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Controlling the unpredictable: bifurcation and backstepping strategies in supply chain dynamics

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Abstract

Bifurcation points in a chaotic system represent critical thresholds where the system undergoes a qualitative change in behavior. In the context of supply chains, bifurcation points may signify shifts in demand patterns, disruptions in the flow of materials, or changes in market conditions. In this paper, we explore the intricacies of complexity within the Supply Chain Management Model (SCMM). The primary objective of this study involves an examination of the stability of the SCMM, revealing Hopf bifurcation, transcritical bifurcation, and double-zero bifurcation within the system. Additionally, we delve into the dynamical characteristics of the SCMM through the utilization of bifurcation diagrams and Lyapunov exponents. The findings indicate that the SCMM exhibits periodic, chaotic, and reverse period-doubling behaviors. To further comprehend the dynamics of the SCMM, we employ backstepping controllers to manage the chaotic SCMM and achieve synchronization between two SCMMs. Through numerical simulations, we demonstrate the effectiveness and applicability of the proposed methodologies.

Keywords: Chaos, SCMM, bifurcation, Hopf bifurcation and backstepping controllers.

2020 MSC: 34H10, 34C23, 93C10, 91B74.

1. Introduction

Supply Chain Management (SCM) involves the coordination and integration of various activities and processes within a network of organizations that collaborate to deliver a product or service to the end customer [3, 12]. The key entities in a supply chain typically include retailers, distributors, and manufacturers [1, 27]. In summary, they play distinct but interconnected roles in the supply chain. Effective

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coordination and collaboration among these entities is essential for achieving efficiency, responsiveness, and customer satisfaction throughout the supply chain [7, 16].

Several attribute properties within the SCMM defy adequate description through basic analysis. Therefore, it becomes imperative to engage in theoretical exploration of the complexity inherent in the supply chain system. This involves employing methods such as bifurcation and chaos control within the framework of nonlinear dynamics. Numerous studies have explored the manifestation of chaos in SCMM. For instance, Xu et al. [31] introduced an ASTSM control algorithm to address chaotic SCMM. The theory control can be extended to novel integration software for operational management within SCMM networks. Kocamaz et al. [14] demonstrated the control of a chaotic SCMM using ANN based controllers. They also presented the synchronization of two identical SCMM with distinct initial conditions using ANFIS technique. Göksu et al. [8] proposed the mathematical model of a control chaotic SCMM system, employing linear feedback controllers with Lyapunov stability theory. Nav et al. [21] investigated SCMM with a smooth ordering policy and introduced a new policy designed based on a proportional-derivative technique. Açıkgöz et al. [2] examined a three-dimensional SCMM, utilizing it to stabilize the system by introducing a linear control to increase production and prevent a collapse leading to dangerous instability. Chen and Zhou [6] developed a dynamic price and advertising game model for an omni-channel SCMM involving online purchases with in-store pickups. They explored the stability and complexity of migration rates for online and traditional consumers, as well as the sharing rate of advertising costs. Tian et al. [28] analyzed a Stackelberg dynamic model for a multi-channel SCMM, involving a manufacturer and two retailers. This study focused on a manufacturer providing a single product to traditional retailers, online retailers, and a direct channel. Peng et al. [22] proposed a novel SCMM sensitive to various uncertainties and exogenous disturbances. They discussed the impulsive synchronization SCMMs with identical structures. Qian et al. [23] investigated a SCMM under technology subsidies, involving a general contractor and two green building material manufacturers. The research examined how individuals involved in the green building material industry make decisions related to technological innovation within the SCMM under a dynamic game. Han and Wang [11] explored the chaotic behaviors of nonlinear SCMM using control theory, nonlinear system theory, chaos theory, and confirmed through the Lyapunov exponent that the behavior of demand information-sharing SCMM members alternates between chaotic and periodic movement.

The primary hurdle in this domain involves formulating control and management strategies that guarantee the systems operate effectively. Control theory has laid a solid groundwork for addressing the complexities of nonlinear dynamics. Studies related to control in supply chain have been widely studied in the field of chaos such as fixed-time super-twisting sliding mode [30], robust $H\infty$ control [20], unidirectional and bidirectional [19], radial basis function [5], nonlinear control [32], ANFIS control [10], fuzzy adaptive control [17], delayed feedback control [18], and fault-tolerant control strategy [4]. Based on this literature, backstepping control is not reported for control strategy for the SCMM. In our study, backstepping control provides a systematic way to design controllers that are robust to uncertainties and disturbances in the system. The method allows for the incorporation of adaptive elements to adjust the controller parameters in real-time, enhancing the system's ability to cope with varying conditions.

In this paper, we assess the stability of the SCMM by identifying and analyzing key bifurcation points, including Hopf bifurcation, transcritical bifurcation, and double-zero bifurcation. Furthermore, we investigate the dynamical characteristics of the SCMM through the use of bifurcation diagrams and Lyapunov exponents, aiming to uncover periodic, chaotic, and reverse period-doubling behaviors. Finally, we proposed backstepping controllers to manage the chaotic behavior within the SCMM and achieve synchronization between two SCMMs.

The rest of this paper are structured as outlined below. Section 2 focuses on the modeling of the SCMM and the exploration of stability through bifurcation techniques. Moving to Section 3, the examination of dynamical analysis is conducted using Lyapunov exponents and bifurcation diagrams. In Section 4, a backstepping control scheme is implemented to manage behavior of the SCMM, and diverse numerical simulations are showcased. Lastly, Section 5 provides concluding remarks.

2. A chaotic supply chain model

In 2022, Hamidzadeh et al. [9] defined a chaotic model consisting of a three-tier supply chain network with retailers, distributors and manufacturers, which can be described as follows:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = a y_1 - b y_2 - y_3 - y_1^2, \end{cases} \tag{2.1}$$

where y_1 is retailers, y_2 is distributors, and y_3 is manufacturers. The first differential equation in (2.1) describes the behavior of retailers that are affected by their sales and the response by distributors to these requests. The second differential equation in (2.1) describes the behavior of distributors who mainly seek to control their inventory levels and who will be constantly ordering from the factory. The third differential equation in (2.1) describes the behavior of manufacturers who are in direct contact with the retailers and distributors and the manufacturers produce the products with safety factors. In this equation, a and b are the satisfaction constants of the retailer and distributor of the manufacturer's products, respectively. System (2.1) exhibits chaotic behavior with parameters $a = 7.5$, $b = 3.8$, and the initial conditions are $y_1(0) = 1, y_2(0) = 1, y_3(0) = 1$. Applying the results reported in [15], we obtain the following bifurcation results.

Proposition 2.1. *Let $b > 0$ fixed. Then system (2.1) displays a Hopf bifurcation at the equilibrium point $O\ (0, 0, 0)$ when the parameter a passes through the critical value $a_0 = -b$.*

In fact, the line $a + b = 0$, $b > 0$ is a Hopf curve of system (2.1) at O.

Proposition 2.2. *Let $b > 0$ fixed. Then system (2.1) displays a Hopf bifurcation at the equilibrium point $E\ (a, 0, 0)$ when the parameter a passes through the critical value $a_0 = b$.*

In fact, the line $a - b = 0$, $b > 0$ is a Hopf curve of system (2.1) at E.

Proposition 2.3. *Let $b > 0$ fixed. Then system (2.1) displays a transcritical bifurcation when the parameter a passes through the critical value $a_0 = 0$.*

In fact, the line $a = 0$, $b > 0$ is a transcritical bifurcation curve of system (2.1). In addition, system (2.1) experiences a codim 2 bifurcation.

Proposition 2.4. *System (2.1) displays a double-zero bifurcation when (a, b) passes through $(0, 0)$.*

3. Dynamical analysis

The dynamic traits of nonlinear systems demonstrate significant fluctuations based on the values assigned to their parameters. Transitioning from one behavior to another, termed bifurcation, may take place upon reaching specific parameter ranges. This study will delve into the complexity responses of the recently introduced system (2.1) through numerical computations, with variations in the parameters a and b. In particular, we will analyze how the occurrence of local maxima in the signal x, denoted as x_{max}, evolves concerning the variations in the parameters a and b. These local maxima represent the highest points reached by the signal x over time for each specific value of a and b, offering insights into the system's dynamic behavior as the parameters are varied.

The Lyapunov exponent of a dynamical system is a measurable quantity that characterizes the rate of separation of infinitesimally close trajectories of the dynamical system in the phase space. A 3-D dynamical system has three Lyapunov exponents which can be arranged in non-increasing order as follows: $LE_1 \geqslant LE_2 \geqslant LE_3$. The 3-D dynamical system can be classified based on the following cases for the Lyapunov exponents.

(a) Stable equilibrium point if all the Lyapunov exponents are negative.
(b) Limit cycle if $LE_1 = 0$ and LE_2, LE_3 are negative.
(c) A chaotic attractor if $LE_1 > 0$, $LE_2 = 0$ and $LE_3 < 0$.

In case (c), when a chaotic attractor exists for the 3-D dynamical system, the Kaplan-Yorke dimension of the system can be defined as

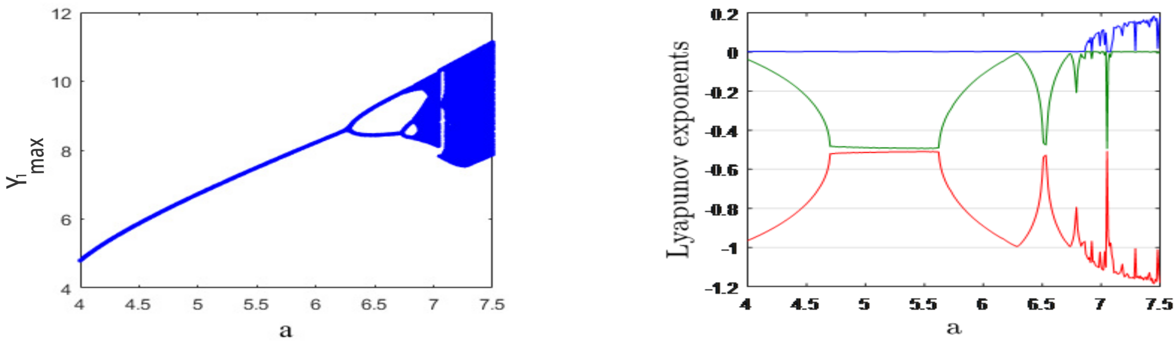
$$D_K = 2 + \frac{LE_1 + LE_2}{|LE_3|}.$$

3.1. *Parameter a varying*

By maintaining the values of b at 3.8, we can examine the impact of adjusting parameter a within the range of 4 to 7.5 on system (2.1). Figure 1 illustrates the Lyapunov exponent spectrum and the associated bifurcation diagram of system (2.1), revealing that the system exhibits both periodic and chaotic behavior as a increases within the [4, 7.5] range.

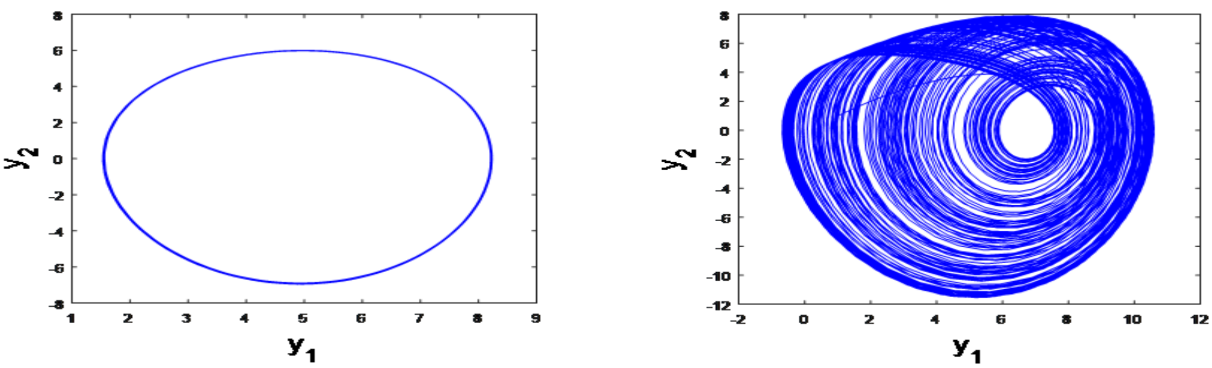
When the parameter a assumes values within the intervals ([4, 6.86], [7.04, 7.11], and 6.92), the dynamics of system (2.1) exhibit periodic behavior, as illustrated in Figure 2a. This is corroborated by the presence of one zero Lyapunov exponent (LE) and two negative LEs. Specifically, these Lyapunov exponent values are as follows: $LE_1 = 0, LE_1 = -0.110$, and $LE_3 = -0.8$ (for $a = 6$).

For values of parameter a within the intervals ([6.86, 7.04], [7.11, 7.5]), the system (2.1) exhibits chaotic behavior, as depicted in Figure 2b, accompanied by one positive Lyapunov exponent (LE). Specifically, the associated Lyapunov exponent values are found as follows: $LE_1 = 0.149, LE_2 = 0$, and $LE_3 = -1.151$ (for $a = 7.2$), and the system's Kaplan-Yorke dimension assumes a non-integer value of $D_K = 2.1295$. Furthermore, the bifurcation diagram illustrated in Figure 1 indicates that the system (2.1) undergoes the well-known period-doubling route to chaos.



(a) (b)

Figure 1: (a) Bifurcation diagram and (b) Lyapunov exponents spectrum of the system (2.1) when $b = 3.8$ and $a \in [4, 7.5]$.



(a) (b)

Figure 2: Phase plots of system (2.1): (a) $a = 6$ and (b) $a = 7.2$.

Period-doubling is a phenomenon in nonlinear dynamic systems where the period (the time it takes for the system to repeat its behavior) doubles as a control parameter is varied [24, 26]. This phenomenon is often observed in chaotic systems and is a type of bifurcation, which represents a qualitative change in the system's behavior as a parameter is adjusted. The presented bifurcation diagram in Figure 1 showcases that the system undergoes successive period-doubling phenomena as the parameter a increases. This leads to the familiar period-doubling route to chaos (period-1 $\rightarrow$ period-2 $\rightarrow$ period-4 $\rightarrow$ period-8 $\rightarrow$ chaos) within specific ranges of the parameter a.

For values of a ranging from 4 to 4.27, the system (2.1) manifests a period-1 attractor. In the interval between 4.27 and 6.73, a period-2 attractor emerges. Within the range of 6.73 to 6.83, the system (2.1) displays a period-4 attractor. Moving further to the interval of 6.83 to 6.86, a period-8 attractor is observed. When falls within the range of 6.86 to 7.04, the system exhibits a chaotic attractor, marking the end of the period-doubling cascade.

Table 1 encapsulates a summary of the diverse attractors observed in numerical simulations, showcasing the period-doubling route to chaos discussed earlier. Furthermore, Figure 3 offers a visual representation of these attractors.

Table 1: Period-doubling route to chaos with parameter a varying.

Parameter a range	Parameter a value	Dynamics	Attractor
$[4, 4.27]$	4	Period-1	Fig. 3a
$[4.27, 6.73]$	6.6	Period-2	Fig. 3b
$[6.73, 6.83]$	6.8	Period-4	Fig. 3c
$[6.83, 6.86]$	6.85	Period-8	Fig. 3d
$[6.86, 7.04]$	6.9	Chaos	Fig. 3e

3.2. Parameter b varying

In order to investigate the impact of changes in b on system (2.1), a is maintained at a constant value of 7.5, while b is systematically adjusted from 3.8 to 5. The Lyapunov exponent spectrum and the bifurcation diagram of system (2.1) are depicted in Figure 4. These visuals suggest that as b increases within this range, the system (2.1) can demonstrate both periodic and chaotic behavior.

System (2.1) exhibits chaotic behavior, as shown in Figure 5a, with a single positive Lyapunov exponent when b falls within the intervals of [3.8, 3.805], [3.810, 3.950], [3.960, 3.995], and [4.025, 4.11]. The Lyapunov exponents for the system are $LE1 = 0.152, LE2 = 0$, and $LE3 = -1.154$ when $b = 3.90$. Furthermore, the system's Kaplan-Yorke dimension is $D_K = 2.1317$, indicating a fractional value.

When b is in the intervals [3.805, 3.810], [3.950, 3.960], [3.995, 4.025], and [4.11, 5], system (2.1) displays periodic behavior, as illustrated in Figure 5b. This is marked by one Lyapunov exponent being zero, and the other two being negative. Specifically, for $b = 4.6$, the associated Lyapunov exponent values are: $LE_1 = 0, LE_2 = -0.110$, and $LE_3 = -0.893$.

Additionally, the bifurcation diagram presented in Figure 4 indicates that system (2.1) undergoes the familiar reverse period-doubling route.

The reverse period-doubling route refers to a phenomenon in nonlinear dynamics where a dynamic system undergoes a sequence of bifurcations that lead to the reduction of the period of oscillations rather than an increase [25]. In a standard period-doubling bifurcation, the period of oscillations doubles as a control parameter is varied, leading to a sequence like 1, 2, 4, 8, etc. In the reverse period-doubling route, the system experiences bifurcations that result in halving the period of oscillations.

The diagram in Figure 4 depicts that the system undergoes a sequence of reverse period-doubling bifurcations with increasing values of the parameter b. Consequently, within specific intervals of b the noteworthy phenomenon of reverse period-doubling unfolds. In this process, the system transitions from chaos to period-16, period-8, period-4, period-2, and ultimately to period-1.

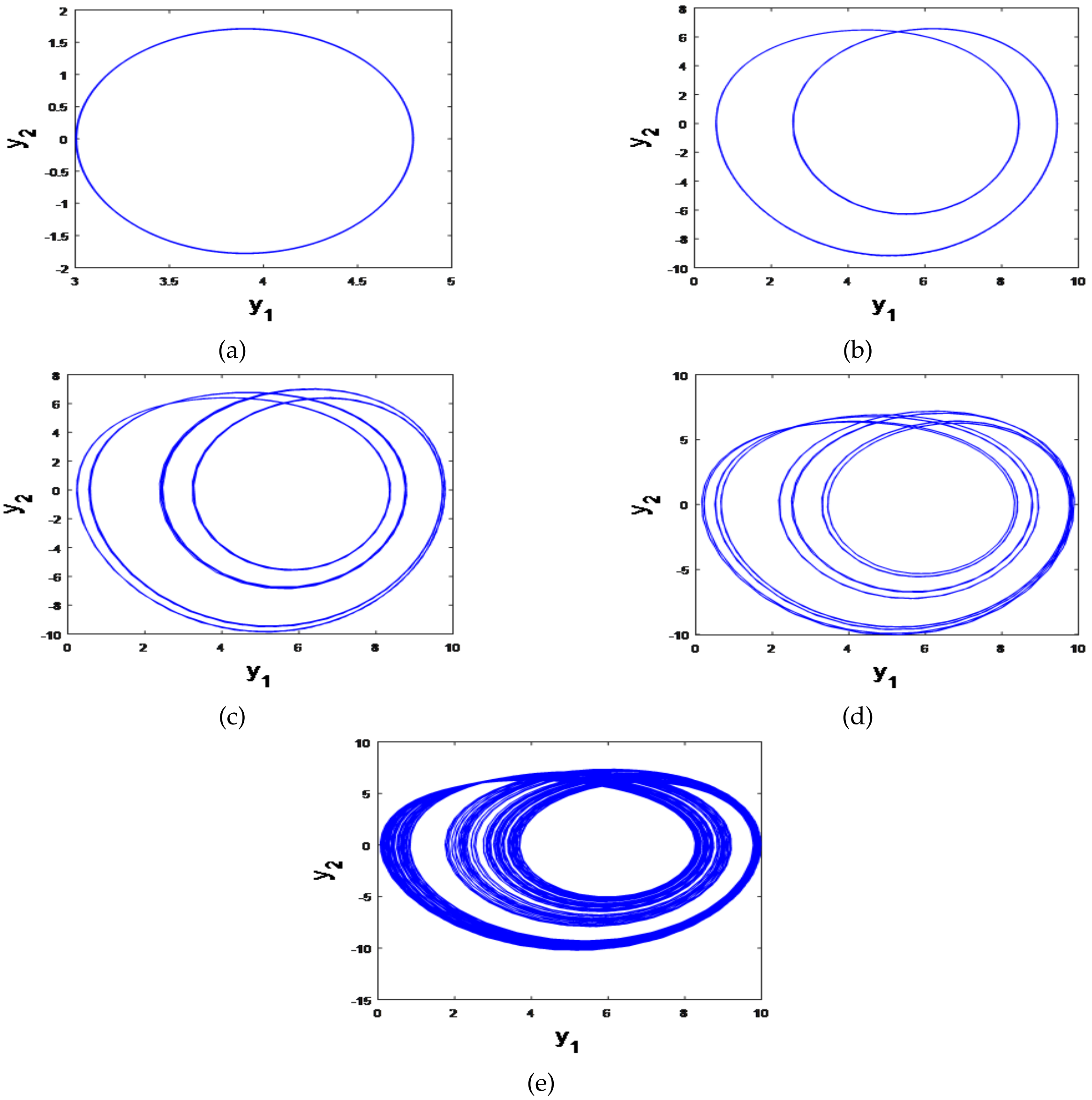


Figure 3: (a): Period-1 for $a = 4$; (b): Period-2 for $a = 6.6$; (c): Period-4 for $a = 6.8$; (d): Period-8 for $a = 6.85$; (e): chaos for $a = 6.9$.

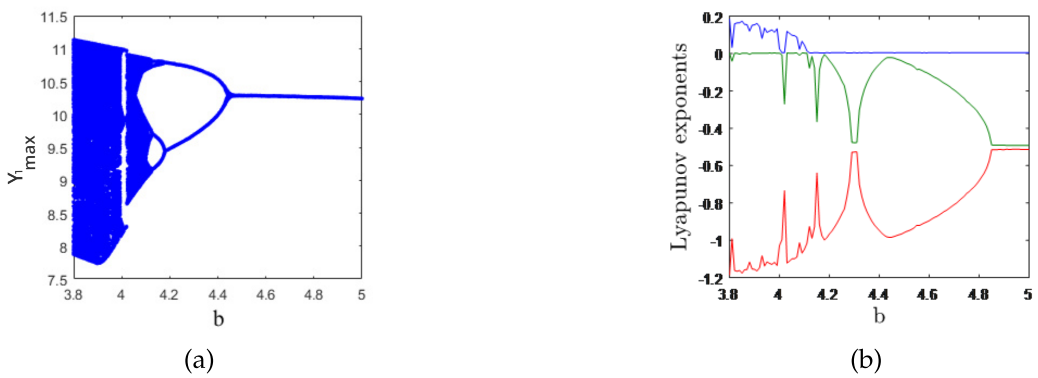
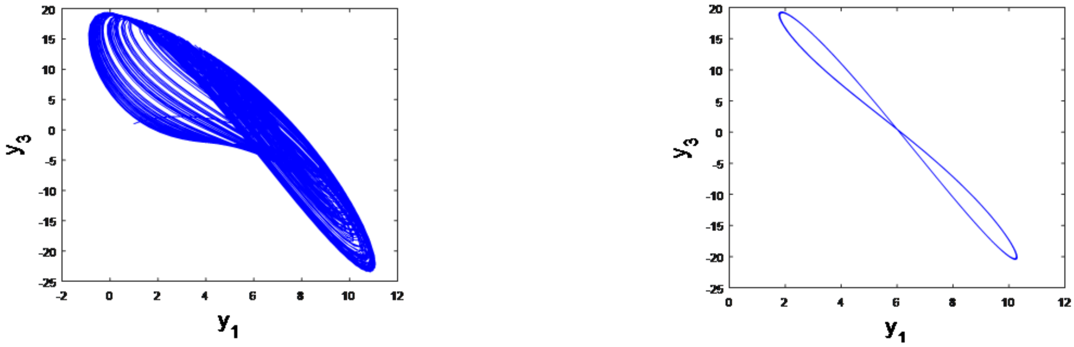


Figure 4: (a) Bifurcation diagram and (b) Lyapunov exponents spectrum of the system (2.1) when: $a = 2.3$ and $b \in [3.8, 5]$.



(a) (b)

Figure 5: Phase plots of system (2.1): (a) $b = 3.9$ and (b) $b = 4.6$.

The values assigned to the parameter b exert a noteworthy influence on the system's behavior. For instance, when b spans from 4.025 to 4.11, the system displays a chaotic attractor. At 4.1125, a period-16 attractor is observed. Similarly, a period-8 attractor occurs when lies within the range of 4.1125 and 4.125, a period-4 attractor from 4.125 to 4.180, a period-2 attractor between 4.18 and 4.45, and a period-1 attractor from 4.45 to 5, marking the end of the reverse period-doubling route.

Table 2 provides a detailed listing of the various attractors derived from numerical simulations, illustrating the reverse period-doubling route discussed earlier. Additionally, Figure 6 offers a visual depiction of these attractors.

Table 2: Reverse period-doubling route with parameter b varying.

Parameter b range	Parameter b value	Dynamics	Attractor
$[4.025, 4.11]$	4.10	Chaos	Fig. 6a
$[4.11, 4.1125]$	4.1125	Period-16	Fig. 6b
$[4.1125, 4.125]$	4.12	Period-8	Fig. 6c
$[4.125, 4.180]$	4.15	Period-4	Fig.6d
$[4.18, 4.45]$	4.25	Period-2	Fig.6e
$[4.45, 5]$	5	Period-1	Fig.6f

4. Complete synchronization of chaotic SCMM using backstepping control

In control systems engineering, backstepping control is a recursive method using Lyapunov stability theory, and this method is very effective for controlling nonlinear dynamical systems in strict-feedback form such as the jerk systems and hyperjerk systems. Backstepping control method has many applications in areas such as robotics, power systems, flight control systems, mechanical oscillators, biological systems, and supply chain models [29].

In view of the special structure of the dynamics for the chaotic supply chain system (2.1), we use backstepping control method for the complete synchronization of a pair of chaotic supply chain models regarded as the *leader* and *follower* systems.

The leader system is depicted by the chaotic supply chain system with the dynamics

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = a y_1 - b y_2 - y_3 - y_1^2. \end{cases} \tag{4.1}$$

It is noted that the leader system (4.1) has the same dynamics as the chaotic supply chain system (2.1).

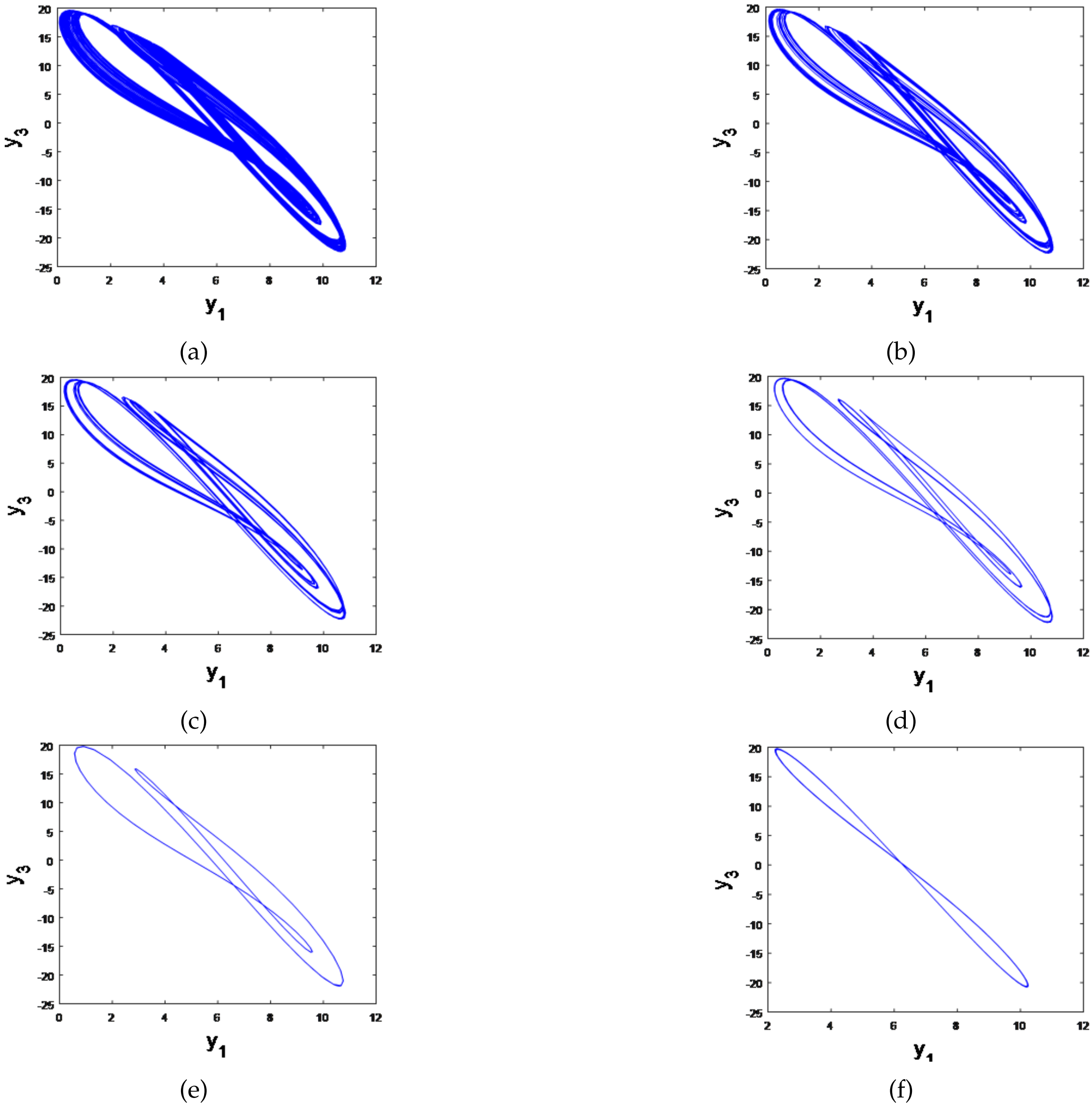


Figure 6: (a): Chaos for $b = 4.10$; (b): Period-16 for $b = 4.1125$; (c): Period-8 for $b = 4.12$; (d): Period-4 for $b = 4.15$; (e): Period-2 for $b = 4.25$; (f): Period-1 for $b = 5$.

In the chaotic supply chain system (4.1), the system states y_1, y_2, and y_3 represent the retailers, distributors, and manufacturers, respectively. The follower system is depicted by the chaotic supply chain system with the dynamics

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = z_3, \\ \dot{z}_3 = a z_1 - b z_2 - z_3 - z_1^2 + v. \end{cases} \tag{4.2}$$

In the chaotic supply chain system (4.2), the system states z_1, z_2, and z_3 represent the retailers, distributors, and manufacturers, respectively. Also, v is an active backstepping control which is to be designed using the recursive design procedure given in the backstepping control theory [29].

The synchronization error between the chaotic supply chain systems (4.1) and (4.2) can be mathematically determined by means of the following equations:

$$\varepsilon_j = z_j - y_j, \quad (j = 1, 2, 3).$$

By performing a simple mathematical calculation, we derive the dynamics for the synchronization error as given below:

$$\begin{cases} \dot{\varepsilon}_1 = \varepsilon_2, \\ \dot{\varepsilon}_2 = \varepsilon_3, \\ \dot{\varepsilon}_3 = a\varepsilon_1 - b\varepsilon_2 - \varepsilon_3 - z_1^2 + y_1^2 + \nu. \end{cases} \tag{4.3}$$

In this control section, we shall use Lyapunov stability theory [13] to prove the following main result for the complete synchronization between the chaotic supply chain systems (4.1) and (4.2). The chaotic supply chain systems (4.1) and (4.2) are said to be completely synchronized if the synchronization error between their respective states converges to zero asymptotically for all values of initial conditions of the chaotic supply chain systems (4.1) and (4.2), i.e., $\varepsilon_i(t) \to 0$ as $t \to \infty$ for all values of $\varepsilon_i(0) \in \mathbb{R}$, where $i = 1, 2, 3$.

Theorem 4.1. *The chaotic supply chain systems given by the equations (4.1) and (4.3) can be completely synchronized for all initial states $y(0), z(0) \in R^3$ by means of implementing the backstepping control law given by*

$$\nu = -(3 + a)\varepsilon_1 - (5 - b)\varepsilon_2 - 2\varepsilon_3 + z_1^2 - y_1^2 - L\varphi_3, \tag{4.4}$$

where the feedback gain $L > 0$ and $\varphi_3 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.

Proof. We establish the result claimed in Theorem 4.1 using the backstepping control theory [29] and the Lyapunov stability theory [13].

We start the proof by considering the Lyapunov function $W_1(\varphi_1)$ defined by

$$W_1(\varphi_1) = \frac{1}{2}\varphi_1^2,$$

where $\varphi_1 = \varepsilon_1$. Then it follows immediately that

$$\dot{W}_1(\varphi_1) = \varepsilon_1\dot{\varepsilon}_1 = \varepsilon_1\varepsilon_2 = -\frac{1}{2}\varphi_1^2 + \varphi_1\varphi_2,$$

where $\varphi_2 = \varepsilon_1 + \varepsilon_2$. Next, we define the Lyapunov function

$$W_2(\varphi_1, \varphi_2) = W_1(\varphi_1) + \frac{1}{2}\varphi_2^2 = \frac{1}{2}\varphi_1^2 + \frac{1}{2}\varphi_2^2.$$

A simple mathematical calculation yields

$$\dot{W}_2(\varphi_1, \varphi_2) = -\frac{1}{2}\varphi_1^2 - \frac{1}{2}\varphi_2^2 + \varphi_2\varphi_3,$$

where $\varphi_3 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$. Finally, we consider the Lyapunov function

$$W(\varphi_1, \varphi_2, \varphi_3) = W_2(\varphi_1, \varphi_2, \varphi_3) + \frac{1}{2}\varphi_3^2 = \frac{1}{2}\varphi_1^2 + \frac{1}{2}\varphi_2^2 + \frac{1}{2}\varphi_3^2.$$

Clearly, W is a quadratic and positive definite function defined on R^3. Moreover,

$$\dot{W} = \dot{W}_2 + \varphi_3\dot{\varphi}_3 = -\varphi_1^2 - \varphi_2^2 - \varphi_3^2 + \varphi_3 Z, \tag{4.5}$$

where

$$Z = \varphi_2 + \varphi_3 + \dot{\varphi}_3 = (3 + a)\varepsilon_1 + (5 - b)\varepsilon_2 + 2\varepsilon_3 - z_1^2 + y_1^2 + \nu. \tag{4.6}$$

Substituting the definition for ν from Eq. (4.4) into Eq. (4.6), we get $Z = -L\varphi_3$. Then Eq. (4.5) can be simplified as follows:

$$\dot{W} = -\varphi_1^2 - \varphi_2^2 - (1 + L)\varphi_3^2,$$

which is negative definite everywhere on R^3. Using Lyapunov stability theory [13], we conclude that the synchronization error dynamics (4.3) is globally asymptotically stable for all initial values of the synchronization error. This completes the proof. □

We present MATLAB simulations for the synchronization result in Theorem 4.1. We choose the parameters of the chaotic supply chain systems (4.1) and (4.2) as $a = 7.5$ and $b = 3.8$. We let $L = 12$. We choose the initial state of the chaotic supply chain system (4.1) as $y(0) = (1.6, 4.5, 2.1)$, and the initial state of the chaotic supply chain system (4.2) as $z(0) = (5.1, 2.3, 0.6)$.

Figure 7 shows the asymptotic convergence of the synchronization errors $(\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t))$ between the chaotic supply chain systems (4.1) and (4.2).

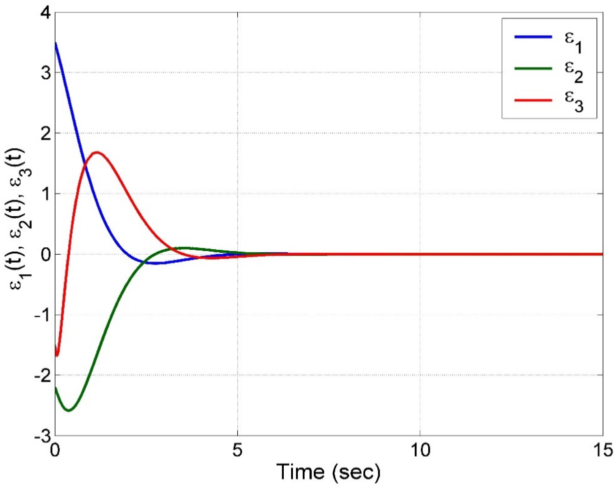


Figure 7: Time-history of the synchronization error between the supply chain systems (4.1) and (4.2) with $y(0) = (1.6, 4.5, 2.1)$ and $z(0) = (5.1, 2.3, 0.6)$.

5. Conclusion

In conclusion, this study has delved into the intricate dynamics and complexity within the supply chain mathematical model (SCMM) developed by Hamidzadeh et al. ([9]) through the lens of bifurcation analysis. Bifurcation points, serving as critical thresholds in chaotic systems, have been examined to understand the qualitative shifts in behavior, particularly within the context of supply chains. The primary objective of this investigation was to scrutinize the stability of the SCMM. Our analysis revealed significant bifurcation phenomena, including Hopf bifurcation, transcritical bifurcation, and double-zero bifurcation. These findings contribute to a deeper understanding of the structural changes that can occur within the SCMM, potentially signifying shifts in demand patterns, material flow disruptions, or alterations in market conditions.

Furthermore, the exploration extended to the dynamical characteristics of the SCMM, utilizing bifurcation diagrams and Lyapunov exponents. The outcomes unveiled a diverse range of behaviors, including periodic, chaotic, and reverse period-doubling, underscoring the complexity inherent in supply chain dynamics. To manage the chaotic nature of the SCMM, backstepping controllers were employed, demonstrating their effectiveness in achieving synchronization between two SCMMs. This strategic application of control methodologies opens avenues for enhancing the stability and coordination of supply chain systems. Numerical simulations were conducted to provide empirical evidence of the practical applicability and effectiveness of the proposed methodologies. The results validate the robustness of the backstepping controllers in taming chaotic SCMM behavior, offering a promising approach for real-world implementation. In essence, this study contributes valuable insights into the dynamics of supply chain systems, emphasizing the significance of bifurcation analysis and control strategies. The methodologies explored herein not only enhance our theoretical understanding but also offer practical solutions for managing the complexities of SCMM in dynamic and unpredictable environments.

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