J. Math. Computer Sci., 35 (2024), 241–255

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

New oscillation results for first-order nonlinear difference equations with retarded arguments

Emad R. Attia

Department of Mathematics, College of Sciences and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia.

Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt.

Abstract

In this paper, the oscillation of the first-order nonlinear delay difference equation

$$\Delta y(l) + a(l)y(l+1) + b(l)f(y(\vartheta(l))) = 0, \quad l \in \mathbb{N}_0,$$

is studied. Some explicit oscillation results of liminf and limsup are given. We obtain many new results using the comparison between both first-order delay linear and nonlinear difference equations. We give an illustrative example to demonstrate the strength and simplicity of our results.

Keywords: Nonlinear difference equations, differential equations, oscillation, nonmonotone delays.

2020 MSC: 34C10, 34K11, 34K05.

1. Introduction

Consider the first-order nonlinear difference equation

$$\Delta y(l) + a(l)y(l+1) + b(l)f(y(\vartheta(l))) = 0, \qquad l \in \mathbb{N}_0, \tag{1.1}$$

where $f \in C[\mathbb{R}, \mathbb{R}]$ such that

$$\liminf_{y \to 0} \frac{f(y)}{y} > \delta > 0 \quad \text{and} \quad yf(y) > 0 \quad \text{for } y \neq 0, \tag{1.2}$$

and $\mathbb{N}_0$ is the set of all nonnegative integers, $\Delta y(l) = y(l+1) - y(l)$, $(a(l))_{l \geqslant 0}$ is a sequence of real numbers such that $a(l) > -1$, and $(b(l))_{l \geqslant 0}$ is a sequence of nonnegative real numbers and $(\vartheta(l))_{l \geqslant 0}$ is a sequence of integers such that $\lim_{l \to \infty} \vartheta(l) = \infty$ and

$$\vartheta(l) \leqslant l - 1, \qquad l \in \mathbb{N}_0.$$

Email address: er.attia@psau.edu.sa; dr_emadr@yahoo.com (Emad R. Attia)

doi: 10.22436/jmcs.035.02.08

Received: 2024-01-10 Revised: 2024-04-07 Accepted: 2024-04-15

By a solution of Eq. (1.1) we mean a sequence of real numbers $(y(l))_{l\geqslant -d}$, $d = -\min_{l_1\geqslant 0}\vartheta(l_1)$ that satisfies (1.1) for $l \geqslant 0$. For a given set of real numbers $u_{-d}, u_{-d+1}, \dots, u_0$, Eq. (1.1) has a unique solution $(y(l))_{l\geqslant -d}$ that satisfies $y(-d) = u_{-d}, y(-d+1) = u_{-d+1}, \dots, y(0) = u_0$. As usual, any solution to Eq. (1.1) that is neither eventually positive nor eventually negative is called oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory; otherwise, it is called nonoscillatory. We use the following notation:

$$\zeta(l) = \sup_{0\leqslant r\leqslant l}\vartheta(r), \qquad \liminf_{l\to\infty} a(l) > \alpha,$$
$$\gamma = \liminf_{l\to\infty}\sum_{r=\zeta(l)}^{l-1} b(r), \qquad D(\eta) = \frac{1-\eta-\sqrt{1-2\eta-\eta^2}}{2}, \quad 0\leqslant\eta\leqslant\frac{1}{e}, \tag{1.3}$$

and

$$\sum_{r_1=s}^{s-1} K(r_1) = 0, \qquad \prod_{r_1=s}^{s-1} K(r_1) = 1.$$

When $a(l) \equiv 0$ and $f(y) = y$, Eq. (1.1) becomes the first-order linear delay difference equation

$$\Delta y(l) + b(l)y(\vartheta(l)) = 0, \qquad l \in \mathbb{N}_0,$$

that has received a great deal of attention from many researchers to study the oscillatory behavior of all its solutions, see [2–4, 9–11, 13, 17]. The two well-known sufficient oscillation conditions

$$\limsup_{l\to\infty}\sum_{r=\zeta(l)}^{l} b(r) > 1, \tag{1.4}$$

and

$$\liminf_{l\to\infty}\sum_{r=\vartheta(l)}^{l-1} b(r) > \frac{1}{e} \quad \text{and} \quad \limsup_{l\to\infty}\sum_{r=\vartheta(l)}^{l-1} b(r) < +\infty, \tag{1.5}$$

are due to [10] and [11], respectively. Many efforts have been made to fill the gap between conditions (1.4) and (1.5), see [2–6, 8–14, 17]. Very recently, Attia et al. [4] established the condition

$$\limsup_{l\to\infty}\left(P(l,m_1) + \sum_{r_1=\zeta(l)}^{l} b(r_1)W_{m_2}^{-1}(\zeta(l),\vartheta(r_1))\right) > 1,$$

where $m_1, m_2 \in \mathbb{N}$, and

$$W_1(l,u) = \prod_{r_1=u}^{l-1}\left(1-\kappa b(r_1)\right), \qquad W_{m+1}(l,u) = \prod_{r_1=u}^{l-1}\left(1-b(r_1)W_m^{-1}(r_1,\vartheta(r_1))\right),$$
$$\kappa = \begin{cases} 1, & q = 0,\\ \lambda(q), & q > 0,\end{cases} \qquad q = \liminf_{l\to\infty}\sum_{r=\vartheta(l)}^{l-1} b(r),$$

and $\lambda(\xi)$ be the smaller real root of $\lambda = e^{\xi\lambda}$, $\xi \geqslant 0$, and

$$h(l) = \min\{r_1 \in \mathbb{N}_0 : r_1 > l, \vartheta(r) > l-1\},$$

and

$$P(l,m_1) = \frac{\sum_{r_1=l+1}^{h(l+1)-1} b(r_1)\sum_{r_2=\zeta(r_1)}^{l} b(r_2)W_{m_1}^{-1}(\nu(l),\zeta(r_2))}{1-\sum_{r_1=l+1}^{h(l+1)-1} b(r_1)}.$$

On the other hand, some authors try to extend the above results for covering more general difference equations, see [1, 5–7, 14, 15, 18–21]. For instance, Yan and Qian [21] studied the oscillation of the first-order difference equation with several delays

$$\Delta y(l)+\sum_{r_1=1}^{m} q_{r_1}(l)y(l-\tau_{r_1})=0, \quad l\in\mathbb{N}_0, \tag{1.6}$$

where $q_{r_1}(l)$ is a sequence of nonnegative real numbers and τ_{r_1}, $r_1=1,2,\dots,m$, are positive integers. The authors [21] applied some of their results to the nonlinear difference equation

$$\Delta y(l)+\sum_{r_1=1}^{m} q_{r_1}(l)f(y(l-\tau_{r_1}))=0, \quad l\in\mathbb{N}_0, \tag{1.7}$$

where $f\in C[\mathbb{R},\mathbb{R}]$ such that $yf(y)>0$, $y\neq 0$. It was shown in [21, Theorem 6] that the oscillation of Eq. (1.6) leads also to the oscillation of Eq. (1.7) when $\liminf_{y\to 0}\frac{f(y)}{y}=1$. Tang and Yu [19] gave a counterexample to show that the assumption $\liminf_{y\to 0}\frac{f(y)}{y}=1$ of the preceding result needs more adjustment.

On the other hand, Tang and Yu [19], and Jiang and Tang [16] studied the oscillation of a special case of Eq. (1.1), i.e., the first-order nonlinear difference equation

$$\Delta y(l)+b(l)f(y(l-\tau))=0, \quad l\in\mathbb{N}_0, \tag{1.8}$$

where $f\in C[\mathbb{R},\mathbb{R}]$ such that $yf(y)>0$, $y\neq 0$. For the nonlinear function $f(y)$ in (1.8) the authors [16, 19] assumed that there exists a nondecreasing continuous function that satisfies some additional assumptions. Further, Jiang and Li [15] obtained many sufficient conditions for the oscillation of the first-order nonlinear difference equation

$$\Delta y(l)+\frac{k^k}{(k+1)^{k+1}}y_{n-\tau}+G(l,y(l-\tau_1),\dots,y(l-\tau_n))=0, \quad l\in\mathbb{N}_0,$$

where $\tau_1,\dots,\tau_n$ are positive integer numbers, the function G satisfies certain conditions.

In this article, we will try to extend the results in [5, 8, 12, 14, 17, 18, 20] to Eq. (1.1). We study the oscillation of Eq. (1.1), taking into account that the oscillation of a first-order nonlinear difference equation and the corresponding linear equation may be different even if the nonlinear function satisfies $\liminf_{y\to 0}\frac{f(y)}{y}=1$. Finally, as we will show in Example 2.11, our results can be easily applied without the need for any additional functions, as in [16, 19].

2. Main results

Consider the first-order difference inequality

$$\Delta y(l)+c(l)y(\upsilon(l))\leqslant 0, \quad l\in\mathbb{N}_0, \tag{2.1}$$

where $(c(l))_{l\geqslant 0}$ is a sequence of nonnegative real numbers and $(\upsilon(l))_{l\geqslant 0}$ is a nondecreasing sequence of integers such that $\upsilon(l)\leqslant l-1$, and $\lim_{l\to\infty}\upsilon(l)=\infty$. Let $(y(l))$ be a positive solution of inequality (2.1), then according to [13, Lemma 3], we have

$$\liminf_{l\to\infty}\frac{y(l+1)}{y(\upsilon(l))}\geqslant D(q_1) \quad \text{for} \quad q_1\leqslant\frac{1}{e}, \tag{2.2}$$

where

$$q_1=\liminf_{l\to\infty}\sum_{r=\upsilon(l)}^{l-1}c(r).$$

Lemma 2.1. *Let $(y(l))$ be a positive solution of Eq. (1.1). If, for some $l_1, l_2 \in \mathbb{N}$,*

$$\sum_{r=l_1}^{\infty} a(r) = \infty \quad or \quad \sum_{r=l_2}^{\infty} b(r) = \infty, \tag{2.3}$$

then $y(l)$ tends to zero as t goes to ∞.

Proof. Since $(y(n))$ is a positive solution of Eq. (1.1). It follows that $(y(l))$ is nonincreasing eventually for all sufficiently large l. Therefore, $y(l)$ is convergent to some $d \geqslant 0$, and hence, there exists a sufficiently large $l_3 \geqslant l_i$, $i = 1, 2$, such that $\frac{d}{2} < y(l), y(l+1), y(\vartheta(l)) < \frac{d}{0.9}$ for all $l \geqslant l_3$, so there exists $y^* \in [\frac{d}{2}, \frac{d}{0.9}]$ such that $f(y(\vartheta(l))) \geqslant f(y^*)$ for all $l \geqslant l_3$. It follows from Eq. (1.1) that,

$$\Delta y(l) = -a(l)y(l+1) - b(l)f(y(\vartheta(l))) \leqslant -a(l)\frac{d}{2} - b(l)f(y^*).$$

In view of (2.3), there exist $l^*, l^{**} > l_3$, such that

$$\sum_{r=l^*}^{l^{**}-1} a(r) > \frac{2}{d}y(l^*) \qquad \text{or} \qquad \sum_{r=l^*}^{l^{**}-1} b(r) > \frac{1}{f(y^*)}y(l^*). \tag{2.4}$$

Taking the sum of Eq. (2.1) from l^* to $l^{**}-1$, and then using (2.4) we obtain

$$\begin{aligned} y(l^{**}) &= y(l^*) - \sum_{r=l^*}^{l_1**-1} a(r)y(r+1) - \sum_{r=l^*}^{l^{**}-1} b(r)f(y(\vartheta(r))) \\ &\leqslant y(l^*) - \frac{d}{2}\sum_{r=l^*}^{l^{**}-1} a(r) - \sum_{r=l^*}^{l^{**}-1} b(r)f(y(\vartheta(r))) < 0, \end{aligned}$$

or

$$\begin{aligned} y(l^{**}) &= y(l^*) - \sum_{r=l^*}^{l_1**-1} a(r)y(r+1) - \sum_{r=l^*}^{l^{**}-1} b(r)f(y(\vartheta(r))) \\ &\leqslant y(l^*) - \sum_{r=l^*}^{l_1**-1} a(r)y(r+1) - f(y^*)\sum_{r=l^*}^{l^{**}-1} b(r) < 0. \end{aligned}$$

This contradiction completes the proof. □

Let the sequence $(\Psi_k(l))_{k\geqslant 1}$ be defined by

$$\Psi_1(l) = \frac{1+a(l)}{1-\delta b(l)},$$

and

$$\Psi_k(l) = \frac{1+a(l)}{1-\delta b(l)\prod_{r_1=\vartheta(l)}^{l-1}\Psi_{k-1}(r_1)}, \quad k = 2, 3, \ldots.$$

Lemma 2.2. *Assume that $k \in \mathbb{N}$ and (2.3) is satisfied, and $(y(l))$ is a positive solution of Eq. (1.1). Then*

$$\frac{y(l)}{y(l+1)} \geqslant \Psi_k(l) \ \textit{ for all sufficiently large } l. \tag{2.5}$$

Proof. Dividing Eq. (1.1) by $y(l)$, we get

$$(1+a(l))\frac{y(l+1)}{y(l)}-1+b(l)\frac{f(y(\vartheta(l)))}{y(l)}=0. \tag{2.6}$$

In view of Lemma 2.1, we obtain

$$\lim_{l\to\infty}\frac{f(y(\vartheta(l)))}{y(\vartheta(l))}=\lim_{y\to 0^+}\frac{f(y)}{y}>\delta.$$

Then

$$f(y(\vartheta(l)))>\delta y(\vartheta(l)) \text{ for all sufficiently large } l. \tag{2.7}$$

Substituting into (2.6), we have

$$(1+a(l))\frac{y(l+1)}{y(l)}-1+\delta b(l)\frac{y(\vartheta(l))}{y(l)}\leqslant 0. \tag{2.8}$$

Therefore,

$$\frac{y(l)}{y(l+1)}\geqslant\frac{1+a(l)}{1-\delta b(l)\prod_{r_1=\vartheta(l)}^{l-1}\frac{y(r_1)}{y(r_1+1)}}. \tag{2.9}$$

Using the fact that $\frac{y(r_1)}{y(r_1+1)}\geqslant 1$, we have

$$\frac{y(l)}{y(l+1)}\geqslant\frac{1+a(l)}{1-\delta b(l)}=\Psi_1(l).$$

Substituting into (2.9), we obtain

$$\frac{y(l)}{y(l+1)}\geqslant\frac{1+a(l)}{\left(1-b(l)\delta\prod_{r_1=\vartheta(l)}^{l-1}\Psi_1(r_1)\right)}=\Psi_2(l).$$

Continuing in this fashion, one can obtain (2.5). □

We aim to extend the iterative criteria for first-order linear difference equations with several delay arguments, proposed by Braverman et al. in [8], to the first-order nonlinear difference equation (1.1). A useful technique for generating oscillation conditions is the following result, which gives an estimate of the quantity $\frac{y(u)}{y(v)}$, $v\geqslant u$, where $(y(l))$ is a positive solution of Eq. (1.1). Let the sequence $\{W_{k_1,k_2}(u,v)\}_{k_1,k_2\geqslant 1}$, $v\geqslant u$, be defined by

$$W_{1,k_2}(u,v)=\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)\prod_{r_1=\vartheta(r)}^{r-1}\Psi_{k_2}(r_1)},\quad k_2=1,2,\ldots$$

and

$$W_{k_1,k_2}(u,v)=\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)W_{k_1-1,k_2}(\vartheta(v),v)},\quad k_1=1,2,\ldots.$$

Lemma 2.3. *Assume that $k_1,k_2\in\mathbb{N}$ and (2.3) is satisfied, and $(y(l))$ is a positive solution of Eq. (1.1). Then*

$$\frac{y(u)}{y(v)}\geqslant W_{k_1,k_2}(u,v)\ \textit{for}\ \ v\geqslant u. \tag{2.10}$$

Proof. Dividing (1.1) by $y(l)$ and taking the product from u to $v-1$, we get

$$\prod_{r=u}^{v-1}\left(\frac{y(r+1)}{y(r)}(1+a(r))\right)=\prod_{r=u}^{v-1}\left(1-b(r)\frac{f(y(\vartheta(r)))}{y(r)}\right).$$

Consequently,

$$\frac{y(u)}{y(v)}=\prod_{r=u}^{v-1}\frac{1+a(r)}{1-b(r)\frac{f(y(\vartheta(r)))}{y(r)}}.$$

Using (2.7), we have

$$\frac{y(u)}{y(v)}\geqslant\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)\frac{y(\vartheta(r))}{y(r)}}. \tag{2.11}$$

That is,

$$\frac{y(u)}{y(v)}\geqslant\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)\prod_{r_1=\vartheta(r)}^{r-1}\frac{y(r_1)}{y(r_1+1)}}.$$

In view of (2.5), it follows that

$$\frac{y(u)}{y(v)}\geqslant\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)\prod_{r_1=\vartheta(r)}^{r-1}\Psi_{k_2}(r_1)}=W_{1,k_2}(u,v).$$

Form this and (2.11), we have

$$\frac{y(u)}{y(v)}\geqslant\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)W_{1,k_2}(\vartheta(r),r)}=W_{2,k_2}(u,v).$$

In the same way, continuing the process k_1 times, we obtain

$$\frac{y(u)}{y(v)}\geqslant\prod_{r=u}^{v-1}\frac{1+a(r)}{1-\delta b(r)W_{k_1-1,k_2}(\vartheta(r),r)}=W_{k_1,k_2}(u,v).$$

The proof is complete. □

The following result is an extension of [8, Theorem 3.3] to Eq. (1.1) when $l-\vartheta(l)<\infty$.

Theorem 2.4. *If* $N_1\leqslant l-\vartheta(l)\leqslant N$ *for all sufficiently large* l, *and*

$$\liminf_{l\to\infty}\sum_{r=\vartheta(l)}^{l-1}b(r)>\frac{1}{\delta}\left(\frac{1}{1+\alpha}\right)^{N_1}\left(\frac{N}{N+1}\right)^{N+1}, \tag{2.12}$$

then, every solution of Eq. (1.1) is oscillatory.

Proof. Let the sequence $(h_1(l))$ be defined by

$$h_1(l)=\frac{(l-\vartheta(l))^{l-\vartheta(l)+1}}{(1+l-\vartheta(l))^{l-\vartheta(l)+1}}.$$

Then

$$\frac{1}{4}\leqslant h_1(l)\leqslant\left(\frac{N}{N+1}\right)^{N+1}.$$

In view of (2.12), it follows for sufficiently small $\epsilon > 0$ that

$$\sum_{r=\vartheta(l)}^{l-1} b(r) > \frac{1}{\delta}\left(\frac{1}{1+\alpha}\right)^{N_1}\left(\left(\frac{N}{N+1}\right)^{N+1}+\epsilon\right) > 0 \text{ for all sufficiently large } l. \tag{2.13}$$

Let

$$u = \left(\frac{N}{N+1}\right)^{-N-1}\left(\left(\frac{N}{N+1}\right)^{N+1}+\epsilon\right) > 1.$$

Then

$$\begin{aligned}\sum_{r=\vartheta(l)}^{l-1} \frac{\delta\,(1+\alpha)^{l-\vartheta(l)}\, b(r)}{h_1(l)} &\geqslant \delta\,(1+\alpha)^{N_1}\left(\frac{N}{N+1}\right)^{-N-1}\sum_{r=\vartheta(l)}^{l-1} b(r)\\ &> \left(\frac{N}{N+1}\right)^{-N-1}\left(\left(\frac{N}{N+1}\right)^{N+1}+\epsilon\right) = u > 1.\end{aligned}$$

Assume the contrary, i.e., let $(y(l))$ be a positive solution of Eq. (1.1). Dividing Eq. (1.1) by $y(l)$ and taking the sum from $\vartheta(l)$ to $l-1$, we have

$$\sum_{r=\vartheta(l)}^{l-1}\left(\frac{y(r+1)}{y(r)}\,(1+a(r))-1\right) = -\sum_{r=\vartheta(l)}^{l-1} b(r)\frac{f(y(\vartheta(r)))}{y(r)}. \tag{2.14}$$

Using (2.13), so condition (2.3) is satisfied. From (1.3), (2.7), and (2.14), we have

$$\sum_{r=\vartheta(l)}^{l-1}\left(\frac{y(r+1)}{y(r)}\,(1+\alpha)-1\right) \leqslant -\delta\sum_{r=\vartheta(l)}^{l-1} b(r)\frac{y(\vartheta(r))}{y(r)}$$

for all sufficiently large l. Using the relation between the arithmetic and the geometric means, we get

$$(l-\vartheta(l))\left(\left(\prod_{r=\vartheta(l)}^{l-1}\frac{y(r+1)}{y(r)}\,(1+\alpha)\right)^{\frac{1}{l-\vartheta(l)}}-1\right) \leqslant -\delta\sum_{r=\vartheta(l)}^{l-1} b(r)\frac{y(\vartheta(r))}{y(r)}.$$

Therefore,

$$\prod_{r=\vartheta(l)}^{l-1}\left(\frac{y(r+1)}{y(r)}\,(1+\alpha)\right) \leqslant \left(1-\frac{\delta}{l-\vartheta(l)}\sum_{r=\vartheta(l)}^{l-1} b(r)\frac{y(\vartheta(r))}{y(r)}\right)^{l-\vartheta(l)}.$$

Then

$$\frac{y(l)}{y(\vartheta(l))} \leqslant \left(\frac{1}{1+\alpha}\right)^{l-\vartheta(l)}\left(1-\frac{\delta}{l-\vartheta(l)}\sum_{r=\vartheta(l)}^{l-1} b(r)\frac{y(\vartheta(r))}{y(r)}\right)^{l-\vartheta(l)}. \tag{2.15}$$

Using the fact that $\frac{y(\vartheta(r))}{y(r)} \geqslant 1$, we have

$$\frac{y(l)}{y(\vartheta(l))} \leqslant \left(\frac{1}{1+\alpha}\right)^{l-\vartheta(l)}\left(1-\frac{\delta}{l-\vartheta(l)}\sum_{r=\vartheta(l)}^{l-1} b(r)\right)^{l-\vartheta(l)}. \tag{2.16}$$

Let $g:(0,1)\to\mathbb{R}$, $g(s)=s\,(1-s)^{\beta}$, $\beta\in\mathbb{N}$. It is easy to see that $g(s)$ takes it's max at $s=\frac{1}{1+\beta}$, and hence

$$g(s)\leqslant\frac{\beta^{\beta}}{(1+\beta)^{1+\beta}}.$$

If we assume that $\beta=l-\vartheta(l)$, $s=\frac{w}{\beta}$, and $w=\delta\sum_{r=\vartheta(l)}^{l-1}b(r)$, then (2.16) leads to

$$\frac{y(l)}{y(\vartheta(l))}\leqslant\left(\frac{1}{1+\alpha}\right)^{\beta}(1-s)^{\beta}=\left(\frac{1}{1+\alpha}\right)^{\beta}\frac{\beta}{\delta\sum_{r=\vartheta(l)}^{l-1}b(r)}s\,(1-s)^{\beta}.$$

Therefore,

$$\frac{y(l)}{y(\vartheta(l))}\leqslant\left(\frac{1}{1+\alpha}\right)^{\beta}\frac{\frac{\beta^{\beta+1}}{(1+\beta)^{1+\beta}}}{\delta\sum_{r=\vartheta(l)}^{l-1}b(r)}=\left(\frac{1}{1+\alpha}\right)^{l-\vartheta(l)}\frac{\frac{(l-\vartheta(l))^{l-\vartheta(l)+1}}{(1+l-\vartheta(l))^{l-\vartheta(l)+1}}}{\delta\sum_{r=\vartheta(l)}^{l-1}b(r)}.$$

Then

$$\frac{y(\vartheta(l))}{y(l)}\geqslant\sum_{r=\vartheta(l)}^{l-1}\frac{\delta\,(1+\alpha)^{l-\vartheta(l)}\,b(r)}{\frac{(l-\vartheta(l))^{l-\vartheta(l)+1}}{(1+l-\vartheta(l))^{l-\vartheta(l)+1}}}=\sum_{r=\vartheta(l)}^{l-1}\frac{\delta\,(1+\alpha)^{l-\vartheta(l)}\,b(r)}{h_1(l)}>u.$$

Substituting into (2.15), we get

$$\frac{y(l)}{y(\vartheta(l))}\leqslant\left(\frac{1}{1+\alpha}\right)^{l-\vartheta(l)}\left(1-u\frac{\delta}{l-\vartheta(l)}\sum_{r=\vartheta(l)}^{l-1}b(r)\right)^{l-\vartheta(l)}.$$

By using a similar argument, we obtain

$$\frac{y(\vartheta(l))}{y(l)}>u\sum_{r=\vartheta(l)}^{l-1}\frac{\delta\,(1+\alpha)^{l-\vartheta(l)}\,b(r)}{h_1(l)}>u^2.$$

Therefore,

$$\frac{y(\vartheta(l))}{y(l)}>u^k,\quad k\in\mathbb{N}.$$

By (2.12), we get

$$\limsup_{l\to\infty}\ b(l)\geqslant d=\frac{1}{N}\frac{1}{\delta}\left(\frac{1}{1+\alpha}\right)^{N_1}\left(\frac{N}{N+1}\right)^{N+1}>0.$$

Then there exists a sequence (l_i) and a sufficiently small ϵ_1, $0<\epsilon_1<d$ such that

$$b(l_i)>d_1=d-\epsilon_1>0\ \text{ for all } i\in\mathbb{N}_0.$$

In view of (2.8), it follows that

$$\delta b(l_i)\frac{y(\vartheta(l_i))}{y(l_i)}<1\ \text{ for all } i\in\mathbb{N}_0.$$

One can choose $k\in\mathbb{N}$ such that $u^k>\frac{1}{\delta d_1}$. Then

$$u^k<\frac{y(\vartheta(l_i))}{y(l_i)}<\frac{1}{\delta b(l_i)}<\frac{1}{\delta d_1}<u^k.$$

This contradiction completes the proof. □

Theorem 2.5. *If $l-\vartheta(l) \geqslant N_1$ for all sufficiently large l, and*

$$\liminf_{l\to\infty} \sum_{r=\vartheta(l)}^{l-1} b(r) > \frac{1}{\delta}\left(\frac{1}{1+\alpha}\right)^{N_1}\frac{1}{e}, \tag{2.17}$$

then, every solution of Eq. (1.1) is oscillatory.

Proof. As in the proof of Theorem 2.4, let the sequence $(h_1(l))$ be defined by

$$h_1(l) = \frac{(l-\vartheta(l))^{l-\vartheta(l)+1}}{(1+l-\vartheta(l))^{l-\vartheta(l)+1}}.$$

Therefore,

$$\frac{1}{4} \leqslant h_1(l) \leqslant \frac{1}{e}.$$

Using (2.17), then for sufficiently small $\epsilon > 0$, we obtain

$$\sum_{r=\vartheta(l)}^{l-1} b(r) > \frac{1}{\delta}\left(\frac{1}{1+\alpha}\right)^{N_1}\left(\frac{1}{e}+\epsilon\right) > 0 \quad \text{for all sufficiently large } l.$$

Let

$$u = e\left(\frac{1}{e}+\epsilon\right) > 1.$$

Then

$$\sum_{r=\vartheta(l)}^{l-1} \frac{\delta\,(1+\alpha)^{l-\vartheta(l)}\,b(r)}{h_1(l)} \geqslant \delta\,(1+\alpha)^{N_1}\,e \sum_{r=\vartheta(l)}^{l-1} b(r) > e\left(\frac{1}{e}+\epsilon\right) = u > 1.$$

The rest of the proof is similar to the proof of Theorem 2.4. The proof is complete. □

Theorem 2.6. *Assume that $k_1, k_2, k_3 \in \mathbb{N}$ and (2.3) is satisfied. If*

$$\limsup_{l\to\infty} \sum_{r=\zeta(l)}^{l} b(r) W_{k_1,k_2}(\vartheta(r),\, \zeta(l)) > \frac{1}{\delta} - \frac{D(\gamma)}{\delta}\left(1+\alpha+\liminf_{l\to\infty}\sum_{r=\zeta(l)}^{l-1} a(r)\prod_{r_1=r+1}^{l}\Psi_{k_3}(r_1)\right), \tag{2.18}$$

then, every solution of Eq. (1.1) is oscillatory.

Proof. Let $(y(l))$ be a positive solution of Eq. (1.1). Then

$$\Delta y(l) + a(l)y(l+1) + b(l)\delta y(\vartheta(l)) \leqslant 0. \tag{2.19}$$

Summing Eq. (1.1) from $\zeta(l)$ to l, it follows that

$$y(l+1) - y(\zeta(l)) + \sum_{r=\zeta(l)}^{l} a(r)y(r+1) + \sum_{r=\zeta(l)}^{l} b(r)\delta y(\vartheta(r)) \leqslant 0.$$

Since $\zeta(l) \geqslant \vartheta(r)$ for $\zeta(l) \leqslant r \leqslant l$, it follows from (2.10) that

$$y(l+1) - y(\zeta(l)) + \sum_{r=\zeta(l)}^{l} a(r)y(r+1) + y(\zeta(l))\sum_{r=\zeta(l)}^{l} b(r)\delta W_{k_1,k_2}(\vartheta(r),\, \zeta(l)) \leqslant 0.$$

Then

$$y(l+1)\,(1+a(l))-y(\zeta(l))+\sum_{r=\zeta(l)}^{l-1}a(r)y(r+1)+y(\zeta(l))\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 0.$$

Therefore,

$$\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 1-\frac{y(l+1)}{y(\zeta(l))}\,(1+a(l))-\sum_{r=\zeta(l)}^{l-1}a(l)\frac{y(r+1)}{y(\zeta(l))}.$$

Consequently,

$$\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 1-\frac{y(l+1)}{y(\zeta(l))}\,(1+a(l))-\sum_{r=\zeta(l)}^{l-1}a(r)\frac{y(l+1)}{y(\zeta(l))}\prod_{r_1=r+1}^{l}\frac{y(r_1)}{y(r_1+1)}. \tag{2.20}$$

Then

$$\begin{aligned}\delta\limsup_{l\to\infty}\sum_{r=\zeta(l)}^{l}b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 1-\liminf_{l\to\infty}\left(\frac{y(l+1)}{y(\zeta(l))}\right)\liminf_{l\to\infty}\,(1+a(l))\\ -\liminf_{l\to\infty}\left(\frac{y(l+1)}{y(\zeta(l))}\right)\liminf_{l\to\infty}\sum_{r=\zeta(l)}^{l-1}a(r)\prod_{r_1=r+1}^{l}\frac{y(r_1)}{y(r_1+1)}.\end{aligned} \tag{2.21}$$

In view of (2.19), it follows that

$$\Delta y(l)+\delta b(l)y(\zeta(l))\leqslant 0.$$

By (2.2), we obtain

$$\liminf_{l\to\infty}\frac{y(l+1)}{y(\zeta(l))}\geqslant D(\gamma). \tag{2.22}$$

This together with (1.3), (2.5), and (2.21) leads to

$$\limsup_{l\to\infty}\sum_{r=\zeta(l)}^{l}b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant\frac{1}{\delta}-\frac{D(\gamma)}{\delta}\left(1+\alpha+\liminf_{l\to\infty}\sum_{r=\zeta(l)}^{l-1}a(r)\prod_{r_1=r+1}^{l}\Psi_{k_3}(r_1)\right).$$

This contradicts (2.18). The proof is complete. □

Theorem 2.7. *Assume that* $k_1,k_2,k_3\in\mathbb{N}$ *and* (2.3) *is satisfied. If*

$$\limsup_{l\to\infty}\left(\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))+(D(\gamma)-\epsilon)\left(1+a(l)+\sum_{r=\zeta(l)}^{l-1}a(r)\prod_{r_1=r+1}^{l}\Psi_{k_3}(r_1)\right)\right)>1,$$

where $\epsilon>0$*, then every solution of Eq.* (1.1) *is oscillatory.*

Proof. Let $(y(l))$ be a positive solution of Eq. (1.1). In view of (2.20) form the proof of Theorem 2.6, we have

$$\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))+\frac{y(l+1)}{y(\zeta(l))}\,(1+a(l))+\sum_{r=\zeta(l)}^{l-1}a(r)\frac{y(l+1)}{y(\zeta(l))}\prod_{r_1=r+1}^{l}\frac{y(r_1)}{y(r_1+1)}\leqslant 1.$$

Using (2.5) and (2.22), then for sufficiently small $\epsilon > 0$, we have

$$\sum_{r=\zeta(l)}^{l} b(r)\delta W_{k_1,k_2}(\vartheta(r),\, \zeta(l)) + (D(\gamma)-\epsilon)\left(1 + a(l) + \sum_{r=\zeta(l)}^{l-1} a(r) \prod_{r_1=r+1}^{l} \Psi_{k_3}(r_1)\right) \leqslant 1.$$

Therefore,

$$\limsup_{l\to\infty}\left(\sum_{r=\zeta(l)}^{l} b(r)\delta W_{k_1,k_2}(\vartheta(r),\, \zeta(l)) + (D(\gamma)-\epsilon)\left(1 + a(l) + \sum_{r=\zeta(l)}^{l-1} a(r) \prod_{r_1=r+1}^{l} \Psi_{k_3}(r_1)\right)\right) \leqslant 1.$$

This contradiction completes the proof. □

Theorem 2.8. *Assume that* $k_1, k_2, k_3 \in \mathbb{N}$ *and* (2.3) *is satisfied. If*

$$\limsup_{l\to\infty}\left(\delta b(l+1) W_{k_1,k_2}(\zeta(l+1), l)\,(1+a(l)) + \delta b(l) \prod_{r_1=\zeta(l)}^{l-1} \Psi_{k_3}(r_1)\right) > 1, \tag{2.23}$$

then every solution of Eq. (1.1) *is oscillatory.*

Proof. As before, assume that $(y(l))$ is a positive solution of Eq. (1.1). Therefore,

$$y(l+1) - y(l) + a(l)y(l+1) + b(l)\delta y(\zeta(l)) \leqslant 0. \tag{2.24}$$

In view of $\zeta(l) \leqslant l-1$, it follows from (2.10) that

$$y(l+1) - y(l) + a(l)y(l+1) + \delta b(l) W_{k_1,k_2}(\zeta(l), l-1) y(l-1) \leqslant 0.$$

Consequently,

$$\frac{y(l)}{y(l-1)} \geqslant (1+a(l))\frac{y(l+1)}{y(l-1)} + \delta b(l) W_{k_1,k_2}(\zeta(l), l-1) > \delta b(l) W_{k_1,k_2}(\zeta(l), l-1).$$

Thus

$$\frac{y(l)}{y(l-1)} > \delta b(l) W_{k_1,k_2}(\zeta(l), l-1).$$

That is,

$$\frac{y(l+1)}{y(l)} > \delta b(l+1) W_{k_1,k_2}(\zeta(l+1), l). \tag{2.25}$$

Using (2.24), we obtain

$$y(l+1) - y(l) + a(l)y(l+1) + \delta b(l) y(l) \prod_{r_1=\zeta(l)}^{l-1} \frac{y(r_1)}{y(r_1+1)} \leqslant 0.$$

From this and (2.5) and (2.25), we have

$$\delta b(l+1) W_{k_1,k_2}(\zeta(l+1), l)\,(1+a(l))\, y(l) + \delta b(l) y(l) \prod_{r_1=\zeta(l)}^{l-1} \Psi_{k_3}(r_1) \leqslant y(l).$$

Therefore,

$$\limsup_{l\to\infty}\left(\delta b(l+1) W_{k_1,k_2}(\zeta(l+1), l)\,(1+a(l)) + \delta b(l) \prod_{r_1=\zeta(l)}^{l-1} \Psi_{k_3}(r_1)\right) \leqslant 1.$$

Contradicting with (2.23). The proof is complete. □

Theorem 2.9. *Assume that* $k_1, k_2, k_3 \in \mathbb{N}$ *and* (2.3) *is satisfied. If*

$$\limsup_{l\to\infty}\Bigg(\delta b(l+1)W_{k_1,k_2}(\zeta(l+1),l)\left[1+a(l)+\delta b(l)\left(1+a(l)\right)\right]$$
$$+\delta b(l)\sum_{r=\zeta(l)}^{l-1}a(r)\prod_{r_1=r+1}^{l-1}\Psi_{k_3}(r_1)+\left(\delta^2 b(l)\sum_{r=\zeta(l)}^{l}b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\right)\prod_{r_1=\zeta(l)}^{l-1}\Psi_{k_3}(r_1)\Bigg)>1.$$

then, every solution of Eq. (1.1) *is oscillatory.*

Proof. Let $(y(l))$ be a positive solution of Eq. (1.1). Then

$$\Delta y(l)+a(l)y(l+1)+b(l)\delta y(\vartheta(l))\leqslant 0.$$

Summing from $\zeta(l)$ to l, it follows that

$$y(l+1)-y(\zeta(l))+\sum_{r=\zeta(l)}^{l}a(r)y(r+1)+\sum_{r=\zeta(l)}^{l}b(r)\delta y(\vartheta(r))\leqslant 0.$$

Since $\zeta(l)\geqslant\vartheta(r)$, it follows from (2.10) that

$$y(l+1)-y(\zeta(l))+\sum_{r=\zeta(l)}^{l}a(r)y(r+1)+y(\zeta(l))\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 0.$$

Multiplying both sides by $b(l)$,

$$b(l)\left(1+a(l)\right)y(l+1)-b(l)y(\zeta(l))+b(l)\sum_{r=\zeta(l)}^{l-1}a(r)y(r+1)$$
$$+y(\zeta(l))b(l)\sum_{r=\zeta(l)}^{l}b(r)\delta W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\leqslant 0.$$

By (1.1), we get

$$\Delta y(l)+\left(a(l)+\delta b(l)\left(1+a(l)\right)\right)y(l+1)+\delta b(l)\sum_{r=\zeta(l)}^{l-1}a(r)y(r+1)$$
$$+\left(\delta^2 b(l)\sum_{r=\zeta(l)}^{l}b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\right)y(\zeta(l))\leqslant 0.$$

Therefore,

$$y(l+1)-y(l)+\left[a(l)+\delta b(l)\left(1+a(l)\right)\right]y(l+1)+y(l)\delta b(l)\sum_{r=\zeta(l)}^{l-1}a(r)\prod_{r_1=r+1}^{l-1}\frac{y(r_1)}{y(r_1+1)}$$
$$+y(l)\left(\delta^2 b(l)\sum_{r=\zeta(l)}^{l}b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\right)\prod_{r_1=\zeta(l)}^{l-1}\frac{y(r_1)}{y(r_1+1)}\leqslant 0.$$

Using (2.25), we get

$$\delta b(l+1)W_{k_1,k_2}(\zeta(l+1),l)\left[1+a(l)+\delta b(l)\left(1+a(l)\right)\right]y(l)-y(l)$$

$$+y(l)\delta b(l)\sum_{r=\zeta(l)}^{l-1} a(r)\prod_{r_1=r+1}^{l-1}\Psi_{k_3}(r_1)+y(l)\left(\delta^2 b(l)\sum_{r=\zeta(l)}^{l} b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\right)\prod_{r_1=\zeta(l)}^{l-1}\Psi_{k_3}(r_1)\leqslant 0.$$

Then

$$\limsup_{l\to\infty}\Bigg(\delta b(l+1)W_{k_1,k_2}(\zeta(l+1),r)\left[1+a(l)+\delta b(l)\left(1+a(l)\right)\right]$$
$$+\delta b(l)\sum_{r=\zeta(l)}^{l-1} a(r)\prod_{r_1=r+1}^{l-1}\Psi_{k_3}(r_1)+\left(\delta^2 b(l)\sum_{r=\zeta(l)}^{l} b(r)W_{k_1,k_2}(\vartheta(r),\,\zeta(l))\right)\prod_{r_1=\zeta(l)}^{l-1}\Psi_{k_3}(r_1)\Bigg)\leqslant 1.$$

This contradiction completes the proof. □

Remark 2.10. It should be noted that Lemma 2.3 can provide a new estimation of a positive solution rate of decay for a differential equation with several retarded arguments, which improves the estimation of [8, Lemma 2.1]. Therefore, all the iterative oscillation results in [8] can be improved.

Example 2.11. Consider the first-order nonlinear difference equation

$$\Delta y(l)+a(l)y(l+1)+b(l)f(y(\vartheta(l)))=0, \tag{2.26}$$

where $a(l)\geqslant \alpha_1>0$, and

$$\vartheta(l)=\begin{cases} l-1, & \text{if } l=2k,\\ l-3, & \text{if } l=2k+1,\end{cases}\qquad k\in\mathbb{N}_0,$$

and

$$b(l)=\begin{cases} \mu, & \text{if } l\in\{2k_i-2,2k_i-1,2k_i,2k_i+1,2k_i+2\},\\ 0, & \text{otherwise},\end{cases}\qquad i\in\mathbb{N}_0,$$

where $(k_i)_{i\geqslant 0}$ is a sequence of positive integers such that $k_{i+1}>k_i+\frac{3}{2}$, for all $i\in\mathbb{N}_0$ and $\lim_{i\to\infty} k_i=\infty$, and

$$f(y)=\begin{cases} 0, & \text{if } y\leqslant -2\epsilon_1,\\ -\frac{(\delta_1\,\epsilon_1-\epsilon_1{}^2)}{\epsilon_1}(y+\epsilon_1)-\delta_1\,\epsilon_1+\epsilon_1{}^2, & \text{if } y\in[-2\epsilon_1,\,-\epsilon_1],\\ y^2+\delta_1 y, & \text{if } y\in[-\epsilon_1,\,\epsilon_1],\\ \frac{(-\delta_1\,\epsilon_1-\epsilon_1{}^2)}{\epsilon_1}(y-\epsilon_1)+\delta_1\,\epsilon_1+\epsilon_1{}^2, & \text{if } y\in[\epsilon_1,\,2\epsilon_1],\\ 0, & \text{if } y\geqslant 2\epsilon_1,\end{cases}$$

where $\epsilon_1,\delta_1>0$. Clearly,

$$\liminf_{y\to 0}\frac{f(y)}{y}=\delta_1,$$

and

$$\zeta(l)=\begin{cases} l-1, & \text{if } l=2k,\\ l-2, & \text{if } l=2k+1,\end{cases}\qquad k\in\mathbb{N}_0.$$

Then one can choose $\delta=\delta_1-\frac{1}{10000}$ and $\alpha=\alpha_1-\frac{1}{10000}$ (that are defined as in (1.2) and (1.3), respectively). Let

$$I_1(l)=\delta b(l+1)W_{k_1,k_2}(\zeta(l+1),r)\,(1+a(l))+\delta b(l)\prod_{r_1=\zeta(l)}^{l-1}\Psi_1(r_1).$$

Therefore,

$$I_1(l)\geqslant \delta b(l+1)\,(1+a(l))+\delta b(l)\prod_{r_1=\zeta(l)}^{l-1}\Psi_1(r_1).$$

Since

$$\Psi_1(l) \geqslant \frac{1+\alpha}{1-\mu\delta} \quad \text{for } l \in \{2k_i-2, 2k_i-1, 2k_i, 2k_i+1, 2k_i+2\},\ i \in \mathbb{N}_0,$$

then

$$I_1(2k_i+1) \geqslant \delta\mu\left((1+\alpha)+\frac{(1+\alpha)^2}{(1-\mu\delta)^2}\right), \quad i \in \mathbb{N}_0.$$

Then

$$\limsup_{l\to\infty} I_1(l) \geqslant \limsup_{i\to\infty} I_1(2k_i+1) \geqslant \delta\mu\left((1+\alpha)+\frac{(1+\alpha)^2}{(1-\mu\delta)^2}\right).$$

Consequently, condition (2.23) with $k_3 = 1$ is satisfied and so every solution of Eq. (2.26) is oscillatory, provided that

$$(1+\alpha)+\frac{(1+\alpha)^2}{(1-\mu\delta)^2} > \frac{1}{\delta\mu}.$$

For example, if $\delta_1 = \frac{1}{2}+\frac{1}{10000}$ and $\alpha_1 = \frac{1}{2}+\frac{1}{10000}$, then Eq. (2.26) is oscillatory for all $\mu \geqslant \frac{4}{10}$.

3. Conclusion

We study the oscillation of a first-order nonlinear difference equation with retarded arguments. We have generalized, extended, and improved some methods used to study the oscillation of first-order linear difference equations with single and several delays to study the oscillation of first-order nonlinear delay difference equations. Many of the methods used in this work can be used to improve many oscillation results for the corresponding linear equations. In Example 2.11, we have demonstrated the simplicity of applying some of our results.

Funding

The author extends their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2023/01/25710).

References

[1] J. Alzabut, M. Bohner, S. R. Grace, *Oscillation of nonlinear third-order difference equations with mixed neutral terms*, Adv. Difference Equ., **2021** (2021), 18 pages. 1

[2] E. R. Attia, G. E. Chatzarakis, *Oscillation tests for difference equations with non-monotone retarded arguments*, Appl. Math. Lett., **123** (2022), 6 pages. 1, 1

[3] E. R. Attia, B. M. El-Matary, *New aspects for the oscillation of first-order difference equations with deviating arguments*, Opuscula Math., **42** (2022), 393–413.

[4] E. R. Attia, B. M. El-Matary, G. E. Chatzarakis, *New oscillation conditions for first-order linear retarded difference equations with non-monotone arguments*, Opuscula Math., **42** (2022), 769–791. 1, 1

[5] L. Berezansky, E. Braverman, *On existence of positive solutions for linear difference equations with several delays*, Adv. Dyn. Syst. Appl., **1** (2006), 29–47. 1, 1

[6] M. Bohner G. E. Chatzarakis, I. P. Stavroulakis, *Oscillation criteria for difference equations with several oscillating coefficients*, Bull. Korean Math. Soc., **52** (2015), 159–172. 1

[7] M. Bohner, H. A. El-Morshedy, S. R. Grace, I. Sağer, *Oscillation of second-order nonlinear difference equations with sublinear neutral term*, Math. Morav., **23** (2019), 1–10. 1

[8] E. Braverman, G. E. Chatzarakis, I. P. Stavroulakis, *Iterative oscillation tests for difference equations with several non-monotone arguments*, J. Difference Equ. Appl., **21** (2015), 854–874. 1, 1, 2, 2, 2.10

[9] G. E. Chatzarakis, I. Jadlovská, *Oscillations in deviating difference equations using an iterative technique*, J. Inequal. Appl., **2017** (2017), 24 pages. 1

[10] G. E. Chatzarakis, R. Koplatadze, I. P. Stavroulakis, *Oscillation criteria of first order linear difference equations with delay argument*, Nonlinear Anal., **68** (2008), 994–1005. 1

[11] G. E. Chatzarakis, R. Koplatadze, I. P. Stavroulakis, *Optimal oscillation criteria for first order difference equations with delay argument*, Pacifc J. Math., **235** (2008), 15–33. 1, 1

[12] G. E. Chatzarakis, S. Pinelas, I. P. Stavroulakis, *Oscillations of difference equations with several deviated arguments*, Aequationes Math., **88** (2014), 105–123. 1
[13] G. E. Chatzarakis, Ch. G. Philos, I. P. Stavroulakis, *An oscillation criterion for linear difference equations with general delay argument*, Port. Math., **66** (2009), 513–533. 1, 2
[14] B. M. El-Matary, H. A. El-Morshedy, V. Benekas, I. P. Stavroulakis, *Oscillation conditions for difference equations with several variable delays*, Opuscula Math., **43** (2022), 789–801. 1, 1
[15] J. C. Jiang, X. Li, X. Tang, *New oscillation criteria for first-Order delay difference equations*, Comput. Math. Appl., **47** (2004), 1875–1884. 1, 1
[16] J. C. Jiang, X. H. Tang, *Oscillation of nonlinear delay difference equations*, J. Comput. Appl. Math., **146** (2002), 395–404. 1, 1
[17] G. Ladas, Ch. G. Philos, Y. G. Sficas, *Sharp conditions for the oscillation of delay difference equations*, J. Appl. Math. Simulation, **2** (1989), 101–111. 1, 1, 1
[18] X. N. Luo, Y. Zhou, C. F. Li, *Oscillations of a nonlinear difference equation with several delays*, Math. Bohem., **128** (2003), 309–317. 1, 1
[19] X. H. Tang, J. S. Yu, *Oscillation of nonlinear delay difference equations*, J. Math. Anal. Appl. **249** (2000), 476–490. 1, 1
[20] W. Yan, Q. Meng, J. Yan, *Oscillation criteria for difference equation of variable delays*, DCDIS Proc., **3** (2005), 641–647. 1
[21] J. Yan, C. Qian, *Oscillation and comparison result for delay difference equations*, J. Math. Anal. Appl. **165** (1992), 346–360. 1, 1, 1