(s, U(s))ds + \int_a^t\mathcal{G}(s, ß_\epsilon(s))dW(s),$$

and hence,

$$ß_{\epsilon_1}(t) + \epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] + \epsilon_1[W(t) - W(a)] \geqslant \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s, U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s, ß_\epsilon(s))dW(s)\right) + \int_a^t\mathcal{F}(s, U(s))ds + \int_a^t\mathcal{G}(s, ß_\epsilon(s))dW(s).$$

In the same way, but for the opposite inequality,

$$ß_{\epsilon_2}(t) - \epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] - \epsilon_1[W(t) - W(a)] \leqslant \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s, U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s, ß_\epsilon(s))dW(s)\right) + \int_a^t\mathcal{F}(s, U(s))ds + \int_a^t\mathcal{G}(s, ß_\epsilon(s))dW(s).$$

This implies that, for $\epsilon_1 > \epsilon_2$,

$$\begin{aligned}ß_{\epsilon_1}(t) &+ \epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] + \epsilon_1[W(t) - W(a)]\\ &\geqslant ß_{\epsilon_2}(t) - \epsilon_2\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] - \epsilon_2[W(t) - W(a)]\\ &\geqslant ß_{\epsilon_2}(t) - \epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] - \epsilon_1[W(t) - W(a)], \quad -\epsilon_2 > -\epsilon_1.\end{aligned}$$

Thus

$$ß_{\epsilon_2}(t) \leqslant ß_{\epsilon_1}(t) + 2\epsilon_1\xi\sum_{k=1}^{m}\xi_k[W(\tau_k) - W(a)] + 2\epsilon_1[W(t) - W(a)].$$

Thus, as $\epsilon_1 \to 0$, $||ß_{\epsilon_2}(t)||_{L_2} \leqslant ||ß_{\epsilon_1}(t)||_{L_2}$ for $\epsilon_2 < \epsilon_1$. By a same way for $\epsilon > \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$,

$$||ß_{\epsilon_n}(t)||_{L_2} \leqslant ||ß_{\epsilon_{n-1}}(t)||_{L_2} \leqslant \cdots \leqslant ||ß_{\epsilon_1}(t)||_{L_2} \leqslant ||ß_\epsilon(t)||_{L_2}.$$

Finally, as shown before in the proof of Theorem 4.1, the family of functions $\text{ß}_\epsilon(t)$ defined by equation (5.2) is uniformly bounded and equi-continuous functions. Hence, by Arzela-Ascoli theorem [21], there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \text{ß}_{\epsilon_n}(t)$ exists uniformly in C and denote this limit by $B(t)$. By the continuity of the function ß_{ϵ_n} in the second argument and applying Lebesgue dominated convergence theorem, we get

$$\text{ß}(t) = \lim_{n\to\infty} \text{ß}_{\epsilon_n}(t),$$

which proves that $\text{ß}(t)$ is a solution of the problem (3.3). Finally, we shall show that $\text{ß}(t)$ is the maximal solution of problem (3.3). To do this, let $\tilde{\text{ß}}(t)$ be any solution of problem (3.3) such that

$$\tilde{\text{ß}}(t) = \xi \left(\text{ß}_a - \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{F}(s, U(s))ds - \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{G}(s, \tilde{\text{ß}}(s))dW(s) \right)$$
$$+ \int_a^t \mathcal{F}(s, U(s))ds + \int_a^t \mathcal{G}(s, \tilde{\text{ß}}(s))dW(s),$$

where $\mathcal{G}(s, \tilde{\text{ß}}(s)) = \mathcal{G}(s, \text{ß}(s)) + \Delta_1$, such that $\Delta_1 > 0$ is a real positive number. We get

$$\| \text{ß}_\epsilon(t) \|_{L_2} > \| \tilde{\text{ß}}(t) \|_{L_2}$$

From the uniqueness of the maximal solution (see [8]), it is clear that $\text{ß}_\epsilon(t)$ tends to $\text{ß}(t)$ uniformly as $\epsilon \to 0$. This finishes the proof. □

The minimal solution of the problem (3.3) can be defined in the same fashion as done above, it means that for $\epsilon > \epsilon_1 > \epsilon_2$,

$$\mathcal{G}(t, \text{ß}_{\epsilon_i}(t)) = \mathcal{G}(t, \text{ß}_\epsilon(t)) - \epsilon_i, \ i = 1, 2, \quad 0 < \epsilon_2 < \epsilon_1.$$

Thus $U_\epsilon(t) > U_{\epsilon_1}(t) > U_{\epsilon_2}(t)$. Also, it is clear that the functions $U_{\epsilon_i}(t)$, $i = 1, 2$ satisfy the conditions (i)-(ii), i.e., the function $\mathcal{G}(t, \text{ß}(t))$ is assumed to satisfy the Definition 5.4.

6. Uniqueness theorem

For discussing the uniqueness of the solution, consider the following assumption.

(iii) The functions $\mathcal{F}(t, \vartheta(t)) : I \times L_2(\Omega) \to L_2(\Omega)$ and $\mathcal{G}(t, \vartheta(t)) : I \times L_2(\Omega) \to L_2(\Omega)$ are Caratheodory and satisfy the second argument, Lipschitz condition

$$\| \mathcal{F}(t, \vartheta(t)) - \mathcal{F}(t, Y(t) \|_{L_2} \leqslant \mu_1 \| \vartheta(t) - Y(t) \|_{L_2}, \ \ \mu_1 > 0,$$
$$\| \mathcal{G}(t, \vartheta(t)) - \mathcal{G}(t, Y(t) \|_{L_2} \leqslant \mu_2 \| \vartheta(t) - Y(t) \|_{L_2}, \ \ \mu_2 > 0,$$

where $\mu = \max\{\mu_1, \mu_2\}$.

Theorem 6.1. *Let assumption* (iii) *be satisfied, then the stochastic integral equation* (3.2) *has a unique solution* $\vartheta(t) \in C(I, L_2(\Omega))$.

Proof. Let $\vartheta_1(t)$ and $\vartheta_2(t)$ be two solutions of (3.2), then

$$\|\vartheta_1 - \vartheta_2\|_C \leqslant \frac{\sqrt{2}}{\Gamma(\alpha)} \left(1 + \eta \sum_{k=1}^{m} \eta_k\right) [\mu(T-a)I^{-\alpha}\|\vartheta_1 - \vartheta_2\|_C] \leqslant \frac{2\sqrt{2}}{\Gamma(\alpha)} [\mu(T-a)I^{-\alpha}\|\vartheta_1 - \vartheta_2\|_C].$$

Thus

$$\|\vartheta_1 - \vartheta_2\|_C = 0.$$

Then the solution of (3.2) is unique. Consequently, the solution $\vartheta(t)$ of the problem (1.3) with the nonlocal initial condition (1.2) is unique. □

Theorem 6.2. *Let the assumption* (iii) *be satisfied, then the stochastic integral equation* (3.3) *has a unique solution* $ß(t) \in C(I, L_2(\Omega))$.

Proof. Let $ß_1(t)$ and $ß_2(t)$ be two solutions of (3.3), then

$$ß_1(t) - ß_2(t) = -\xi \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} [\mathcal{G}(s, ß_1(s)) - \mathcal{G}(s, ß_2(s))] dW(s) + \int_a^t [\mathcal{G}(s, ß_1(s)) - \mathcal{G}(s, ß_2(s))] dW(s).$$

So,

$$\|ß_1 - ß_2\|_C \leqslant \sqrt{2}\left(1 + \xi \sum_{k=1}^{m} \xi_k\right) \mu \|ß_1 - ß_2\|_C \sqrt{T-a} \leqslant 2\sqrt{2}\mu \|ß_1 - ß_2\|_C \sqrt{T-a}.$$

Hence

$$\|ß_1 - ß_2\|_C = 0.$$

Then the solution of (3.3) is unique. Consequently, the solution $ß(t)$ of the problem (1.1)-(1.3) is unique. □

7. Continuous dependence

Firstly, we discuss the continuous dependence of the solution of the stochastic integral equation (3.2) on ϑ_a and η_k.

Theorem 7.1. *The unique solution of the stochastic integral equation* (3.2) *depends continuously on* ϑ_a.

Proof. Let $\| \vartheta_a - \vartheta^*_a \|_{L_2} \leqslant \delta_1$, $\vartheta(t)$ is the solution of (3.2) and $\vartheta^*(t)$ be the solution of

$$\vartheta^*(t) = \eta\left(\frac{\vartheta^*_a (t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds\right) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} U^*(s) ds.$$

So

$$\begin{aligned}
\|\vartheta - \vartheta^*\|_C &\leqslant \eta\left(\sqrt{2}\frac{\|\vartheta_a - \vartheta^*_a\|_C (t-a)^{\gamma-1}}{\Gamma(\gamma)} + 2\|\vartheta - \vartheta^*\|_C \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} I^{-\alpha}(1) ds\right) \\
&\quad + 2\|\vartheta - \vartheta^*\|_C \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} I^{-\alpha}(1) ds \\
&\leqslant \eta\left(\sqrt{2}\delta_1 A_1 + 2\|\vartheta - \vartheta^*\|_C \sum_{k=1}^{m} \eta_k \frac{(T-a)^{\alpha}(T-a)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\right) + 2\|\vartheta - \vartheta^*\|_C \frac{(T-a)^{\alpha}(T-a)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \\
&\leqslant \sqrt{2}\delta_1 A_1 \eta + \|\vartheta - \vartheta^*\|_C \frac{2\left(1 + \eta \sum_{k=1}^{m} \eta_k\right)}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \leqslant \sqrt{2}\delta_1 A_1 \eta + \|\vartheta - \vartheta^*\|_C \frac{4}{\alpha\pi \csc(\alpha\pi)}.
\end{aligned}$$

Thus

$$\|\vartheta - \vartheta^*\|_C \leqslant \left(\frac{\sqrt{2}A_1\eta}{1 - \frac{4}{\alpha\pi\csc(\alpha\pi)}}\right)\delta_1,$$

where $1 \geqslant \frac{4}{\alpha\pi\csc(\alpha\pi)}$. This finishes the proof. □

Theorem 7.2. *The unique solution of the stochastic integral equation* (3.2) *depends continuously on* η_k.

Proof. Let $|\eta_k - \eta_k^*| \leqslant \delta_2$, $\vartheta(t)$ be the solution of (3.2), and $\vartheta^*(t)$ be the solution of

$$\vartheta^*(t) = \eta^* \left(\frac{\vartheta_a (t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k^* \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds \right) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} U^*(s) ds.$$

Now,

$$\vartheta(t) - \vartheta^*(t) = [\eta - \eta^*] \frac{\vartheta_a (t-a)^{\gamma-1}}{\Gamma(\gamma)} - \eta \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U(s) ds$$
$$+ \eta^* \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \eta_k^* \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} [U(s) - U^*(s)] ds.$$

Since

$$1 + \sum_{k=1}^{m} \eta_k \geqslant 1 \quad \to \quad \frac{1}{1 + \sum_{k=1}^{m} \eta_k} \leqslant 1,$$

so

$$|\eta - \eta^*| = \left| \frac{1}{1 + \sum_{k=1}^{m} \eta_k} - \frac{1}{1 + \sum_{k=1}^{m} \eta_k^*} \right| = \left| \frac{\sum_{k=1}^{m} (\eta_k^* - \eta_k)}{\left(1 + \sum_{k=1}^{m} \eta_k\right)\left(1 + \sum_{k=1}^{m} \eta_k^*\right)} \right| \leqslant \left| \sum_{k=1}^{m} (\eta_k^* - \eta_k) \right| \leqslant m\delta_2,$$

also

$$\left| \eta \sum_{k=1}^{m} \eta_k - \eta^* \sum_{k=1}^{m} \eta_k^* \right| = \left| \frac{\sum_{k=1}^{m} \eta_k}{1 + \sum_{k=1}^{m} \eta_k} - \frac{\sum_{k=1}^{m} \eta_k^*}{1 + \sum_{k=1}^{m} \eta_k^*} \right|$$
$$= \left| \frac{\sum_{k=1}^{m} (\eta_k^* - \eta_k)}{\left(1 + \sum_{k=1}^{m} \eta_k\right)\left(1 + \sum_{k=1}^{m} \eta_k^*\right)} \right| \leqslant \left| \sum_{k=1}^{m} (\eta_k^* - \eta_k) \right| \leqslant m\delta_2,$$

and

$$\eta^* \sum_{k=1}^{m} \eta_k^* \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds - \eta \sum_{k=1}^{m} \eta_k \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U(s) ds$$
$$= \eta^* \left(1 + \sum_{k=1}^{m} \eta_k^*\right) \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds - \eta \left(1 + \sum_{k=1}^{m} \eta_k\right) \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U(s) ds$$
$$- \eta^* \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds + \eta \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U(s) ds$$
$$= \eta^* \eta^{*-1} \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U^*(s) ds - \eta \eta^{-1} \int_a^{\tau_k} (\tau_k - s)^{\alpha-1} U(s) ds$$

$$-\eta^*\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U^*(s)ds+\eta\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds$$
$$=-\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}[U(s)-U^*(s)]ds+\eta\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds$$
$$-\eta^*\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U^*(s)ds-\eta^*\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds+\eta^*\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds$$
$$=-\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}[U(s)-U^*(s)]ds+[\eta-\eta^*]\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}U(s)ds+\eta^*\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}[U(s)-U^*(s)]ds.$$

Then

$$\|\vartheta-\vartheta^*\|_C\leqslant\sqrt{2}\frac{|\eta-\eta^*|\|\vartheta_a\|_{L_2}(t-a)^{\gamma-1}}{\Gamma(\gamma)}+2\frac{|\eta-\eta^*|}{\Gamma(\alpha)}\|U\|_C\int_a^{\tau_k}(\tau_k-s)^{\alpha-1}ds$$
$$+\frac{2}{\Gamma(\alpha)}\|U-U^*\|_C\,[|\eta^*|+2]\int_a^{t}(t-s)^{\alpha-1}ds$$
$$\leqslant\sqrt{2}\delta_2\|\vartheta_a\|_CA_1+2\delta_2\|U\|_CA_2+\frac{2\,[|\eta^*|+2]}{\pi\csc(\alpha\pi)}\|\vartheta-\vartheta^*\|_C.$$

Hence

$$\|\vartheta-\vartheta^*\|_C\leqslant\left(\frac{2\|\vartheta_a\|_CA_1+4\|U\|_CA_2}{1-\frac{2[|\eta^*|+2]}{\pi\csc(\alpha\pi)}}\right)\delta_2,$$

where $1\geqslant\frac{2[|\eta^*|+2]}{\pi\csc(\alpha\pi)}$. This finishes the proof. □

Now, we consider some continuous dependencies of the solution of equation (3.3) on $ß_a$, $W(t)$, ξ_k, and also on $U(t)$.

Theorem 7.3. *The unique solution of the stochastic integral equation* (3.3) *depends continuously on* $ß_a$.

Proof. Let $\|ß_a-ß_a^*\|_{L_2}\leqslant\delta_3$, $ß(t)$ be the solution of (3.3) and $ß^*(t)$ be the solution of

$$ß^*(t)=\xi\left(ß_a^*-\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds-\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW(s)\right)$$
$$+\int_a^{t}\mathcal{F}(s,U(s))ds+\int_a^{t}\mathcal{G}(s,ß^*(s))dW(s).$$

Thus

$$ß(t)-ß^*(t)=\xi\left([ß_a-ß_a^*]-\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}[\mathcal{G}(s,ß(s))-\mathcal{G}(s,ß^*(s))]dW(s)\right)+\int_a^{t}[\mathcal{G}(s,ß(s))-\mathcal{G}(s,ß^*(s))]dW(s).$$

So

$$\|ß - ß^*\|_C \leqslant \sqrt{2}\xi\delta_3 + 2\xi\sum_{k=1}^{m}\xi_k(\tau_k - a)\mu\|ß - ß^*\|_C + 2(t-a)\mu\|ß - ß^*\|_C \leqslant \sqrt{2}\xi\delta_3 + 4(T-a)\mu\|ß - ß^*\|_C.$$

Then

$$\|ß - ß^*\|_C \leqslant \left(\frac{\sqrt{2}\xi}{1-4(T-a)\mu}\right)\delta_3,$$

where $1 \geqslant 4(T-a)\mu$. This finishes the proof. □

Theorem 7.4. *The unique solution of the stochastic integral equation* (3.3) *depends continuously on* $W(t)$.

Proof. Let $\|W(t) - W^*(t)\|_{L_2} \leqslant \delta_4$, $ß(t)$ be the solution of (3.3), and $ß^*(t)$ be the solution of

$$ß^*(t) = \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW^*(s)\right) + \int_a^t\mathcal{F}(s,U(s))ds + \int_a^t\mathcal{G}(s,ß^*(s))dW^*(s).$$

Here,

$$\begin{aligned}
ß(t) - ß^*(t) &= -\xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß(s))dW(s) + \xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW^*(s)\\
&\quad + \xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW(s) - \xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW(s) + \int_a^t\mathcal{G}(s,ß(s))dW(s)\\
&\quad - \int_a^t\mathcal{G}(s,ß^*(s))dW^*(s) + \int_a^t\mathcal{G}(s,ß^*(s))dW(s) - \int_a^t\mathcal{G}(s,ß^*(s))dW(s)\\
&= -\xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}[\mathcal{G}(s,ß(s)) - \mathcal{G}(s,ß^*(s))]dW(s) + \xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))d[W^*(s) - W(s)]\\
&\quad + \int_a^t[\mathcal{G}(s,ß(s)) - \mathcal{G}(s,ß^*(s))]dW(s) - \int_a^t\mathcal{G}(s,ß^*(s))d[W^*(s) - W(s)].
\end{aligned}$$

Then

$$\begin{aligned}
\|ß - ß^*\|_C &\leqslant \sqrt{2}\xi\sum_{k=1}^{m}\xi_k(\tau_k - a)\mu\|ß - ß^*\|_C + 2\xi\sum_{k=1}^{m}\xi_k[\nu + \mu\|ß^*\|_C][W^*(\tau_k) - W(\tau_k)]\\
&\quad + 2\sqrt{2}(t-a)\mu\|ß - ß^*\|_C + 2\sqrt{2}[\nu + \mu\|ß^*\|_C][W^*(t) - W(t)]\\
&\leqslant \sqrt{2}(T-a)\mu\|ß - ß^*\|_C + \sqrt{2}[\nu + \mu\|ß^*\|_C]\delta_4.
\end{aligned}$$

Thus

$$\|ß - ß^*\|_C \leqslant \left(\frac{\sqrt{2}[\nu + \mu\|ß^*\|_C]}{1-\sqrt{2}\mu(T-a)}\right)\delta_4,$$

where $1 \geqslant \sqrt{2}\mu(T-a)$. This finishes the proof. □

Theorem 7.5. *The unique solution of the stochastic integral equation* (3.3) *depends continuously on* ξ_k.

Proof. Let $|\xi_k - \xi_k^*(t)| \leqslant \delta_5$, $\text{ß}(t)$ be the solution of (3.3), and $\text{ß}^*(t)$ be the solution of

$$\text{ß}^*(t) = \xi^*\left(\text{ß}_a - \sum_{k=1}^{m} \xi_k^* \int_a^{\tau_k} \mathcal{F}(s, U(s))ds - \sum_{k=1}^{m} \xi_k^* \int_a^{\tau_k} \mathcal{G}(s, \text{ß}^*(s))dW(s)\right) + \int_a^{t} \mathcal{F}(s, U(s))ds + \int_a^{t} \mathcal{G}(s, \text{ß}^*(s))dW(s).$$

Now,

$$\text{ß}(t) - \text{ß}^*(t) = [\xi - \xi^*]\text{ß}_a - \left[\xi \sum_{k=1}^{m} \xi_k - \xi^* \sum_{k=1}^{m} \xi_k^*\right] \int_a^{\tau_k} \mathcal{F}(s, U(s))ds - \xi \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{G}(s, \text{ß}(s))dW(s) + \xi^* \sum_{k=1}^{m} \xi_k^* \int_a^{\tau_k} \mathcal{G}(s, \text{ß}^*(s))dW(s) + \int_a^{t} [\mathcal{G}(s, \text{ß}(s)) - \mathcal{G}(s, \text{ß}^*(s))]dW(s).$$

Since

$$1 + \sum_{k=1}^{m} \xi_k \geqslant 1 \quad \to \quad \frac{1}{1 + \sum_{k=1}^{m} \xi_k} \leqslant 1,$$

as the steps followed in Theorem 7.2,

$$|\ \xi - \xi^*\ | \leqslant m\delta_5 \quad \text{and} \quad \left|\xi \sum_{k=1}^{m} \xi_k - \xi^* \sum_{k=1}^{m} \xi_k^*\right| \leqslant m\delta_5,$$

and

$$\xi^* \sum_{k=1}^{m} \xi_k^* \int_a^{\tau_k} \mathcal{G}(s, \text{ß}^*(s))ds - \xi \sum_{k=1}^{m} \xi_k \int_a^{\tau_k} \mathcal{G}(s, \text{ß}(s))ds = -\int_a^{\tau_k} [\mathcal{G}(s, \text{ß}(s)) - \mathcal{G}(s, \text{ß}^*(s))]ds + [\xi - \xi^*] \int_a^{\tau_k} \mathcal{G}(s, \text{ß}(s))ds + \xi^* \int_a^{\tau_k} [\mathcal{G}(s, \text{ß}(s)) - \mathcal{G}(s, \text{ß}^*(s))]ds.$$

Then

$$\text{ß}(t) - \text{ß}^*(t) = [\xi - \xi^*]\text{ß}_a - \left[\xi \sum_{k=1}^{m} \xi_k - \xi^* \sum_{k=1}^{m} \xi_k^*\right] \int_a^{\tau_k} \mathcal{F}(s, U(s))ds - \int_a^{\tau_k} [\mathcal{G}(s, \text{ß}(s)) - \mathcal{G}(s, \text{ß}^*(s))]ds + [\xi - \xi^*] \int_a^{\tau_k} \mathcal{G}(s, \text{ß}(s))ds + \xi^* \int_a^{\tau_k} [\mathcal{G}(s, B(s)) - \mathcal{G}(s, \text{ß}^*(s))]ds + \int_a^{t} [\mathcal{G}(s, \text{ß}(s)) - \mathcal{G}(s, \text{ß}^*(s))]dW(s).$$

So

$$\begin{aligned} \|\text{ß} - \text{ß}^*\|_C &\leqslant \sqrt{2}\delta_5 \|\text{ß}_a\|_{L_2} + 2m\delta_5[\nu + \mu\|U\|_C](\tau_k - a) + 2\sqrt{2}\mu\,(1 + |\xi^*|)\,\|\text{ß} - \text{ß}^*\|_C(\tau_k - a) \\ &\quad + 4\delta_5[\nu + \mu\|U\|_C](\tau_k - a) + 4\mu\|\text{ß} - \text{ß}^*\|_C\sqrt{W(t) - W(a)} \\ &\leqslant \delta_5\left[\sqrt{2}\|\text{ß}_a\|_{L_2} + (4 + 2m)[\nu + \mu\|U\|_C](T - a)\right] + 2\sqrt{2}\mu\,(1 + |\xi^*|)\,\|\text{ß} - \text{ß}^*\|_C(T - a) \end{aligned}$$

$$+4\mu||ß-ß^*||_C\sqrt{W(T)-W(a)}.$$

Thus

$$||ß-ß^*||_C \leqslant \left(\frac{\sqrt{2}||ß_a||_{L_2}+(4+2m)[\nu+\mu||U||_C](T-a)}{1-2\sqrt{2}\mu\,(1+|\xi^*|)\,(T-a)-4\mu\sqrt{W(T)-W(a)}}\right)\delta_5,$$

where $1 \geqslant 2\sqrt{2}\mu\,(1+|\xi^*|)\,(T-a)+4\mu\sqrt{W(T)-W(a)}$. This finishes the proof. □

Theorem 7.6. *The unique solution of the stochastic integral equation* (3.3) *depends continuously on* $U(t)$.

Proof. Let $||U(t)-U^*(t)||_{L_2} \leqslant \delta_6$, $ß(t)$ be the solution of (3.3), and $ß^*(t)$ be the solution of

$$ß^*(t) = \xi\left(ß_a - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{F}(s,U^*(s))ds - \sum_{k=1}^{m}\xi_k\int_a^{\tau_k}\mathcal{G}(s,ß^*(s))dW(s)\right)$$
$$+\int_a^t\mathcal{F}(s,U^*(s))ds+\int_a^t\mathcal{G}(s,ß^*(s))dW(s).$$

Now

$$ß(t)-ß^*(t) = -\xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}[\mathcal{F}(s,U(s))-\mathcal{F}(s,U^*(s))]ds-\xi\sum_{k=1}^{m}\xi_k\int_a^{\tau_k}[\mathcal{G}(s,ß(s))-\mathcal{G}(s,ß^*(s))]dW(s)$$
$$+\int_a^t[\mathcal{F}(s,U(s))-\mathcal{F}(s,U^*(s))]ds+\int_a^t[\mathcal{G}(s,ß(s))-\mathcal{G}(s,ß^*(s))]dW(s).$$

So

$$||ß-ß^*||_C \leqslant \sqrt{2}\left(1+\xi\sum_{k=1}^{m}\xi_k\right)\mu||U-U^*||_C(T-a)+\sqrt{2}\left(1+\xi\sum_{k=1}^{m}\xi_k\right)\mu||ß-ß^*||_C\sqrt{W(T)-W(a)}.$$

Thus

$$||ß-ß^*||_C \leqslant \left(\frac{2\sqrt{2}\mu(T-a)}{1-2\sqrt{2}\mu\sqrt{W(T)-W(a)}}\right)\delta_6,$$

where $1 \geqslant 2\sqrt{2}\mu\sqrt{W(T)-W(a)}$. This finishes the proof. □

8. Application

Consider as an application, the stochastic problem

$$dß(t) = D_{0^+}^{1,\frac{1}{2}}\vartheta(t)+\mathcal{G}(t,ß(t))dW(t), \quad t\in(0,T], \tag{8.1}$$

with the random initial condition

$$ß(0)+2ß(2)=ß_0 \tag{8.2}$$

and

$$\vartheta(0)+2\vartheta(2)+3\vartheta(3)=\vartheta_0. \tag{8.3}$$

For getting the solution, let

$$U(t) = D_{0^+}^{1,\frac{1}{2}}\vartheta(t). \tag{8.4}$$

We see that the problem (8.1)-(8.3) is as the same as our problem (1.1)-(1.3), where $\alpha = \gamma = 1,\ \beta = \frac{1}{2}, a = 0$. Thus the solution of the stochastic fractional differential equation (8.1) with the initial condition (8.3) can be transformed by Lemma 3.1 as

$$\vartheta(t) = \frac{1}{6}\left(\vartheta_0 - 2\int_0^2 U(s)ds - 3\int_0^3 U(s)ds\right) + \int_0^t U(s)ds \tag{8.5}$$

and the solution of the stochastic problem (8.1)-(8.3) with (8.4) is gotten by Lemma 3.2,

$$ß(t) = \frac{1}{3}\left(ß_0 - 2\int_0^2 U(s)ds - 2\int_0^2 \mathcal{G}(t, ß(t))dW(s)\right) + \int_0^t U(s)ds + \int_0^t \mathcal{G}(t, ß(t))dW(s). \tag{8.6}$$

In this example if

$$\vartheta(t) = \frac{1}{6}\left(\vartheta_0 - \frac{64}{3}\sqrt{\frac{2}{\pi}} - 48\sqrt{\frac{3}{\pi}}\right) + \frac{16t^{\frac{3}{2}}}{3\sqrt{\pi}},$$

and $\mathcal{G}(t, ß(t)) = \mathcal{G}(\iota + W(t))$ is a function of non-standard Brownian motion called the Brownian motion started at $\iota \in L_2(\Omega)$ (see [17]), we will get

$$U(t) = D_{0^+}^{1,\frac{1}{2}}\left(\frac{1}{6}\left(\vartheta_0 - \frac{64}{3}\sqrt{\frac{2}{\pi}} - 48\sqrt{\frac{3}{\pi}}\right) + \frac{16t^{\frac{3}{2}}}{3\sqrt{\pi}}\right) = \frac{8}{\sqrt{\pi}}t^{\frac{1}{2}}.$$

Thus, we finally get

$$ß(t) = \frac{1}{3}\left(ß_0 - \frac{64}{3}\sqrt{\frac{2}{\pi}} - 2\int_0^2 \mathcal{G}(\iota + W(s))dW(s)\right) + \frac{16s^{\frac{3}{2}}}{3\sqrt{\pi}} + \int_0^t \mathcal{G}(\iota + W(s))dW(s).$$

It is clearly that the assumptions (i)-(ii) of Theorems 4.1 and 4.2 are satisfied with $4(T-a)^\alpha < \Gamma(\alpha+1)$ and $\mu < \frac{1}{4\sqrt{2}\sqrt{T-a}}$, these conditions tend to $T < \frac{1}{4}$ and $\mu < \frac{1}{4\sqrt{2T}}$, respectively. By the assumption of Theorems 5.5 and 5.6, the maximal solution of equations (8.5) and (8.6) can be gotten. The unique solution is so trivial using Caratheodory condition (iii), all continuous dependencies discussed before can be proved.

9. Conclusions

In this paper, in Theorems 4.1 and 4.2, the existence of solutions $\vartheta(t)$ and $ß(t) \in C([a, T], L_2(\Omega))$ of the non-local stochastic fractional differential equation (3.1) with the non-local condition (1.3) and the generalized stochastic problem (1.1)-(1.3), respectively are proved. In Definitions 5.1-5.4, the meanings of stochastically decreasing functions, stochastically increasing functions, maximal solution, and minimal solution of the stochastic problem are all discussed. After that, in Theorems 5.5 and 5.6, the assumptions for the solution to be a maximal solution of equation (3.2) and equation (3.3) are discussed.

In the second part of the paper, the sufficient conditions for the uniqueness of the solution of (3.2) and (3.3) has been given in Theorems 6.1 and 6.2, respectively. The continuous dependence of the solution on ϑ_a and η_k, of the solution of equation (3.2) and the continuous dependence on $ß_a$, ξ_k, $W(t)$, also on $U(t)$ of the solution of equation (3.3) are all proved.

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