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Modified inertial subgradient extragradient with auxiliary parameters and parallel viscosity algorithm for minimization problem induced by bounded linear operator over common solution of fixed points of nonexpansive mappings and pseudomonotone equilibrium problems

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Abstract

This paper introduces a modification to the inertial subgradient extragradient algorithm by incorporating auxiliary parameters for updating, along with dynamic regularization coefficient, including the parallel viscosity algorithm. The aim is to find an element in the common solution set of fixed points in a finite family of nonexpansive mappings and Lipschitz-type continuous pseudomonotone equilibrium problems. This element also serves as the unique solution to a minimization problem induced by a bounded linear operator and contraction mapping in the context of a real Hilbert space. The efficiency of the proposed algorithm is influenced by the introduced auxiliary parameters, which are intended to leverage the value of the considered objective bifunction at each iteration, along with the advantages of the designed regularization coefficient, which is self-adaptive and utilizes a straightforward rule for automatic updates. The update rule avoids enforcing monotonic behavior on the dynamic regularization coefficient and does not require prior knowledge of the Lipschitz constants of the bifunction. This flexibility increases the algorithm's applicability for solving a wider range of practical problems. The discussions on the numerical experiments for Nash-Cournot models and image restoration problems are also provided to illustrate the computational effectiveness of the introduced algorithm.

Keywords: Equilibrium problems, fixed point problems, pseudomonotone bifunction, nonexpansive mapping, inertial method, subgradient extragradient method.

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1. Introduction

The fixed point problem and the equilibrium problem have intensively been studied and the spans of the concept of these problems have a broad range of applications in mathematical problems, such as

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variational inequalities problems, minimax problems, null point problems, saddle point problems, optimization problems, and Nash equilibrium problems, see [14, 25, 28], and the references therein. They also have an influence on the development of other branches such as finance, economics, transportation, and image restoration, see [15, 34, 35], and the references therein. One topic interesting is image restoration which plays an important part in many fields of applied sciences including medical and astronomical imaging. Image restoration is the process of recovering a reasonably clear image from a noisy or distorted image. As a result, improving image quality, which is the aim of image restoration, is worth contemplating. For related topics in this work, see [16, 31, 33], and the references therein.

Firstly, the fixed point problem is a problem of finding a point $x \in H$ such that $Sx = x$, where H is a real Hilbert space and $S : H \to H$ is a mapping. The set of fixed points of the mapping S will be denoted by $F(S)$. In order to find fixed points of a nonexpansive mapping S, Moudafi [24] proposed the following so-called viscosity method:

$$\begin{cases} x_0 \in H, \\ x_{k+1} = (1-\alpha_k)Sx_k + \alpha_k h(x_k), \end{cases} \tag{1.1}$$

where $\{\alpha_k\} \subset (0,1)$ and $h\colon H \to H$ is a contraction mapping. Under certain appropriate conditions, the author proved that the sequence $\{x_k\}$ generated by Algorithm (1.1) converges strongly to $p^* \in F(S)$, which is a solution of the variational inequality

$$\langle (I-h)p^*, x-p^* \rangle \geqslant 0, \quad \forall x \in F(S),$$

where I is an identity mapping. It is important to note that the iteration methods for finding fixed points of nonexpansive mappings have been applied to solve convex minimization problems, see [38], and references therein. A typical convex minimization problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping S:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where $A : H \to H$ is a bounded linear operator and b is a point in H. Furthermore, by using the idea of the viscosity method, Marino and Xu [21] proposed the following algorithm for finding fixed points of a nonexpansive mapping S:

$$\begin{cases} x_0 \in H, \\ x_{k+1} = (I-\alpha_k A)Sx_k + \alpha_k \gamma h(x_k), \end{cases} \tag{1.2}$$

where $\{\alpha_k\} \subset (0,1)$, $h\colon H \to H$ is a contraction mapping with coefficient $\rho \in (0,1)$, and $A\colon H \to H$ is a strongly positive bounded linear mapping with coefficient $\beta > 0$ such that $0 < \gamma < \frac{\beta}{\rho}$. Under certain appropriate conditions, they proved that the sequence $\{x_k\}$ generated by Algorithm (1.2) converges strongly to $p^* \in F(S)$, where p^* is also a solution of the variational inequality

$$\langle (A-\beta h)p^*, x-p^* \rangle \geqslant 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - g(x),$$

when g is a potential function for βh (i.e., $g'(x) = \beta h(x)$, $\forall x \in H$). On the other hand, the equilibrium problem introduced by Blum and Oettli [4] is stated in the following manner:

$$\text{Find a point} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geqslant 0, \quad \forall y \in C, \tag{1.3}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f\colon H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.3) will be represented by $EP(f, C)$. A famous method for solving the equilibrium problem (1.3) is the proximal point method. This method was first proposed by Martinet [22] for solving the variational inequality problem and further investigated by Moudafi [23] to the monotone equilibrium problem. It is worth noting that the proximal point method cannot be applied to solve the equilibrium problem if the bifunction f satisfies a weaker assumption, like pseudomonotone, see [11]. To surmount this limitation, the extragradient method was introduced for solving the pseudomonotone equilibrium problem instead of the proximal point method. The extragradient method was early proposed by Korpelevich [19] for solving the saddle point problem and later expanded by Noor [26] to the pseudomonotone variational inequality problem. Afterward, Tran et al. [35] proposed the following extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\left\{\lambda f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min\left\{\lambda f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\right\}, \end{cases} \tag{1.4}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by Algorithm (1.4) converges weakly to a solution of the equilibrium problem (1.3). It is emphasized that the extragradient method is a two-step iteration method and requires to solve the optimization problems on the feasible set C twice for finding y_k and x_{k+1} in each iteration, which affects the computational efficiency of such algorithm when the structure of the feasible set C is complex. To overcome this drawback, Hieu [12] extended the following so-called subgradient extragradient method, which was proposed by Censor et al. [7] in context of the variational inequality problem, for solving the equilibrium problem when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in H, \\ y_k = \arg\min\left\{\lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\right\}, \\ T_k = \{z \in H : \langle x_k - \lambda_k r_k - y_k, z - y_k\rangle \leqslant 0\}, r_k \in \partial_2 f(x_k, y_k), \\ z_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in T_k\right\}, \\ x_{k+1} = \alpha_k x_0 + (1 - \alpha_k) z_k, \end{cases} \tag{1.5}$$

where $0 < \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\alpha_k\} \subset (0, 1)$ such that $\sum_{k=0}^{\infty} \alpha_k = +\infty$ and $\lim_{k\to\infty} \alpha_k = 0$, and $\partial_2 f(x_k, y_k)$ is the subdifferential of $f(x_k, \cdot)$ at y_k. The author proved that the sequence $\{x_k\}$ generated by Algorithm (1.5) converges strongly to $P_{EP(f,C)}(x_0)$. It is worth noting that the subgradient extragradient method converts the optimization problem on the feasible set C in the second step to an optimization problem on the half-space T_k for finding z_k in each iteration. Consequently, the subgradient extragradient method improves the computational efficiency of the extragradient method because it only needs to solve the optimization problem on the feasible set C once for finding y_k. Notice that the step sizes of the aforementioned algorithms depend on the Lipschitz constants of the bifunction f. This means that these algorithms need to know the prior information of the Lipschitz constants of the bifunction f. However, this information is usually not easily available in practical applications.

Meanwhile, the inertial method has received a lot of attention from many researchers, for instance, see [13, 36] and the references therein. This method is regarded to speed up the convergence properties of the algorithm and was used in the implicit discretization algorithm of the heavy ball with friction system [1, 2] which was first studied by Polyak [29]. The main feature of this method is that the next iterate point is determined through the previous two iterates.

In 2022, Xie et al. [37] proposed the following algorithm by using the techniques of inertial and subgradient extragradient methods together with the viscosity-type method for solving the equilibrium and fixed point problems when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous and the mapping S is nonexpansive:

$$\begin{cases} x_0, x_1 \in H, \\ w_k = x_k + \theta_k(x_k - x_{k-1}), \\ y_k = \arg\min\left\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}, \\ T_k = \{z \in H : \langle w_k - \lambda_k r_k - y_k, z - y_k\rangle \leqslant 0\}, r_k \in \partial_2 f(w_k, y_k), \\ z_k = \arg\min\left\{\sigma\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\right\}, \\ \lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \frac{\mu(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, & \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, \\ \lambda_k, & \text{otherwise}, \end{cases} \\ x_{k+1} = \alpha_k h(x_k) + \psi_k x_k + (1 - \psi_k - \alpha_k) S z_k, \end{cases} \tag{1.6}$$

where $\lambda_1 > 0$, $\mu \in (0,1)$, $\sigma \in (0,1]$, $\{\alpha_k\} \subset (0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, $0 < \liminf_{k\to\infty} \psi_k \leqslant \limsup_{k\to\infty} \psi_k < 1$, and $\theta_k \subset [0,\theta)$ for some $\theta > 0$ such that $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| = 0$. They proved that the sequence $\{x_k\}$ generated by Algorithm (1.6) converges strongly to $\tilde{p} = P_{EP(f,C)\cap F(S)} h(\tilde{p})$. It is evident that Algorithm (1.6) used the adaptive step size to deal with the unknown knowledge of the Lipschitz constants of the bifunction f. Moreover, the adaptive step size criteria that update the step size of each iteration with a simple computation by using the previously known information is presented. However, this adaptive step size is a non-increasing sequence, which may affect the efficient computation of Algorithm (1.6).

In this paper, we will focus on the algorithm for solving the equilibrium and fixed point problems. That is, we introduce a new iterative algorithm for finding the common solution of the pseudomonotone equilibrium problem and the fixed point problem of a finite family of nonexpansive mappings by using the adaptive dynamic regularization coefficients. Applications to Nash-Cournot models and image restoration problems demonstrated the efficiency of the proposed algorithm via numerical experiments.

This paper is organized as follows. Section 2 includes some basic definitions and relevant properties to be used in subsequent sections. Section 3 contains the modified inertial subgradient extragradient with auxiliary parameters and parallel viscosity algorithm and the corresponding strong convergence theorem. In Section 4, we will discuss the numerical behavior of the introduced algorithm in comparison with respect to the aforementioned interesting algorithms on test problems including Nash-Cournot models and image restoration problems.

2. Preliminaries

This section will present some necessary definitions and results that will be used in the sequel. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and its corresponding $\|\cdot\|$. The symbols $\to$ and $\rightharpoonup$ will be denoted for the strong convergence and the weak convergence in H, respectively. The notation $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of the real numbers and the natural numbers, respectively.

First, we will collect some definitions and properties that will be used in this paper.

Definition 2.1. Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is said to be:

(i) monotone on C if $f(x,y) + f(y,x) \leqslant 0$, $\forall x, y \in C$;
(ii) pseudomonotone on C if $f(x,y) \geqslant 0 \Rightarrow f(y,x) \leqslant 0$, $\forall x, y \in C$;
(iii) Lipschitz-type continuous on H if there exists two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geqslant f(x,z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \quad \forall x, y, z \in H.$$

Remark 2.2. A monotone bifunction is a pseudomonotone bifunction, but the converse is not true in general, for instance, see [17].

Definition 2.3. A mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in H.$$

Remark 2.4. It is well-known that $F(T)$ is closed and convex when T is a nonexpansive mapping, see [9].

Definition 2.5 ([5]). A mapping $T : H \to H$ is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup x^* \in H$ and $Tx_k \to y$ imply $Tx^* = y$.

Lemma 2.6 ([9]). *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, $I - T$ demiclosed at zero.*

Definition 2.7. A mapping $A : H \to H$ is said to be strongly positive bounded linear mapping with coefficient β, if there exists a constant $\beta > 0$ such that

$$\langle Ax, x\rangle \geqslant \beta\|x\|^2, \quad \forall x \in H.$$

Lemma 2.8 ([21]). *Let $A : H \to H$ be a strongly positive bounded linear mapping with coefficient $\beta > 0$ and $0 < \alpha < \|A\|^{-1}$. Then, $\|I - \alpha A\| \leqslant 1 - \alpha\beta$.*

For each $x \in H$, we denote the metric projection of x onto a nonempty closed convex subset C of H by $P_C(x)$, that is

$$\|x - P_C(x)\| \leqslant \|x - y\|, \quad \forall y \in C.$$

Lemma 2.9 ([6, 10]). *Let C be a nonempty closed convex subset of H. Then,*

(i) *$P_C(x)$ is singleton and well-defined for each $x \in H$;*
(ii) *$z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leqslant 0$, $\forall y \in C$.*

For a function $f : H \to \mathbb{R}$, the subdifferential of f at $x \in H$ is defined by

$$\partial f(x) = \{z \in H : f(y) - f(x) \geqslant \langle z, y - x\rangle, \quad \forall y \in H\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$.

Lemma 2.10 ([6]). *For any $x \in H$, the subdifferentiable $\partial f(x)$ of a continuous convex function f is a weakly closed and bounded convex set.*

Lemma 2.11 ([8]). *Let C be a convex subset of H and $f : C \to \mathbb{R}$ be subdifferentiable on C. Then, x^* is a solution to the following convex problem: $\min\{f(x) : x \in C\}$ if and only if $0 \in \partial f(x^*) + N_C(x^*)$, where $N_C(x^*) := \{z \in H : \langle z, y - x^*\rangle \leqslant 0, \ \forall y \in C\}$ is the normal cone of C at x^*.*

We end this section by recalling some important results for proving the convergence theorems.

Lemma 2.12 ([27]). *Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of non-negative real numbers such that $a_{k+1} \leqslant a_k b_k + c_k$, $\forall k \in \mathbb{N}$. If $\{b_k\} \subset [1, \infty)$, $\sum\limits_{k=0}^{\infty} (b_k - 1) < \infty$, and $\sum\limits_{k=1}^{\infty} c_k < \infty$, then $\lim\limits_{k\to\infty} a_k$ exists.*

Lemma 2.13 ([38]). *Let $\{a_k\}$ and $\{c_k\}$ be sequences of non-negative real numbers such that*

$$a_{k+1} \leqslant (1 - \alpha_k)a_k + \alpha_k b_k + c_k, \quad \forall k \in \mathbb{N} \cup \{0\},$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{b_k\}$ is a sequence in $\mathbb{R}$. Assume that $\sum\limits_{k=0}^{\infty} c_k < \infty$. If $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$ and $\limsup\limits_{k\to\infty} b_k \leqslant 0$, then $\lim\limits_{k\to\infty} a_k = 0$.

Lemma 2.14 ([20]). *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j} < a_{k_j+1}$, for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_n\}$ of positive integers such that $\lim\limits_{n\to\infty} m_n = \infty$ and the following properties hold:*

$$a_{m_n} \leqslant a_{m_n+1} \text{ and } a_n \leqslant a_{m_n+1},$$

for all (sufficiently large) numbers $n \in \mathbb{N}$. Indeed, m_n is the largest number k in the set $\{1, 2, \ldots, n\}$ such that

$$a_k < a_{k+1}.$$

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Now, we will consider the following problem:

$$\text{Find a point } x^* \in C \text{ such that } S_i x^* = x^*, i = 1, \dots, M, \text{ and } f(x^*, y) \geqslant 0, \ \forall y \in C, \tag{3.1}$$

where $\{S_i\}_{i=1}^M \colon H \to H$ is a finite family of nonexpansive mappings and $f \colon H \times H \to \mathbb{R}$ is a bifunction. From now on, the solution set of problem (3.1) will be denoted by Ω. That is:

$$\Omega := (\cap_{i=1}^M F(S_i)) \cap EP(f, C).$$

For the bifunction $f \colon H \times H \to \mathbb{R}$, we are concerned with the following assumptions in this work.

(A1) $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C, for each fixed $y \in C$, that is if $\{x_k\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup_{k\to\infty} f(x_k, y) \leqslant f(x, y)$;

(A2) $f(x, \cdot)$ is convex, subdifferentiable and lower semicontinuous on H, for each fixed $x \in H$;

(A3) f is psuedomonotone on C;

(A4) f is Lipschitz-type continuous on H.

Remark 3.1.

(i) If the bifunction f satisfies the assumptions (A1)-(A3), then the solution set $EP(f, C)$ is closed and convex, see [30, 35] for more detail.

(ii) If the bifunction f satisfies the assumptions (A3) and (A4), then $f(x, x) = 0$, for each $x \in C$, see [36].

Next, we introduce the following modified inertial subgradient extragradient with auxiliary parameters and parallel viscosity algorithm for solving the problem (3.1), when $A \colon H \to H$ is a strongly positive bounded linear mapping with coefficient $\beta > 0$ and $h \colon H \to H$ is a contraction mapping with coefficient $\rho \in (0, 1)$ such that $0 < \gamma < \frac{\beta}{\rho}$.

Algorithm 3.2 (Modified inertial subgradient extragradient with auxiliary parameters and parallel viscosity algorithm)**.**

Initialization: Choose parameters $\lambda_1 > 0$, $\mu \in [0, 1)$, $\varphi \in (0, 1)$, $\tau \in (0, 1)$, $\sigma \in \left(0, \frac{1}{2\tau}\right)$, $\eta \in \left[\sigma, \frac{1}{\tau}\right)$, $\{\xi_k\} \subset [1, \infty)$ with $\sum_{k=0}^{\infty} (\xi_k - 1) < \infty$, $\{\rho_k\} \subset [0, \infty)$ with $\sum_{k=0}^{\infty} \rho_k < \infty$, $\{\delta_k\} \subset (\varphi, 1]$ with $\lim_{k\to\infty} \delta_k = 1$, $\{\epsilon_k\} \subset [0, \infty)$, and $\alpha_k \subset (0, 1)$ such that $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, and $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$. Pick $x_0, x_1 \in H$ and set $k = 1$.

Step 1: Choose θ_k such that $0 \leqslant \theta_k \leqslant \bar{\theta}_k$, where

$$\bar{\theta}_k = \begin{cases} \min\left\{\mu, \frac{\epsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \mu, & \text{otherwise}, \end{cases}$$

and compute

$$w_k = x_k + \theta_k (x_k - x_{k-1}).$$

Step 2: Solve the strongly convex program

$$y_k = \arg\min\left\{\eta\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\right\}.$$

Step 3. Construct a half-space

$$T_k = \{z \in H : \langle w_k - \eta\lambda_k r_k - y_k, z - y_k\rangle \leqslant 0\},$$

where $r_k \in \partial_2 f(w_k, y_k)$.

Step 4: Solve the strongly convex program

$$z_k = \arg\min\left\{\sigma\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in T_k\right\}.$$

Step 5: Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{\xi_k\lambda_k + \rho_k, \dfrac{\tau\delta_k(\|w_k - y_k\|^2 + \|y_k - z_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]}\right\}, & \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, \\ \xi_k\lambda_k + \rho_k, & \text{otherwise.} \end{cases}$$

Step 6: Compute

$$u_k^i = \alpha_k\gamma h(w_k) + (I - \alpha_k A)S_i z_k, \quad i = 1, 2, \ldots, M.$$

Step 7: The next approximation x_{k+1} is defined as the farthest element from w_k among u_k^i, $i = 1, 2, \ldots, M$, i.e.,

$$x_{k+1} = \arg\max\left\{\|u_k^i - w_k\| : i = 1, 2, \ldots, M\right\}.$$

Step 8: Put $k := k + 1$ and return to Step 1.

Remark 3.3.

(i) The auxiliary parameters η and σ in Algorithm 3.2 are proposed to modify the subgradient extragradient method presented in [12]. We emphasize that the choices of parameters η and σ can significantly impact the superior numerical performance of Algorithm 3.2. Notably, based on the choice of parameter τ, we observe that η and σ can be selected as values strictly greater than 1. Consequently, the presence of these two parameters introduces bias to the objective bifunction f, especially when considering steps 2 and 4. Notice that if $\eta = \sigma = 1$ or $\eta = 1$, $\sigma \in (0, 1]$, then the subgradient extragradient method in Algorithm 3.2 reduces to a situation as presented in [12], and the subgradient extragradient method presented in [37], respectively.

(ii) Observe that in the case of $\xi_k = 1$, $\rho_k = 0$, and $\delta_k = 1$, the step size in Algorithm 3.2 reduces to the non-increasing step size presented in [37]. We emphasize that the property of the auxiliary parameter δ_k, a sequence of real numbers converging to 1 from the left, introduces bias to the current value of the objective bifunction f, along with the relationships among w_k, y_k, and z_k. This bias plays a crucial role in determining the regularized parameter λ_{k+1}, see Section 4 for discussion and experiments. Furthermore, one sees that the regularization coefficient λ_k may increase from iteration to iteration and so Algorithm 3.2 reduces the dependence on the initial step size λ_1. Meanwhile, the advantages of the regularization coefficient λ_k are self-adaptive which uses a simple rule to automatically update the iteration regularization coefficient, and does not necessitate to know the Lipschitz constants of the bifunction in advance.

The following lemma is quite helpful in analyzing the convergence of Algorithm 3.2.

Lemma 3.4. *Let* $f\colon H \times H \to \mathbb{R}$ *be a bifunction which satisfies* (A1)-(A4). *Suppose that the solution set* $EP(f, C)$ *is nonempty. Let* $w_k \in H$. *If* y_k, z_k, *and* λ_{k+1} *are constructed as in the process of Algorithm* 3.2, *then the following result holds:*

$$\|z_k - p\|^2 \leqslant \|w_k - p\|^2 - \left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2, \quad \forall p \in EP(f, C).$$

Proof. Firstly, we will claim that $C \subset T_k$ for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed and $y \in C$. From the definition of y_k and Lemma 2.11, we get

$$0 \in \partial_2\left\{\eta\lambda_k f(w_k, y_k) + \frac{1}{2}\|y_k - w_k\|^2\right\} + N_C(y_k).$$

Thus, there exists $r_k \in \partial_2 f(w_k, y_k)$ and $s_k \in N_C(y_k)$ such that

$$\eta\lambda_k r_k + y_k - w_k + s_k = 0.$$

It follows from $s_k \in N_C(y_k)$ that

$$\langle w_k - \eta\lambda_k r_k - y_k, y - y_k\rangle = \langle s_k, y - y_k\rangle \leqslant 0.$$

This implies that $y \in T_k$. Then, we had shown that $C \subset T_k$, for each $k \in \mathbb{N}$. Consequently, this one guarantees that Algorithm 3.2 is well-defined.

Next, we will assert the result of the Lemma by applying the above facts. Let $p \in EP(f, C)$. By the subdifferentiability of f and $r_k \in \partial_2 f(w_k, y_k)$, we have

$$f(w_k, y) - f(w_k, y_k) \geqslant \langle r_k, y - y_k\rangle, \quad \forall y \in H.$$

Indeed, from $z_k \in T_k \subset H$, we have

$$f(w_k, z_k) - f(w_k, y_k) \geqslant \langle r_k, z_k - y_k\rangle. \tag{3.2}$$

Also, by using the definition of T_k and $z_k \in T_k$, we get

$$\langle w_k - \eta\lambda_k r_k - y_k, z_k - y_k\rangle \leqslant 0.$$

It follows from the inequality (3.2) that

$$\eta\lambda_k[f(w_k, z_k) - f(w_k, y_k)] \geqslant \langle y_k - w_k, y_k - z_k\rangle. \tag{3.3}$$

In addition, from the definition of z_k and Lemma 2.11, we have

$$0 \in \partial_2\left\{\sigma\lambda_k f(y_k, z_k) + \frac{1}{2}\|z_k - w_k\|^2\right\} + N_{T_k}(z_k).$$

Thus, there exists $r \in \partial_2 f(y_k, z_k)$ and $s \in N_{T_k}(z_k)$ such that

$$\sigma\lambda_k r + z_k - w_k + s = 0. \tag{3.4}$$

It follows from the subdifferentiability of f that

$$f(y_k, y) - f(y_k, z_k) \geqslant \langle r, y - z_k\rangle, \quad \forall y \in H. \tag{3.5}$$

So, from $s \in N_{T_k}(z_k)$, we obtain

$$\langle s, z_k - y\rangle \geqslant 0, \quad \forall y \in T_k,$$

which together with the equality (3.4) implies that

$$\langle w_k - z_k, z_k - y\rangle \geqslant \sigma\lambda_k\langle r, z_k - y\rangle, \quad \forall y \in T_k.$$

Combining with the inequality (3.5), we get

$$\langle w_k - z_k, z_k - y\rangle \geqslant \sigma\lambda_k[f(y_k, z_k) - f(y_k, y)], \quad \forall y \in T_k. \tag{3.6}$$

In particular, since $p \in C \subset T_k$, we have

$$\langle w_k - z_k, z_k - p\rangle \geqslant \sigma\lambda_k[f(y_k, z_k) - f(y_k, p)].$$

This together with the pseudomonotonic of f yields that

$$\langle w_k - z_k, z_k - p\rangle \geqslant \sigma\lambda_k f(y_k, z_k).$$

It follows from the relation (3.3) that

$$\eta\sigma\lambda_k[f(w_k,z_k)-f(w_k,y_k)-f(y_k,z_k)] \geqslant \eta\langle z_k-w_k,z_k-p\rangle+\sigma\langle y_k-w_k,y_k-z_k\rangle. \tag{3.7}$$

On the other hand, from the definition of λ_{k+1}, we note that

$$f(w_k,z_k)-f(w_k,y_k)-f(y_k,z_k) \leqslant \frac{\tau\delta_k(\|w_k-y_k\|^2+\|y_k-z_k\|^2)}{2\lambda_{k+1}}. \tag{3.8}$$

Combining with the inequality (3.7), we get

$$\eta\langle w_k-z_k,z_k-p\rangle \geqslant \sigma\langle y_k-w_k,y_k-z_k\rangle-\frac{\tau\eta\sigma\delta_k\lambda_k(\|w_k-y_k\|^2+\|y_k-z_k\|^2)}{2\lambda_{k+1}}.$$

Due to the above expression, we have the following relations:

$$\begin{aligned}&\eta\left(\|w_k-p\|^2-\|w_k-z_k\|^2-\|z_k-p\|^2\right)\\ &=2\eta\langle w_k-z_k,z_k-p\rangle \geqslant 2\sigma\langle y_k-w_k,y_k-z_k\rangle-\frac{\tau\eta\sigma\delta_k\lambda_k(\|w_k-y_k\|^2+\|y_k-z_k\|^2)}{\lambda_{k+1}}.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_k-p\|^2 &\leqslant \|w_k-p\|^2-\|w_k-z_k\|^2-\frac{2\sigma}{\eta}\langle y_k-w_k,y_k-z_k\rangle+\frac{\tau\sigma\delta_k\lambda_k(\|w_k-y_k\|^2+\|y_k-z_k\|^2)}{\lambda_{k+1}}\\ &=\|w_k-p\|^2-\|w_k-z_k\|^2+\frac{\sigma}{\eta}\|w_k-z_k\|^2-\frac{\sigma}{\eta}\|w_k-y_k\|^2-\frac{\sigma}{\eta}\|y_k-z_k\|^2\\ &\quad+\frac{\tau\sigma\delta_k\lambda_k(\|w_k-y_k\|^2+\|y_k-z_k\|^2)}{\lambda_{k+1}}\\ &=\|w_k-p\|^2-\left(\frac{\sigma}{\eta}-\frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|w_k-y_k\|^2-\left(\frac{\sigma}{\eta}-\frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|y_k-z_k\|^2-(1-\frac{\sigma}{\eta})\|w_k-z_k\|^2.\end{aligned}$$

Then, by using the conditions of the parameters σ and η (observing that $\frac{\sigma}{\eta}\in(0,1]$), we conclude that

$$\|z_k-p\|^2 \leqslant \|w_k-p\|^2-\left(\frac{\sigma}{\eta}-\frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|w_k-y_k\|^2-\left(\frac{\sigma}{\eta}-\frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|y_k-z_k\|^2.$$

This completes the proof. □

Now, we are ready to analyze the strong convergence theorem of Algorithm 3.2.

Theorem 3.5. *Let $f\colon H\times H\to\mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and $\{S_i\}_{i=1}^M\colon H\to H$ be a finite family of nonexpansive mappings. Assume that $A\colon H\to H$ is a strongly positive bounded linear mapping with coefficient $\beta>0$, and $h\colon H\to H$ is a contraction mapping with coefficient $\rho\in(0,1)$ such that $0<\gamma<\frac{\beta}{\rho}$. Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 converges strongly to $\tilde{p}=P_\Omega(I-A+\gamma h)(\tilde{p})$.*

Proof. Let $p\in\Omega$. From the Lipschitz-type continuity of f on H, there exists two positive constants c_1 and c_2 such that

$$f(w_k,z_k)-f(w_k,y_k)-f(y_k,z_k) \leqslant \max\{c_1,c_2\}(\|w_k-y_k\|^2+\|y_k-z_k\|^2).$$

Combining with the definition of λ_{k+1} and the assumptions on the sequences $\{\xi_k\}$, $\{\rho_k\}$, $\{\delta_k\}$, we obtain that

$$\lambda_{k+1} \geqslant \min\left\{\xi_k\lambda_k+\rho_k,\frac{\tau\delta_k}{2\max\{c_1,c_2\}}\right\} \geqslant \min\left\{\lambda_k,\frac{\tau\varphi}{2\max\{c_1,c_2\}}\right\}.$$

By induction, we get that the sequence $\{\lambda_k\}$ has a lower bound as $\min\left\{\lambda_1, \frac{\tau\varphi}{2\max\{c_1, c_2\}}\right\}$.

On the other hand, from the definition of λ_{k+1}, one sees that $\lambda_{k+1} \leqslant \xi_k\lambda_k + \rho_k$, for each $k \in \mathbb{N}$. So, by applying Lemma 2.12 and the conditions on the sequences $\{\xi_k\}$ and $\{\rho_k\}$, we have the limit of $\{\lambda_k\}$ exists. It follows from the choices of the parameters $\tau \in (0, 1)$, $\sigma \in \left(0, \frac{1}{2\tau}\right)$, $\eta \in \left[\sigma, \frac{1}{\tau}\right)$, and $\lim_{k\to\infty} \delta_k = 1$ that

$$\lim_{k\to\infty}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right) = \sigma\left(\frac{1}{\eta} - \tau\right) > 0.$$

Thus, there exists $k_0 \in \mathbb{N}$ such that

$$\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}} > 0, \quad \forall k \geqslant k_0.$$

This together with the results of Lemma 3.4 yields that

$$\|z_k - p\| \leqslant \|w_k - p\|, \tag{3.9}$$

for each $k \geqslant k_0$. Now, since $\lim_{k\to\infty} \alpha_k = 0$, we may assume, with no loss of generality, that $\alpha_k < \|A\|^{-1}$, for each $k \in \mathbb{N}$. Furthermore, from the definition of x_{k+1}, we suppose that $i_k \in \{1, 2, \ldots, M\}$ such that $u_k^{i_k} = x_{k+1} = \arg\max\left\{\|u_k^i - w_k\| : i = 1, 2, \ldots, M\right\}$. Using this one together with the expression (3.9), the nonexpansivity of S_{i_k}, $i_k \in \{1, 2, \ldots, M\}$, and the facts of Lemma 2.8, we obtain that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|\alpha_k(\gamma h(w_k) - Ap) + (I - \alpha_k A)(S_{i_k}z_k - p)\| \\
&\leqslant \alpha_k\|\gamma h(w_k) - Ap\| + \|I - \alpha_k A\|\|S_{i_k}z_k - p\| \\
&\leqslant \alpha_k\|\gamma h(w_k) - Ap\| + (1 - \alpha_k\beta)\|z_k - p\| \\
&\leqslant \gamma\rho\alpha_k\|w_k - p\| + \alpha_k\|\gamma h(p) - Ap\| + (1 - \alpha_k\beta)\|w_k - p\| \\
&= (1 - (\beta - \gamma\rho)\alpha_k)\,\|w_k - p\| + \alpha_k\|\gamma h(p) - Ap\|,
\end{aligned}$$

for each $k \geqslant k_0$. It follows from the definition of w_k that, for each $k \geqslant k_0$, we have

$$\begin{aligned}
\|x_{k+1} - p\| &\leqslant (1 - (\beta - \gamma\rho)\alpha_k)\,\|x_k - p\| + (1 - (\beta - \gamma\rho)\alpha_k)\,\theta_k\|x_k - x_{k-1}\| + \alpha_k\|\gamma h(p) - Ap\| \\
&= (1 - (\beta - \gamma\rho)\alpha_k)\,\|x_k - p\| + (\beta - \gamma\rho)\alpha_k\left(\psi_k + \frac{\|\gamma h(p) - Ap\|}{\beta - \gamma\rho}\right),
\end{aligned} \tag{3.10}$$

where $\psi_k = \left(\frac{1 - (\beta - \gamma\rho)\alpha_k}{\beta - \gamma\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\|$. Combining with the choices of the sequences $\{\theta_k\}$, we obtain that

$$\psi_k = \left(\frac{1 - (\beta - \gamma\rho)\alpha_k}{\beta - \gamma\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leqslant \left(\frac{1 - (\beta - \gamma\rho)\alpha_k}{\beta - \gamma\rho}\right)\frac{\epsilon_k}{\alpha_k},$$

for each $k \geqslant k_0$. Due to the facts that $\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0$ and $\lim_{k\to\infty} \alpha_k = 0$, we have

$$\lim_{k\to\infty} \psi_k = 0.$$

Thus, there exists a constant $M_1 > 0$ such that

$$\psi_k = \left(\frac{1 - (\beta - \gamma\rho)\alpha_k}{\beta - \gamma\rho}\right)\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leqslant M_1,$$

for each $k \geqslant k_0$. This together with the inequality (3.10) yields that

$$\|x_{k+1} - p\| \leqslant (1 - (\beta - \gamma\rho)\alpha_k)\,\|x_k - p\| + (\beta - \gamma\rho)\alpha_k\left(M_1 + \frac{\|\gamma h(p) - Ap\|}{\beta - \gamma\rho}\right)$$

$$\leqslant \max\left\{\|x_k - p\|, M_1 + \frac{\|\gamma h(p) - Ap\|}{\beta - \gamma\rho}\right\}$$
$$\vdots$$
$$\leqslant \max\left\{\|x_{k_0} - p\|, M_1 + \frac{\|\gamma h(p) - Ap\|}{\beta - \gamma\rho}\right\}.$$

This implies that the sequence $\{\|x_k - p\|\}$ is bounded. Consequently, $\{x_k\}$ is a bounded sequence.

Furthermore, from the definition of w_k, we provide the following:

$$\begin{aligned}
\|w_k - p\|^2 &= \|(1+\theta_k)(x_k - p) - \theta_k(x_{k-1} - p)\|^2 \\
&= (1+\theta_k)\|x_k - p\|^2 - \theta_k\|x_{k-1} - p\|^2 + \theta_k(1+\theta_k)\|x_k - x_{k-1}\|^2 \\
&\leqslant (1+\theta_k)\|x_k - p\|^2 - \theta_k\|x_{k-1} - p\|^2 + 2\theta_k\|x_k - x_{k-1}\|^2 \\
&= \|x_k - p\|^2 + \theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) + 2\theta_k\|x_k - x_{k-1}\|^2,
\end{aligned}$$

for each $k \geqslant k_0$. Thus, applying Lemma 3.4 to the above relation, we have

$$\begin{aligned}
\|z_k - p\|^2 \leqslant\ & \|x_k - p\|^2 + \theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) + 2\theta_k\|x_k - x_{k-1}\|^2 \\
& - \left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 - \left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2,
\end{aligned}$$

for each $k \geqslant k_0$. Using this one together with the definition of x_{k+1} and the nonexpansivity of S_{i_k}, $i_k \in \{1, 2, \ldots, M\}$, we get

$$\begin{aligned}
\|x_{k+1} - p\|^2 &= \|\alpha_k(\gamma h(w_k) - Ap) + (I - \alpha_k A)(S_{i_k} z_k - p)\|^2 \\
&= \left\|\alpha_k(\gamma h(w_k) - Ap) + (1-\alpha_k)\frac{I - \alpha_k A}{1-\alpha_k}(S_{i_k} z_k - p)\right\|^2 \\
&\leqslant \alpha_k\|\gamma h(w_k) - Ap\|^2 + (1-\alpha_k)\left\|\frac{I - \alpha_k A}{1-\alpha_k}(S_{i_k} z_k - p)\right\|^2 \\
&\leqslant \alpha_k\|\gamma h(w_k) - Ap\|^2 + \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\|z_k - p\|^2 \\
&\leqslant \alpha_k\|\gamma h(w_k) - Ap\|^2 + \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\|x_k - p\|^2 + 2\frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\theta_k\|x_k - x_{k-1}\|^2 \\
&\quad + \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) \\
&\quad - \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\left(\|w_k - y_k\|^2 + \|y_k - z_k\|^2\right),
\end{aligned}$$

for each $k \geqslant k_0$. This implies that

$$\begin{aligned}
&\frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2 + \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_k\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|^2 \\
&\leqslant \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\theta_k(\|x_k - p\|^2 - \|x_{k-1} - p\|^2) \\
&\quad + 2\frac{(1-\alpha_k\beta)^2}{1-\alpha_k}\theta_k\|x_k - x_{k-1}\|^2 + \alpha_k\|\gamma h(w_k) - Ap\|^2 + \left(\frac{(1-\alpha_k\beta)^2}{1-\alpha_k} - 1\right)\|x_k - p\|^2,
\end{aligned} \tag{3.11}$$

for each $k \geqslant k_0$. On the other hand, we know that $P_\Omega(I - A + \gamma h)$ is a contraction on H. Indeed, by applying Lemma 2.8, one sees that

$$\|P_\Omega(I - A + \gamma h)(x) - P_\Omega(I - A + \gamma h)(y)\| \leqslant \|(I - A + \gamma h)(x) - (I - A + \gamma h)(y)\|$$

$$\leqslant \|(I-A)(x)-(I-A)(y)\|+\gamma\|h(x)-h(y)\|$$
$$\leqslant (1-\beta)\|x-y\|+\gamma\rho\|x-y\| = (1-(\beta-\gamma\rho))\|x-y\|,$$

for each $x, y \in H$, thus, $P_\Omega(I-A+\gamma h)$ is a contraction on H and so we know that there exists $\tilde{p} \in \Omega$ such that $\tilde{p} = P_\Omega(I-A+\gamma h)(\tilde{p})$. Now, we are in a position to show that the sequence $\{x_k\}$ converges strongly to $\tilde{p} = P_\Omega(I-A+\gamma h)(\tilde{p})$ by considering the following two possible cases.

Case 1. Suppose that $\|x_{k+1}-\tilde{p}\| \leqslant \|x_k-\tilde{p}\|$, for all $k \geqslant k_0$. This means that $\{\|x_k-\tilde{p}\|\}_{k\geqslant k_0}$ is a nonincreasing sequence. Consequently, by utilizing this fact together with the boundness property of $\{\|x_k-\tilde{p}\|\}$, we obtain that the limit of $\|x_k-\tilde{p}\|$ exists. Using this one together with the relation (3.11), the fact that $\lim_{k\to\infty} \theta_k\|x_k-x_{k-1}\|^2 = 0$, and the properties of the control sequences $\{\alpha_k\}$ and $\{\theta_k\}$, we have

$$\lim_{k\to\infty} \|w_k-y_k\| = 0, \tag{3.12}$$

and

$$\lim_{k\to\infty} \|y_k-z_k\| = 0. \tag{3.13}$$

These imply that

$$\lim_{k\to\infty} \|w_k-z_k\| = 0. \tag{3.14}$$

Additionally, from the definition of w_k and the fact that $\lim_{k\to\infty} \theta_k\|x_k-x_{k-1}\| = 0$, we get

$$\lim_{k\to\infty} \|x_k-w_k\| = 0. \tag{3.15}$$

It follows from (3.12) that

$$\lim_{k\to\infty} \|x_k-y_k\| = 0. \tag{3.16}$$

This together with (3.13) yields that

$$\lim_{k\to\infty} \|x_k-z_k\| = 0. \tag{3.17}$$

On the other hand, from the definition of x_{k+1} and the expression (3.9), we have

$$\begin{aligned}
&\|x_{k+1}-\tilde{p}\|^2\\
&= \|\alpha_k(\gamma h(w_k)-A\tilde{p})+(I-\alpha_k A)(S_{i_k}z_k-\tilde{p})\|^2\\
&\leqslant (1-\alpha_k\beta)^2\|z_k-\tilde{p}\|^2+2\alpha_k\langle\gamma h(w_k)-A\tilde{p}, x_{k+1}-\tilde{p}\rangle\\
&\leqslant (1-\alpha_k\beta)^2\|w_k-\tilde{p}\|^2+2\alpha_k\langle\gamma h(w_k)-A\tilde{p}, x_{k+1}-\tilde{p}\rangle\\
&= (1-\alpha_k\beta)^2\|w_k-\tilde{p}\|^2+2\alpha_k\langle\gamma h(w_k)-\gamma h(\tilde{p}), x_{k+1}-\tilde{p}\rangle+2\alpha_k\langle\gamma h(\tilde{p})-A\tilde{p}, x_{k+1}-\tilde{p}\rangle\\
&\leqslant (1-\alpha_k\beta)^2\|w_k-\tilde{p}\|^2+2\gamma\rho\alpha_k\|w_k-\tilde{p}\|\|x_{k+1}-\tilde{p}\|+2\alpha_k\langle\gamma h(\tilde{p})-A\tilde{p}, x_{k+1}-\tilde{p}\rangle\\
&\leqslant (1-\alpha_k\beta)^2\|w_k-\tilde{p}\|^2+\gamma\rho\alpha_k\left(\|w_k-\tilde{p}\|^2+\|x_{k+1}-\tilde{p}\|^2\right)+2\alpha_k\langle\gamma h(\tilde{p})-A\tilde{p}, x_{k+1}-\tilde{p}\rangle\\
&= \left((1-\alpha_k\beta)^2+\gamma\rho\alpha_k\right)\|w_k-\tilde{p}\|^2+\gamma\rho\alpha_k\|x_{k+1}-\tilde{p}\|^2+2\alpha_k\langle\gamma h(\tilde{p})-A\tilde{p}, x_{k+1}-\tilde{p}\rangle,
\end{aligned} \tag{3.18}$$

for each $k \geqslant k_0$. Besides, in view of the definition of w_k, we observe that

$$\begin{aligned}
\|w_k-\tilde{p}\|^2 &\leqslant (\|x_k-\tilde{p}\|+\theta_k\|x_k-x_{k-1}\|)^2\\
&\leqslant \|x_k-\tilde{p}\|^2+2\theta_k\|x_k-\tilde{p}\|\|x_k-x_{k-1}\|+\theta_k\|x_k-x_{k-1}\|^2\\
&\leqslant \|x_k-\tilde{p}\|^2+3M_2\theta_k\|x_k-x_{k-1}\|,
\end{aligned}$$

where $M_2 = \sup_{k\geqslant k_0}\{\|x_k-\tilde{p}\|, \|x_k-x_{k-1}\|\}$. Combining with the relation (3.18), we get

$$\|x_{k+1}-\tilde{p}\|^2 \leqslant \left(\frac{(1-\alpha_k\beta)^2+\gamma\rho\alpha_k}{1-\gamma\rho\alpha_k}\right)\|x_k-\tilde{p}\|^2+3M_2\left(\frac{(1-\alpha_k\beta)^2+\gamma\rho\alpha_k}{1-\gamma\rho\alpha_k}\right)\theta_k\|x_k-x_{k-1}\|$$

$$
\begin{aligned}
&+\left(\frac{2\alpha_k}{1-\gamma\rho\alpha_k}\right)\langle\gamma h(\tilde{p})-A\tilde{p},x_{k+1}-\tilde{p}\rangle\\
&\leqslant\left(1-\frac{2(\beta-\gamma\rho)\alpha_k}{1-\gamma\rho\alpha_k}\right)\|x_k-\tilde{p}\|^2+\frac{2(\beta-\gamma\rho)\alpha_k}{1-\gamma\rho\alpha_k}\left(\frac{\beta^2\alpha_k\|x_k-\tilde{p}\|^2}{2(\beta-\gamma\rho)}\right.\\
&\left.+3M_2\left(\frac{(1-\alpha_k\beta)^2+\gamma\rho\alpha_k}{2(\beta-\gamma\rho)}\right)\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\|+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-A\tilde{p},x_{k+1}-\tilde{p}\rangle\right)\\
&\leqslant\left(1-\frac{2(\beta-\gamma\rho)\alpha_k}{1-\gamma\rho\alpha_k}\right)\|x_k-\tilde{p}\|^2+\frac{2(\beta-\gamma\rho)\alpha_k}{1-\gamma\rho\alpha_k}\left(\frac{\beta^2\alpha_k M_3}{2(\beta-\gamma\rho)}\right.\\
&\left.+3M_2\left(\frac{(1-\alpha_k\beta)^2+\gamma\rho\alpha_k}{2(\beta-\gamma\rho)}\right)\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\|+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-A\tilde{p},x_{k+1}-\tilde{p}\rangle\right),
\end{aligned}
$$

where $M_3=\sup_{k\geqslant k_0}\{\|x_k-\tilde{p}\|^2\}$. Put $\zeta_k=\frac{2(\beta-\gamma\rho)\alpha_k}{1-\gamma\rho\alpha_k}$. This together with the above inequality yields that

$$
\begin{aligned}
\|x_{k+1}-\tilde{p}\|^2\leqslant(1-\zeta_k)\|x_k-\tilde{p}\|^2+\zeta_k&\left(3M_2\left(\frac{(1-\alpha_k\beta)^2+\gamma\rho\alpha_k}{2(\beta-\gamma\rho)}\right)\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\|\right.\\
&\left.+\frac{\beta^2\alpha_k M_3}{2(\beta-\gamma\rho)}+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-A\tilde{p},x_{k+1}-\tilde{p}\rangle\right),
\end{aligned}
\tag{3.19}
$$

for each $k\geqslant k_0$. Furthermore, by the assumption on the sequence $\{\alpha_k\}$, one sees that

$$
\sum_{k=1}^{\infty}\zeta_k=\infty. \tag{3.20}
$$

Now, let $x^*\in\omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n}\rightharpoonup x^*$, as $n\to\infty$. We know that, by utilizing (3.16) and (3.17), we also have $y_{k_n}\rightharpoonup x^*$ and $z_{k_n}\rightharpoonup x^*$, as $n\to\infty$. Since C is closed and convex set, so C is weakly closed, therefore, $x^*\in C$.

Next, due to the relations (3.3), (3.6), and (3.8), we get

$$
\begin{aligned}
\sigma\lambda_{k_n}f(y_{k_n},y)&\geqslant\sigma\lambda_{k_n}f(y_{k_n},z_{k_n})+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle\\
&\geqslant\sigma\lambda_{k_n}f(w_{k_n},z_{k_n})-\sigma\lambda_{k_n}f(w_{k_n},y_{k_n})-\frac{\sigma\tau\delta_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n}-y_{k_n}\|^2\\
&\quad-\frac{\sigma\tau\delta_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n}-z_{k_n}\|^2+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle\\
&\geqslant\frac{\sigma}{\eta}\langle y_{k_n}-w_{k_n},y_{k_n}-z_{k_n}\rangle-\frac{\sigma\tau\delta_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|w_{k_n}-y_{k_n}\|^2-\frac{\sigma\tau\delta_{k_n}\lambda_{k_n}}{2\lambda_{k_n+1}}\|y_{k_n}-z_{k_n}\|^2\\
&\quad+\langle w_{k_n}-z_{k_n},y-z_{k_n}\rangle,
\end{aligned}
$$

for each $y\in C$. Using this one together with (3.12), (3.13), (3.14), and the boundedness of $\{z_k\}$, we have the right-hand side of the above inequality tends to zero. Thus, by applying the sequentially weakly upper semicontinuity of f and the parameters $\sigma,\lambda_{k_n}>0$, we obtain that

$$
0\leqslant\limsup_{n\to\infty}f(y_{k_n},y)\leqslant f(x^*,y),\quad\forall y\in C.
$$

This means that $x^*\in EP(f,C)$. On the other hand, from the definition of w_k, one sees that

$$
\theta_k\|x_k-x_{k-1}\|=\|w_k-x_k\|.
$$

Combining with the fact (3.15), we have

$$\lim_{k\to\infty}\|x_k - x_{k-1}\| = 0. \tag{3.21}$$

In addition, since $\|x_{k+1} - w_k\| \leqslant \|x_{k+1} - x_k\| + \|x_k - w_k\|$, it follows from (3.15) and (3.21) that

$$\lim_{k\to\infty}\|x_{k+1} - w_k\| = 0.$$

This together with the definition of x_{k+1} yields that

$$\lim_{k\to\infty}\|u_k^i - w_k\| = 0, \tag{3.22}$$

for each $i = 1, 2, \dots, M$. Furthermore, by the definition of u_k^i, we get

$$\|u_k^i - S_i z_k\| = \alpha_k\|\gamma h(w_k) - AS_i z_k\|, \tag{3.23}$$

for each $i = 1, 2, \dots, M$. Using this one together with the boundedness of $\{AS_i z_k\}$, $\{h(w_k)\}$, and the condition that $\lim_{k\to\infty}\alpha_k = 0$, we have

$$\lim_{k\to\infty}\|u_k^i - S_i z_k\| = 0,$$

for each $i = 1, 2, \dots, M$. Moreover, since $\|S_i z_k - z_k\| \leqslant \|S_i z_k - u_k^i\| + \|u_k^i - w_k\| + \|w_k - z_k\|$, it follows from (3.14), (3.22), and (3.23) that

$$\lim_{k\to\infty}\|S_i z_k - z_k\| = 0,$$

for each $i = 1, 2, \dots, M$. This together with the demiclosedness at 0 of $I - S_i, i = 1, 2, \dots, M$, and $z_{k_n} \rightharpoonup x^*$, as $n \to \infty$, gives $x^* \in F(S_i)$, for each $i = 1, 2, \dots, M$. Then, we had shown that $x^* \in \Omega$, and so $\omega_w(x_k) \subset \Omega$. Finally, by the properties that $\tilde{p} = P_\Omega(I - A + \gamma h)(\tilde{p})$ and $x^* \in \omega_w(x_k) \subset \Omega$, we have

$$\limsup_{k\to\infty}\langle A\tilde{p} - \gamma h(\tilde{p}), \tilde{p} - x_{k+1}\rangle = \lim_{n\to\infty}\langle A\tilde{p} - \gamma h(\tilde{p}), \tilde{p} - x_{k_n+1}\rangle = \langle A\tilde{p} - \gamma h(\tilde{p}), \tilde{p} - x^*\rangle \leqslant 0. \tag{3.24}$$

Hence, by using (3.19), (3.20), (3.24), and Lemma 2.13, we obtain that

$$\lim_{k\to\infty}\|x_k - \tilde{p}\| = 0,$$

This completes the proof for the first case.

Case 2. Suppose that there exists a subsequence $\{\|x_{k_j} - \tilde{p}\|\}$ of $\{\|x_k - \tilde{p}\|\}$ such that

$$\|x_{k_j} - \tilde{p}\| < \|x_{k_j+1} - \tilde{p}\|, \ \forall j \in \mathbb{N}.$$

According to Lemma 2.14, there exists a nondecreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} m_n = \infty$, and

$$\|x_{m_n} - \tilde{p}\| \leqslant \|x_{m_n+1} - \tilde{p}\| \ \text{ and } \ \|x_n - \tilde{p}\| \leqslant \|x_{m_n+1} - \tilde{p}\|, \ \forall n \in \mathbb{N}. \tag{3.25}$$

Using this one together with the relation (3.11), we get

$$\begin{aligned}
&\frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_{m_n}\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|w_{m_n} - y_{m_n}\|^2 + \frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\left(\frac{\sigma}{\eta} - \frac{\tau\sigma\delta_{m_n}\lambda_{m_n}}{\lambda_{m_n+1}}\right)\|y_{m_n} - z_{m_n}\|^2 \\
&\leqslant \|x_{m_n} - p\|^2 - \|x_{m_n+1} - p\|^2 + \frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\theta_{m_n}(\|x_{m_n} - p\|^2 - \|x_{m_n-1} - p\|^2) \\
&\quad + 2\frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|^2 + \alpha_{m_n}\|\gamma h(w_{m_n}) - Ap\|^2 + \left(\frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}} - 1\right)\|x_{m_n} - p\|^2
\end{aligned}$$

$$\leqslant \frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\theta_{m_n}(\|x_{m_n}-p\|^2-\|x_{m_n-1}-p\|^2)+\left(\frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}-1\right)\|x_{m_n}-p\|^2$$
$$+2\frac{(1-\alpha_{m_n}\beta)^2}{1-\alpha_{m_n}}\theta_{m_n}\|x_{m_n}-x_{m_n-1}\|^2+\alpha_{m_n}\|\gamma h(w_{m_n})-Ap\|^2.$$

Following the proof of Case 1, we can show that for each $i=1,2,\dots,M$,

$$\begin{aligned}\lim_{n\to\infty}\|w_{m_n}-y_{m_n}\|=0,\quad \lim_{n\to\infty}\|y_{m_n}-z_{m_n}\|=0,\quad \lim_{n\to\infty}\|w_{m_n}-z_{m_n}\|=0,\\ \lim_{n\to\infty}\|x_{m_n}-y_{m_n}\|=0,\quad \lim_{n\to\infty}\|x_{m_n}-z_{m_n}\|=0,\quad \lim_{n\to\infty}\|S_i z_{m_n}-z_{m_n}\|=0,\end{aligned} \tag{3.26}$$

$$\limsup_{n\to\infty}\langle A\tilde{p}-\gamma h(\tilde{p}),\tilde{p}-x_{m_n+1}\rangle=\langle A\tilde{p}-\gamma h(\tilde{p}),\tilde{p}-x^*\rangle\leqslant 0,\quad \forall x^*\in\omega_w(x_n)\subset\Omega, \tag{3.27}$$

and

$$\|x_{m_n+1}-\tilde{p}\|^2\leqslant(1-\zeta_{m_n})\|x_{m_n}-\tilde{p}\|^2+\zeta_{m_n}\left(3M_2\left(\frac{(1-\alpha_{m_n}\beta)^2+\gamma\rho\alpha_{m_n}}{2(\beta-\gamma\rho)}\right)\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|\right.$$
$$\left.+\frac{\beta^2\alpha_{m_n}M_3}{2(\beta-\gamma\rho)}+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-\tilde{p},x_{m_n+1}-\tilde{p}\rangle\right).$$

where $M_2=\sup_{n\in\mathbb{N}}\{\|x_{m_n}-\tilde{p}\|,\|x_{m_n}-x_{m_n-1}\|\}$ and $M_3=\sup_{n\in\mathbb{N}}\left\{\|x_{m_n}-\tilde{p}\|^2\right\}$. It follows from the expressions (3.25) that

$$\|x_{m_n+1}-\tilde{p}\|^2\leqslant(1-\zeta_{m_n})\|x_{m_n+1}-\tilde{p}\|^2+\zeta_{m_n}\left(3M_2\left(\frac{(1-\alpha_{m_n}\beta)^2+\gamma\rho\alpha_{m_n}}{2(\beta-\gamma\rho)}\right)\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|\right.$$
$$\left.+\frac{\beta^2\alpha_{m_n}M_3}{2(\beta-\gamma\rho)}+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-\tilde{p},x_{m_n+1}-\tilde{p}\rangle\right).$$

Combining with the expressions (3.25) again, we obtain

$$\|x_n-\tilde{p}\|^2\leqslant 3M_2\left(\frac{(1-\alpha_{m_n}\beta)^2+\gamma\rho\alpha_{m_n}}{2(\beta-\gamma\rho)}\right)\frac{\theta_{m_n}}{\alpha_{m_n}}\|x_{m_n}-x_{m_n-1}\|+\frac{\beta^2\alpha_{m_n}M_3}{2(\beta-\gamma\rho)}$$
$$+\frac{1}{\beta-\gamma\rho}\langle\gamma h(\tilde{p})-\tilde{p},x_{m_n+1}-\tilde{p}\rangle.$$

Then, by using the relation (3.27), the choices of the sequences $\{\theta_k\}$, and the condition that $\lim_{k\to\infty}\alpha_k=0$, we have

$$\limsup_{n\to\infty}\|x_n-\tilde{p}\|^2\leqslant 0.$$

Hence, we can conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{p}=P_\Omega(I-A+\gamma h)(\tilde{p})$. This completes the proof. □

4. Numerical experiments

This section will consider some examples and numerical results to verify and justify the presented Theorem 3.5. In Example 4.1, we will provide to demonstrate the effectiveness of the proposed Algorithm 3.2. In the case $M=1$, we will compare the introduced Algorithm 3.2 with Algorithm (1.6) in Example 4.2. All the numerical experiments are carried out using Matlab R2021b and executed on an Apple M1 with 8.00 GB RAM. In both two Examples 4.1 and 4.2, the $\|\cdot\|$ represents the spectral norm for each considered matrix.

Example 4.1. Let $H = \mathbb{R}^n$ be n-dimensional vector space equipped with the Euclidean norm. For the constrained box $C = \{x \in \mathbb{R}^n : -5 \leqslant x_j \leqslant 5, \ \forall j = 1, 2, \ldots, n\}$, we will consider a classical form of the bifunction which arises from the Nash-Cournot models, see [18],

$$g(x,y) = \langle Px + r^n(y+x), y-x\rangle, \ \forall x,y \in \mathbb{R}^n,$$

where

$$P = \begin{pmatrix} 0 & r & r & \cdots & r \\ r & 0 & r & \cdots & r \\ r & r & 0 & \cdots & r \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ r & r & \cdot & \cdots & 0 \end{pmatrix}_{n\times n},$$

when r is a positive real number. Following [3], we know that the bifunctions g is pseudomonotone and it is not monotone on C. Next, we consider the bifunction f which is given by

$$f(x,y) = \begin{cases} g(x,y), & \text{if } (x,y) \in C \times C, \\ 0, & \text{otherwise,} \end{cases}$$

see [32]. Observe that f satisfies Lipschitz-type continuous. On the other hand, for the boxes Q_i, $i = 1, 2, \ldots, M$, which are taken by

$$Q_i = \{x \in \mathbb{R}^n : -q_i \leqslant x_j \leqslant q_i, \ \ \forall j = 1, 2, \ldots, n\}, \ i = 1, 2, \ldots, M,$$

where q_i are the positive real numbers, the nonexpansive mappings S_i, $i = 1, 2, \ldots, M$, are generated by

$$S_i = P_{Q_i}, \ i = 1, 2, \ldots, M.$$

Furthermore, we set $A(x) = \frac{x}{3}$ and $h(x) = \frac{x}{5}$ which are the strongly positive bounded linear mapping with coefficient $\frac{1}{3}$ and the contraction mapping with coefficient $\frac{1}{5}$, respectively and choose $\gamma = 0.9$.

The numerical experiments are considered under the following details regarding control parameters setting: $\lambda_1 = 0.9$, $\mu = 0.5$, $\varphi = 0.4$, $\tau = 0.3$, $\alpha_k = \frac{1}{k+1}$, $\epsilon_k = \frac{1}{(k+1)^2}$, and $\theta_k = \overline{\theta}_k$. Besides, the positive real number r is generated randomly in the interval $(1, 1.001)$ and the positive real numbers q_i, $i = 1, 2, \ldots, M$, are generated randomly in the interval $(0, 3)$. The starting points $x_0 = x_1 \in \mathbb{R}^n$ are generated randomly with its elements being in the interval $[-5, 5]$. Also, by randomly 10 starting points and the reported results are average. The stopping criterion used for the numerical computation of Algorithm 3.2 is $\frac{\|x_{k+1}-x_k\|}{\|x_k\|+1} < 10^{-6}$ when $n = 10$ and $M = 50$.

To see the optimum values of the control parameters, the first experiment was carried out taking into account the variation of the control parameters $\sigma = 1, 1.2, 1.4, 1.6$ and $\eta = 1, 1.2, 1.4, 1.6, 1.8$ by fixing the control parameters $\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$, $\rho_k = \frac{1}{(k+1)^{1.2}}$, and $\delta_k = \frac{k}{k+1}$. We omit the combinations that do not satisfy the assumption in Theorem 3.5 and label it by −.

Table 1: Numerical behavior of Algorithm 3.2 with different parameters σ and η, where $\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$, $\rho_k = \frac{1}{(k+1)^{1.2}}$, and $\delta_k = \frac{k}{k+1}$ in Example 4.1.

	$\eta = 1$		$\eta = 1.2$		$\eta = 1.4$		$\eta = 1.6$		$\eta = 1.8$	
σ	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
1	47.1	1.08	50.9	0.87	55.2	0.95	60.1	1.05	63.7	1.08
1.2	-	-	43.6	0.78	47.7	0.84	50.7	0.89	55.1	0.96
1.4	-	-	-	-	43.5	0.74	45.6	0.78	48.1	0.85
1.6	-	-	-	-	-	-	40.5	0.69	43.8	0.74

From Table 1, the number of iterations (Iter) and the CPU time (Time) in seconds are presented. It is important to highlight that the careful selection of relevant parameters in Algorithm 3.2, specifically the values of σ and η, hold significance. The optimal choices for these parameters are found to be $\sigma = 1.6$ and $\eta = 1.6$, considering both the number of iterations and CPU time across all examined cases. We note that choosing $\tau = 0.3$ allows σ and η to take on values greater than 1. One can observe that, for each fixed η, both the iteration number and CPU time efficiency of the algorithm show improved behavior with respect to σ. However, the situation appears to reverse when we fix each σ and consider the efficiency with respect to η. This observation supports the overall choice of σ and η mentioned above, which establishes the superiority of the algorithm. Nevertheless, the results presented in Table 1 indicate that the algorithm's superiority does not exhibit a monotonic trend correlated to τ, as $\tau = 0.27$ gives $\frac{1}{2\tau}$ approximately to 1.85. This suggests that just roughly reducing the parameter value of τ to a positive real number less than 0.3 and then choosing the possible highest value of σ, may not ensure an improved behavior of the constructed sequence.

In the upcoming experiment, we examine the numerical behavior of the control parameters ξ_k, ρ_k, and δ_k while maintaining fixed values for the control parameters $\sigma = 1.6$ and $\eta = 1.6$. The numerical results are presented for various values of control parameters $\xi_k = 1, 1 + \frac{1}{(k+1)^{1.2}}$, and $\rho_k = 0, \frac{1}{(k+1)^{1.2}}$, and $\delta_k = 1, \frac{k}{k+1}$. We observe that when $\xi_k = 1$, $\rho_k = 0$, and $\delta_k = 1$, the step size considered in Algorithm 3.2 reduces to a form equivalent to Algorithm (1.6).

Table 2: Numerical behavior of Algorithm 3.2 with different parameters ξ_k, ρ_k, and δ_k,with fixed values for $\sigma = 1.6$ and $\eta = 1.6$.

	$\delta_k = 1$				$\delta_k = \frac{k}{k+1}$			
	$\xi_k = 1$		$\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$		$\xi_k = 1$		$\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$	
ρ_k	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0	68.0	1.38	44.4	0.79	75.4	1.35	43.8	0.82
$\frac{1}{(k+1)^{1.2}}$	41.6	0.74	41.0	0.74	40.2	0.71	39.0	0.69

From Table 2, for each fixed parameter ξ_k, ρ_k, and δ_k, we observe that the chosen parameters $\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$, $\rho_k = \frac{1}{(k+1)^{1.2}}$, and $\delta_k = \frac{k}{k+1}$ exhibit better performance than the parameters $\xi_k = 1, \rho_k = 0$, and $\delta_k = 1$, respectively, in terms of both iteration number and CPU time. This demonstrates that opting for the suggested alternatives of parameters ξ_k, ρ_k, and δ_k, in Algorithm 3.2, as proposed in this paper, leads to improved performance in solving this kind of problem. Indeed, it is evident that choosing $\xi_k = 1 + \frac{1}{(k+1)^{1.2}}, \rho_k = \frac{1}{(k+1)^{1.2}}$, and $\delta_k = \frac{k}{k+1}$ results in the highest performance for Algorithm 3.2.

Example 4.2. In the case $M = 1$, we perform some computational experiments to solve the image restoration problems by comparing Algorithm 3.2 with Algorithm (1.6). It is known that all images have $n := N_1 \times N_2$ pixels and each pixel value is in the range $[0, 255]$. Here, let $H = \mathbb{R}^n$ be a real Hilbert space equipped with the Euclidean norm and $C = \{x \in \mathbb{R}^n : 0 \leqslant x_j \leqslant 255, \ \forall j = 1, 2, \ldots, n\}$ be a constrained box.

Let us consider the image restoration problem, which can be modeled by the following linear equation system:

$$v = Ux + w, \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the original image, $v \in \mathbb{R}^n$ is the degraded image, $w \in \mathbb{R}^n$ is additive noise, and $U \in \mathbb{R}^{n \times n}$ is the blurring matrix. In order to solve (4.1), we aim to approximate the original image, vector x, by minimizing the additive noise, by using the following minimization problem:

$$\min_{x \in C} \frac{1}{2} \|Ux - v\|^2,$$

see [34]. To be considered here is the bifunction f, which is defined by

$$f(x, y) = g(y) - g(x), \ \forall x, y \in \mathbb{R}^n,$$

where $g(x) := \frac{1}{2}\|Ux - v\|^2$. It is clear that

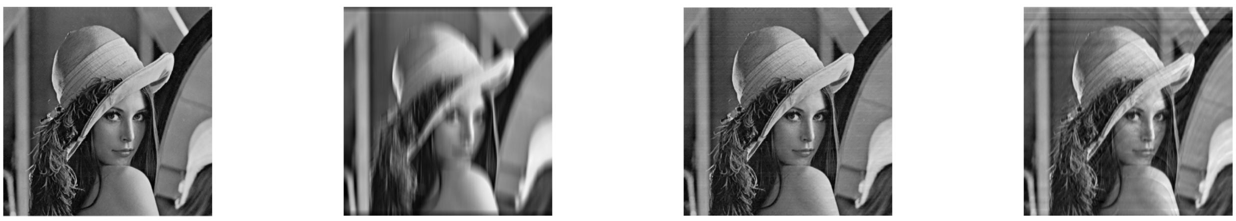
$$f(x, y) + f(y, x) = 0, \ \forall x, y \in \mathbb{R}^n.$$

Thus, the bifunction f is a monotone. Observe that the bifunction f satisfies Lipschitz-type continuous. On the other hand, the nonexpansive mapping S is given by $Sx = x$, $\forall x \in \mathbb{R}^n$.

Consider the following details regarding control parameters setting: $\lambda_1 = 0.9$, $\gamma = 0.9$, $\mu = 0.5$, $\varphi = 0.4$, $\tau = 0.3$, $\sigma = 1.6$, $\eta = 1.6$, $\alpha_k = \frac{1}{k+1}$, $\epsilon_k = \frac{1}{(k+1)^2}$, $\xi_k = 1 + \frac{1}{(k+1)^{1.2}}$, $\rho_k = \frac{1}{(k+1)^{1.2}}$, $\delta_k = \frac{k}{k+1}$, and $\theta_k = \overline{\theta}_k$ by using the strongly positive bounded linear mapping A and the contraction mapping h as in Example 4.1. The starting points $x_0 = x_1 \in \mathbb{R}^n$ are generated randomly with its elements being in the interval $[0, 1]$. Algorithm 3.2 was tested along with Algorithm (1.6) by using the stopping criteria as the number of iterations 1000. In all comparisons, we will work for two grayscale images, Lena and Barbara with sizes of 343×343 and 512×512, respectively as the original images. The degraded images are obtained by adding motion blur with a motion length of 15 pixels and motion orientation 60° to the original images. The quality of the restored image is measured by the signal-to-noise ratio (SNR) in decibel (dB), which is defined by

$$\mathrm{SNR} = 20 \log_{10} \frac{\|x\|}{\|x - x_k\|},$$

where x is the original image and x_k is the restored image at iteration k. A higher SNR value means that the restored image is of higher quality. That is, the SNR value increases when the restored image x_k tends to the original image x. The restored images of the 1000^{th} iteration are shown in Figures 1 and 2, respectively. Meanwhile, the SNR values are presented in Figure 3.

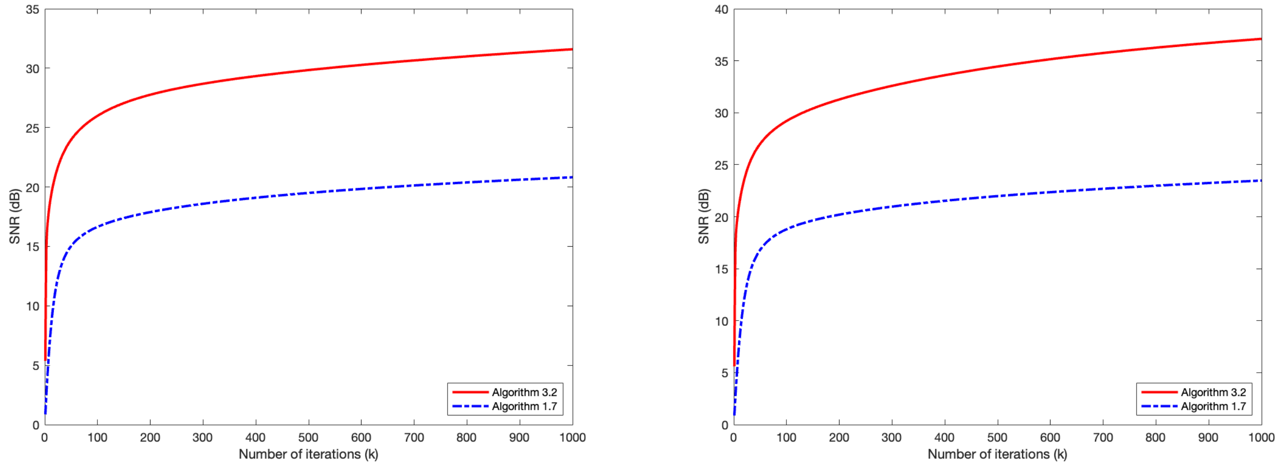


(a) Original image (b) Degraded image (c) Algorithm 3.2 SNR = 31.59 dB (d) Algorithm (1.6) SNR = 20.82 dB

Figure 1: Comparison of the restored images of Lena at 1000^{th} iteration in Example 4.2.

(a) Original image (b) Degraded image (c) Algorithm 3.2 SNR = 37.10 dB (d) Algorithm (1.6) SNR = 23.47 dB

Figure 2: Comparison of the restored images of Barbara at 1000^{th} iteration in Example 4.2



(a) The Lena image with size 343 × 343

(b) The Barbara image with size 512 × 512

Figure 3: The behavior of SNR values of the images with different sizes in Example 4.2.

From Figures 1 and 2, one can observe that Algorithm 3.2 yields higher SNR values compared to Algorithm (1.6) for both considered images. Furthermore, the plots in Figures 3 show that Algorithm 3.2 provides a more efficient solution than Algorithm (1.6) at each iteration. These observations lead to the conclusion that making alternative choices for auxiliary parameters τ, σ, η, and ξ_k, ρ_k, δ_k, as allowed in Algorithm 3.2, can enhance the performance in solving problems such as restoring images in this example.

5. Conclusions

We propose an algorithm aimed at identifying a unique solution to a minimization problem induced by a bounded linear operator and contraction mapping. This problem involves common elements found in the set of fixed points of a finite family of nonexpansive mappings and the solution set of a pseudomonotone equilibrium problem within the context of a real Hilbert space. By incorporating auxiliary parameters and utilizing the parallel viscosity concept in the modification of the inertial and subgradient extragradient methods, we establish a sequence that strongly converges to the unique solution of the aforementioned minimization problem. Numerical experiments demonstrate that auxiliary parameters, such as τ, σ, η, along with the sequences ξ_k, ρ_k, and δ_k, enhance the efficiency of Algorithm 3.2 itself, as well as improving other relevant algorithms. We observe that the efficiency of the constructed sequence appears to follow the choice of these auxiliary parameters, as discussed in Section 4. Hence, it would be interesting in future research papers to explore the behavior of these auxiliary parameters that lead to the superior convergence behavior of the sequence induced by Algorithm 3.2. Another aspect of future research directions involves refining the auxiliary parameters to enhance convergence speed and investigating the convergence rate of the iteration. Additionally, extending the scope of this work to encompass applications in signal processing and other domains is also a promising avenue for further exploration.

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