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Various generalizations of uncertainty principles related to the linear canonical ambiguity function

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Abstract

The uncertainty principle is a fundamental result of the Fourier transform and is currently one of the most rapidly developing areas of mathematics due to its application in various transformations. This paper deals with the linear canonical ambiguity function. It combines the classical ambiguity function and the linear canonical transform. We derive in detail various uncertainty principles related to the proposed transformation.

Keywords: Linear canonical ambiguity function, uncertainty principle, Pitt's inequality, Matolcsi-Szücs inequality.

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1. Introduction

The linear canonical transform (LCT) [9, 14, 20, 22–24] represents a class of linear integral transforms with three free parameters that has found many applications in various fields, including signal processing and optics. It can be looked as extension of many transformations such as the Fourier transform, Laplace transform, the fractional Fourier transform, the Fresnel transform and the other transforms. Many essential properties of this transformation have been known, including shifting, modulation, and uncertainty principles. In recent years, the research on generalization of various kinds of transformations using the linear canonical transform has developed rapidly. In [2, 4, 5, 13, 17, 21], the authors presented Wigner-Ville distribution associated with the linear canonical transforms. Some inequalities related to this transformation were demonstrated in detail. The authors of [1, 11, 16] proposed the windowed linear canonical transform and the uncertainty principles concerning this transform is also demonstrated. The authors of [3, 6, 8, 10] studied the linear canonical ambiguity function, which is a generalization of the ambiguity function in the LCT space. Several essential properties and an application of the generalized transform is also discussed. However, they have not published the uncertainty principles related to this generalized transformation.

Therefore, the purpose of this paper is to propose several versions of the uncertainty principles concerning the linear canonical ambiguity function (LCAF). In that regard, we recall definition of the linear

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canonical ambiguity function and its useful properties. We make a natural relation between the linear canonical ambiguity function and the classical ambiguity function. Based on these facts, we construct several uncertainty principles associated with the linear canonical ambiguity function. The uncertainty inequalities play an important role in understanding behavioral signals associated with the proposed linear canonical ambiguity functions.

Remaining part of the paper is organized as follows. In Section 2, we present some preliminaries related to the linear canonical transform, that will be useful in sequel. The definition of the linear canonical ambiguity function (LCAF) and its useful properties in Section 3. Section 4 is devoted to derivation of several versions of the uncertainty principles related to the linear canonical ambiguity function. Section 5 focuses on the Heisenberg-type inequality in the context of the LCAF. Subsequently, Section 6 examines a generalized version of Pitt's inequality specifically tailored to the LCAF. Lastly, Section 7 draws conclusions.

2. Linear canonical transform

This part contains a brief introduction of the linear canonical transform (LCT) [9, 18] and a discussion of its basic properties.

Definition 2.1 (LCT definition)**.** Let $A = \big(a, b, c, d\big) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(\mathbb{R}, 2)$. The linear canonical transform of any function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$L_A\{f\}(\omega) = \begin{cases} \int_{\mathbb{R}} f(x) K_A(x,\omega)dx, & b \neq 0, \\ \sqrt{d} e^{i\left(\frac{cd}{2}\right)\omega^2} f\big(d\omega\big), & b = 0. \end{cases}$$

Here kernel $K_A(x, \omega)$ is given by

$$K_A(x,\omega) = \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\ \omega^2 - \frac{\pi}{2}\right)}. \tag{2.1}$$

It should be remembered, when $b = 0$, the LCT of a signal is essentially a chirp multiplication. Therefore, in this study we always consider $b \neq 0$. The reconstruction formula of the LCT is given by

$$f(x) = \int_{\mathbb{R}} L_A\{f\}(\omega)\overline{K_A(x,\omega)}d\omega = \frac{1}{\sqrt{2\pi b}} \int_{\mathbb{R}} L_A\{f\}(\omega) e^{-\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} d\omega, \tag{2.2}$$

where $\overline{K_A(x,\omega)}$ is the complex conjugate of $K_A(x, \omega)$. An important property of the LCT is Parseval's formula expressed as

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} L_A(f)(\omega)\overline{L_A(g)}(\omega)d\omega = (L_A(f), L_A(g))$$

for all $f, g \in L^2(\mathbb{R})$. Especially, for $f = g$ we obtain the Plancherel's formula for the LCT as

$$\|f\|_{L^2(\mathbb{R})} = \|L_A(f)\|_{L^2(\mathbb{R})}.$$

3. Linear canonical ambiguity function with properties

This part provides a definition of the linear canonical ambiguity function (LCAF) and summarizes its properties. We also make a relation between the linear canonical ambiguity function and the Fourier transform.

3.1. Definition of LCAF

We start by giving the definition of the linear canonical ambiguity function (LCAF) and state some of its basic properties. This definition is constituted by substituting the kernel of the LCT described by (2.1) with the kernel Fourier in the definition of the classical ambiguity function.

Definition 3.1. Let f, g be two functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The linear canonical ambiguity function (LCAF) of functions f and g is defined as

$$\mathcal{A}_{f,g}^{A}(t,\omega) = \frac{1}{\sqrt{2\pi b}} \int_{\mathbb{R}} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)} e^{\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} dx. \tag{3.1}$$

It is easily seen that equation (3.1) may be rewritten in the form

$$\mathcal{A}_{f,g}^{A}(t,\omega) = \int_{\mathbb{R}} h_{f,g}(x,t) K_A(x,\omega)\, dx = L_A\left\{h_{f,g}(x,t)\right\}(\omega),$$

for any fixed t. Here

$$h_{f,g}(x,t) = f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}.$$

In the specific case, when $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, equation (3.1) boils down to the classical ambiguity function definition, that is,

$$\mathcal{A}_{f,g}^{A}(t,\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{A}_{f,g}(t,\omega),$$

where

$$\mathcal{A}_{f,g}(t,\omega) = \int_{\mathbb{R}} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)} e^{ix\omega} dx.$$

We first observe that for $f, g \in L^2(\mathbb{R})$, the LCAF defined in (3.1) is bounded on $L^2(\mathbb{R})$, that is,

$$\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^2 \leqslant \frac{1}{2\pi|b|} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.$$

This equation easily can be seen from

$$\begin{aligned}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^2 &\leqslant \int_{\mathbb{R}} \left| f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)} \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} \right|^2 dx \\ &= \frac{1}{2\pi|b|} \int_{\mathbb{R}} \left| f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)} \right|^2 dx.\end{aligned}$$

According to Cauchy-Scwarz inequality, we obtain

$$\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^2 \leqslant \frac{1}{2\pi|b|} \int_{\mathbb{R}} \left| f\left(x+\frac{t}{2}\right)\right|^2 dx \int_{\mathbb{R}} \left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx = \frac{1}{2\pi|b|} \int_{\mathbb{R}} |f(y)|^2 dy \int_{\mathbb{R}} \left|\overline{g(y)}\right|^2 dy.$$

Hence the result follows.

3.2. Useful properties of LCAF

The properties of the LCAF is listed below. It is seen that the most of them are extensions of the corresponding version of the classical ambiguity function (AF) with some changes.

Theorem 3.2 (Complex conjugation)**.** *For any function* $f, g \in L^2(\mathbb{R})$*, we have* ([3])

$$\overline{\mathcal{A}_{g,f}^{A^{-1}}(t,\omega)} = \mathcal{A}_{g,f}^{A^{-1}}(-t,\omega),$$

where $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ *is inverse matrix of* A.

Theorem 3.3 (Shifting)**.** *For every* $f, g \in L^2(\mathbb{R})$*, one has* ([3])

$$\mathcal{A}_{\tau_k f, \tau_k g}^{A}(t,\omega) = e^{ick\omega} e^{-i\frac{ack^2}{2}} \mathcal{A}_{f,g}^{A}(t, \omega - ka).$$

Here, τ_k *is the translation operator defined by*

$$\tau_k f(t) = f(t-k), \quad \forall t \in \mathbb{R}.$$

Theorem 3.4 (Modulation)**.** *For any function* $f, g \in L^2(\mathbb{R})$*, we have* ([3])

$$\mathcal{A}_{\mathbb{M}_{\omega_0} f, \mathbb{M}_{\omega_0} g,}^{A}(t,\omega) = e^{i\omega_0 t} \mathcal{A}_{f,g}^{A}(t,\omega).$$

Here, $\mathbb{M}_{\omega_0}$ *is the modulation operator defined by*

$$\mathbb{M}_{\omega_0} f(t) = e^{i\omega_0 t} f(t), \quad \forall t \in \mathbb{R}.$$

Theorem 3.5 (Modulation and translation)**.** *Let* $f, g \in L^2(\mathbb{R})$ *be two complex functions. Then we get*

$$\mathcal{A}_{\mathbb{M}_{\omega_0}\tau_k f, \mathbb{M}_{\omega_0}\tau_k g}^{A}(t,\omega) = e^{i\omega_0 t} \mathcal{A}_{f,g}^{A}(t, \omega - ka).$$

Theorem 3.6 (Reconstruction formula)**.** *For any function* $f, g \in L^2(\mathbb{R})$*, we have*

$$f(t) = \frac{1}{\overline{g(0)}} \int_{\mathbb{R}} \mathcal{A}_{f,g}^{A}(t,\omega)\, \overline{K_A\left(\frac{t}{2}, \omega\right)} d\omega.$$

Proof. Using the inverse of the LCT (2.2) we easily obtain

$$f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} = \int_{\mathbb{R}} \mathcal{A}_{f,g}^{A}(t,\omega) \overline{K_A(x,\omega)} d\omega.$$

Taking the specific value $x = \frac{t}{2}$ and the above yields

$$f(t)\,\overline{g(0)} = \int_{\mathbb{R}} \mathcal{A}_{f,g}^{A}(t,\omega) \overline{K_A\left(\frac{t}{2}, \omega\right)} d\omega.$$

This is equal to

$$f(t) = \frac{1}{\overline{g(0)}} \int_{\mathbb{R}} \mathcal{A}_{f,g}^{A}(t,\omega)\, \overline{K_A\left(\frac{t}{2}, \omega\right)} d\omega.$$

The proof is complete. □

Theorem 3.7 (Moyal's formula)**.** *For all complex functions* $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$*, then the following result is satisfied:*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{A}_{f_1,g_1}^{A}(t,\omega) \overline{\mathcal{A}_{f_2,g_2}^{A}(t,\omega)} d\omega dt = 2\ (f_1, f_2)\, \overline{(g_1, g_2)}.$$

Especially, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^2 d\omega dt = 2\|f\|_{L^2(\mathbb{R})}^2 \|g\|_{L^2(\mathbb{R})}^2,$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{A}_{f}^{A}(t,\omega) \overline{\mathcal{A}_{g}^{A}(t,\omega)} d\omega dt = 2\ |(f,g)|^2.$$

3.3. *Relationship between LCAF and FT*

In what follows, we describe a relationship between the LCAF and the FT. This fact is very useful in deriving several uncertainty principles associated with the LCAF. The relationship between them is the following:

$$\sqrt{2\pi b}\mathcal{A}_{f,g}^{A}(t,\omega)e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)} = \mathcal{F}\left\{R_{f,g}(x,t)\right\}\left(\frac{\omega}{b}\right), \tag{3.2}$$

where

$$R_{f,g}(x,t) = f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{\frac{ia}{2b}x^2}, \tag{3.3}$$

for a fixed t and $\mathcal{F}\{f\}$ is the Fourier transform of $f \in L^1(\mathbb{R})$ given by ([7, 12, 15])

$$\mathcal{F}\{f(x)\}(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x}dx.$$

To verify this, it follows from (3.1) that

$$\begin{aligned}\mathcal{A}_{f,g}^{A}(t,\omega) &= \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{\frac{i}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}dx\\ &= \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{\frac{ia}{2b}x^2}e^{-i\frac{\omega}{b}x}e^{\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}dx.\end{aligned}$$

The above identity may be expressed as

$$\mathcal{A}_{f,g}^{A}(t,\omega) = \frac{1}{\sqrt{2\pi b}}\int_{\mathbb{R}} R_{f,g}(x,t)e^{\frac{ia}{2b}x^2-i\frac{\omega}{b}x}e^{\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}dx.$$

Hence,

$$\sqrt{2\pi b}\mathcal{A}_{f,g}^{A}(t,\omega)e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)} = \int_{\mathbb{R}} R_{f,g}(x,t)e^{-i\frac{\omega}{b}x}dx,$$

which gives (3.2).

4. New inequalities for LCAF

The uncertainty principle is one of the fundamental results in the linear canonical ambiguity function. In this part, we will establish some new versions of these uncertainty principles involving the proposed transformation.

4.1. *Nazarov's inequality*

We explore Nazarov's inequality for LCAF, which is a direct expansion of Nazarov's inequality for the Fourier transform.

Theorem 4.1. *Let $f, g \in L^2(\mathbb{R})$ and A, B be two measurable subsets of $\mathbb{R}$. Suppose that for some $C > 0$, one has*

$$\|f\|_{L^2(\mathbb{R})}^2\|g\|_{L^2(\mathbb{R})}^2 \leqslant Ce^{|A||B|}\left(\|f\|_{L^2(\mathbb{R})}^2\int_{\mathbb{R}\setminus A}\left|g\left(x-\frac{t}{2}\right)\right|^2dx + \pi\int_{\mathbb{R}}\int_{\mathbb{R}\setminus bB}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^2d\omega dt\right).$$

Proof. Due to Nazarov's inequality for the Fourier transform, we obtain

$$\int_{\mathbb{R}} |f(x)|^2\, dx \leqslant Ce^{|A||B|}\left(\int_{\mathbb{R}\setminus A} |f(x)|^2\, dx + \int_{\mathbb{R}\setminus B} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega\right). \tag{4.1}$$

Substituting $f(x)$ by $R_{f,g}(x,t)$ into (4.1), we obtain

$$\int_{\mathbb{R}} \left|R_{f,g}(x,t)\right|^2 dx \leqslant Ce^{|A||B|}\left(\int_{\mathbb{R}\setminus A} \left|R_{f,g}(x,t)\right|^2 dx + \int_{\mathbb{R}\setminus B} \left|\mathcal{F}\left\{R_{f,g}(x,t)\right\}(\omega)\right|^2 d\omega\right).$$

Letting $\omega = \frac{\omega}{b}$, we have

$$\int_{\mathbb{R}} \left|R_{f,g}(x,t)\right|^2 dx \leqslant Ce^{|A||B|}\left(\int_{\mathbb{R}\setminus A} \left|R_{f,g}(x,t)\right|^2 dx + \int_{\mathbb{R}\setminus bB} \left|\mathcal{F}\left\{R_{f,g}(x,t)\right\}\left(\frac{\omega}{b}\right)\right|^2 d\frac{\omega}{b}\right).$$

An application of equations (3.2) and (3.3) results in

$$\begin{aligned}\int_{\mathbb{R}} \left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx \leqslant Ce^{|A||B|}\Bigg(&\int_{\mathbb{R}\setminus A} \left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\\ &+\frac{1}{|b|}\int_{\mathbb{R}\setminus bB} \left|\sqrt{2\pi b}\mathcal{A}^{A}_{f,g}(t,\omega)e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}\right|^2 d\omega\Bigg).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^2 dx \leqslant Ce^{|A||B|}\Bigg(&\int_{\mathbb{R}\setminus A} \left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^2 dx\\ &+2\pi\int_{\mathbb{R}\setminus bB} \left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega\Bigg).\end{aligned} \tag{4.2}$$

Integrating (4.2) with respect to t on both sides yields

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^2 dxdt \leqslant Ce^{|A||B|}\Bigg(&\int_{\mathbb{R}}\int_{\mathbb{R}\setminus A} \left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^2 dxdt\\ &+2\pi\int_{\mathbb{R}}\int_{\mathbb{R}\setminus bB} \left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega dt\Bigg).\end{aligned}$$

This equation will lead to

$$\begin{aligned}\int_{\mathbb{R}} \left|f\left(x+\frac{t}{2}\right)\right|^2 dt\int_{\mathbb{R}} \left|g\left(x-\frac{t}{2}\right)\right|^2 dx \leqslant Ce^{|A||B|}\Bigg(&\int_{\mathbb{R}}\int_{\mathbb{R}\setminus A} \left|f\left(x+\frac{t}{2}\right)\right|^2\left|g\left(x-\frac{t}{2}\right)\right|^2 dxdt\\ &+2\pi\int_{\mathbb{R}}\int_{\mathbb{R}\setminus bB} \left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega dt\Bigg),\end{aligned}$$

which is, putting $x+\frac{t}{2}=y$ and $x-\frac{t}{2}=z$, we get

$$2\int_{\mathbb{R}} |f(y)|^2 dy\int_{\mathbb{R}} |g(z)|^2 dz \leqslant Ce^{|A||B|}\Bigg(2\int_{\mathbb{R}} |f(y)|^2 dy\int_{\mathbb{R}\setminus A} \left|g\left(x-\frac{t}{2}\right)\right|^2 dx$$

$$+2\pi\int_{\mathbb{R}}\int_{\mathbb{R}\setminus bB}\left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega dt\Bigg).$$

Finally, we obtain

$$\|f\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}\leqslant Ce^{|A||B|}\left(\|f\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}\setminus A}\left|g\left(x-\frac{t}{2}\right)\right|^2 dx+\pi\int_{\mathbb{R}}\int_{\mathbb{R}\setminus bB}\left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega dt\right).$$

The proof is complete. □

4.2. Pitt's inequality

In this part, we present Pitt's inequality for the LCAF, which is an extension of Pitt's inequality in the LCAF domain.

Theorem 4.2. *For all* $f,g\in\mathcal{S}(\mathbb{R})$ *and* $0\leqslant\alpha<1$, *there holds*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^{-\alpha}|\mathcal{A}^{A}_{f,g}(t,\omega)|^2 d\omega dt\leqslant\frac{c_\alpha}{\pi|b|^\alpha}\|f\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}}|x|^\alpha\left|g\left(x-\frac{t}{2}\right)\right|^2 dx,$$

where

$$c_\alpha=\pi^\alpha\left[\frac{\Gamma\left(\frac{1-\alpha}{4}\right)}{\Gamma\left(\frac{1+\alpha}{4}\right)}\right]^2. \tag{4.3}$$

Here, $\mathcal{S}(\mathbb{R})$ *is the Schwartz space on* $\mathbb{R}$ *given by*

$$\mathcal{S}(\mathbb{R})=\left\{f(x)\in C^\infty(\mathbb{R}):\sup_{x\in\mathbb{R}}\left|x^m\frac{d^n}{dx^n}f(x)\right|<\infty,\ \forall m,n\in\mathbb{N}\right\},$$

where $C^\infty(\mathbb{R})$ *is the set of smooth function on* $\mathbb{R}$.

Proof. Based on Pitt's inequality for the Fourier transform, it follows that

$$\int_{\mathbb{R}}\left|\omega\right|^{-\alpha}\left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega\leqslant c_\alpha\int_{\mathbb{R}}\left|x\right|^\alpha\left|f(x)\right|^2 dx. \tag{4.4}$$

Replacing $f(x)$ with $R_{f,g}(x,t)$ in equation (4.4) results in

$$\int_{\mathbb{R}}\left|\omega\right|^{-\alpha}\left|\mathcal{F}\{R_{f,g}(x,t)\}(\omega)\right|^2 d\omega\leqslant c_\alpha\int_{\mathbb{R}}\left|x\right|^\alpha\left|R_{f,g}(x,t)\right|^2 dx.$$

It is obvious to see that

$$\int_{\mathbb{R}}\left|\frac{\omega}{b}\right|^{-\alpha}\left|\mathcal{F}\{R_{f,g}(x,t)\}\left(\frac{\omega}{b}\right)\right|^2 d\left(\frac{\omega}{b}\right)\leqslant c_\alpha\int_{\mathbb{R}}|x|^\alpha\left|R_{f,g}(x,t)\right|^2 dx.$$

Due to equation (3.2), we obtain

$$\frac{1}{|b|^{-\alpha}|b|}\int_{\mathbb{R}}|\omega|^{-\alpha}\left|\sqrt{2\pi b}\mathcal{A}^{A}_{f,g}(t,\omega)e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}\right|^2 d\omega\leqslant c_\alpha\int_{\mathbb{R}}|x|^\alpha\left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx.$$

Simplifying it gives

$$\frac{2\pi}{|b|^{-\alpha}}\int_{\mathbb{R}}|\omega|^{-\alpha}\left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega\leqslant c_\alpha\int_{\mathbb{R}}|x|^\alpha\left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx. \tag{4.5}$$

Integrating both sides of equation (4.5) with respect to t, we see that

$$\frac{2\pi}{|b|^{-\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^{-\alpha}\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega dt \leqslant c_\alpha \int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)\right|^2 dt\int_{\mathbb{R}}|x|^\alpha\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx.$$

Hence,

$$\frac{2\pi}{|b|^{-\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^{-\alpha}|\mathcal{A}^A_{f,g}(t,\omega)|^2 d\omega dt \leqslant 2c_\alpha\int_{\mathbb{R}}|f(t)|^2 dt\int_{\mathbb{R}}|x|^\alpha\left|g\left(x-\frac{t}{2}\right)\right|^2 dx.$$

This equation is the same as

$$\frac{2\pi}{|b|^{-\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^{-\alpha}|\mathcal{A}^A_{f,g}(t,\omega)|^2 d\omega dt \leqslant 2c_\alpha\,\|f\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}}|x|^\alpha\left|g\left(x-\frac{t}{2}\right)\right|^2 dx.$$

The proof is complete. □

4.3. Sharp Hausdorff-Young inequality

In this subsection, we derive sharp Hausdorff-Young inequality related to the LCAF, which generalizes sharp Hausdorff-Young inequality for the FT in LCAF domains.

Theorem 4.3. *Let* $1 < r \leqslant 2$, *such that* $\frac{1}{r}+\frac{1}{s}=1$, *then for any* $f, g \in L^r(\mathbb{R})$, *the following inequality holds*

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^s d\omega dt\right)^{\frac{1}{s}} \leqslant 2c^2(r)\frac{(2\pi)^{-\frac{1}{2}}}{|b|^{\frac{1}{2}-\frac{1}{s}}}\|f\|_{L^r(\mathbb{R})}\|g\|_{L^r(\mathbb{R})}, \tag{4.6}$$

where

$$c(r) = \left(r^{\frac{1}{r}}s^{-\frac{1}{s}}\right)^{\frac{1}{2}}.$$

Proof. By virtue of sharp Hausdorff-Young inequality for the Fourier transform, we immediately obtain

$$\left(\int_{\mathbb{R}}|\mathcal{F}\{f(x)\}(\omega)|^s d\omega\right)^{\frac{1}{s}} \leqslant c^2(r)\left(\int_{\mathbb{R}}|f(x)|^r dx\right)^{\frac{1}{r}}. \tag{4.7}$$

Replacing $f(x)$ by $R_{f,g}(x,t)$ in equation (4.7) we acquire

$$\left(\int_{\mathbb{R}}|\mathcal{F}\{R_{f,g}(x,t)\}(\omega)|^s d\omega\right)^{\frac{1}{s}} \leqslant c^2(r)\left(\int_{\mathbb{R}}|R_{f,g}(x,t)|^r dx\right)^{\frac{1}{r}}.$$

We further find that

$$\left(\int_{\mathbb{R}}\left|\mathcal{F}\{R_{f,g}(x,t)\}\left(\frac{\omega}{b}\right)\right|^s d\left(\frac{\omega}{b}\right)\right)^{\frac{1}{s}} \leqslant c^2(r)\left(\int_{\mathbb{R}}|R_{f,g}(x,t)|^r dx\right)^{\frac{1}{r}}.$$

Therefore,

$$\frac{1}{|b|^{\frac{1}{s}}}\left(\int_{\mathbb{R}}\left|\mathcal{F}\{R_{f,g}(x,t)\}\left(\frac{\omega}{b}\right)\right|^s d\omega\right)^{\frac{1}{s}} \leqslant c^2(r)\left(\int_{\mathbb{R}}|R_{f,g}(x,t)|^r dx\right)^{\frac{1}{r}}.$$

Using relation (3.2), it reduces to

$$\frac{1}{|b|^{\frac{1}{s}}}\left(\int_{\mathbb{R}}\left|\sqrt{2\pi b}\mathcal{A}^A_{f,g}(t,\omega)e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}\right|^s d\omega\right)^{\frac{1}{s}} \leqslant c^2(r)\left(\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{\frac{ia}{2b}x^2}\right|^r dx\right)^{\frac{1}{r}}.$$

This equation can be further simplified to

$$\frac{(2\pi)^{\frac{1}{2}}}{|b|^{\frac{1}{s}-\frac{1}{2}}}\left(\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s}d\omega\right)\leqslant c^{2s}(r)\left(\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^{r}dx\right)^{\frac{s}{r}}. \tag{4.8}$$

Integrating both sides of (4.8) with respect to t we have

$$\frac{(2\pi)^{\frac{1}{2}}}{|b|^{\frac{1}{s}-\frac{1}{2}}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s}d\omega dt\right)\leqslant \int_{\mathbb{R}}\left(c^{2s}(r)\left(\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)g\left(x-\frac{t}{2}\right)\right|^{r}dx\right)^{\frac{s}{r}}\right)dt.$$

This equation can be expressed as

$$\frac{(2\pi)^{\frac{1}{2}}}{|b|^{\frac{1}{s}-\frac{1}{2}}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s}d\omega dt\right)\leqslant c^{2s}(r)\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)\right|^{r}\left|g\left(x-\frac{t}{2}\right)\right|^{r}dxdt\right)^{\frac{s}{r}}.$$

Fubini's theorem allows us to obtain

$$\frac{(2\pi)^{\frac{1}{2}}}{|b|^{\frac{1}{s}-\frac{1}{2}}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s}d\omega dt\right)^{\frac{1}{s}}\leqslant 2c^{2}(r)\left(\int_{\mathbb{R}}|f(t)|^{r}dt\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}}|g(x)|^{r}dx\right)^{\frac{1}{r}},$$

which is equal to

$$\frac{(2\pi)^{\frac{1}{2}}}{|b|^{\frac{1}{s}-\frac{1}{2}}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s}d\omega dt\right)^{\frac{1}{s}}\leqslant 2c^{2}(r)\|f\|_{L^{r}(\mathbb{R})}\|g\|_{L^{r}(\mathbb{R})}.$$

This proves the theorem. □

4.4. Matolcsi-Szücs inequality

The utility of sharp Hausdorff-Young inequality for the LCAF will lead to the following theorem.

Theorem 4.4. *Let* f, g *be in* $L^2(\mathbb{R})$. *The following inequality is satisfied*

$$\begin{aligned}\left\|\mathcal{A}_{f,g}^{A}(t,\omega)\right\|_{L^{s_2}(\mathbb{R}\times\mathbb{R})}\leqslant&\left|\operatorname{supp}\left(\mathcal{A}_{f,g}^{A}(t,\omega)\right)\right|^{\frac{s_1-s_2}{s_1s_2}}2c^{2}(r_1)\frac{(2\pi)^{-\frac{1}{2}}}{|b|^{\frac{1}{2}-\frac{s_1}{2}}}\\&\times\left|\operatorname{supp}(f)\right|^{\frac{r_2-r_1}{r_1r_2}}\|f\|_{L^{r_2}(\mathbb{R})}\left|\operatorname{supp}(g)\right|^{\frac{r_2-r_1}{r_1r_2}}\|g\|_{L^{r_2}(\mathbb{R})},\end{aligned}$$

where

$$\frac{1}{r_1}+\frac{1}{s_1}=1,\quad \frac{1}{r_2}+\frac{1}{s_2}=1,\quad \text{and}\quad 1<r_1\leqslant r_2\leqslant 2.$$

Proof. An application of Hölder's inequality will lead to

$$\begin{aligned}\left\|\mathcal{A}_{f,g}^{A}(t,\omega)\right\|_{L^{s_2}(\mathbb{R}\times\mathbb{R})}&=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}_{f,g}^{A}(t,\omega)\right|^{s_2}dxdt\right)^{\frac{1}{s_2}}\\&\leqslant\left|\operatorname{supp}\left(\mathcal{A}_{f,g}^{A}(t,\omega)\right)\right|^{\frac{s_1-s_2}{s_1s_2}}\left\|\mathcal{A}_{f,g}^{A}(t,\omega)\right\|_{L^{s_1}(\mathbb{R}\times\mathbb{R})}.\end{aligned}$$

By virtue of Hausdorff-Young inequality for LCAF described by (4.6), we get

$$\left\|\mathcal{A}_{f,g}^{A}(t,\omega)\right\|_{L^{s_2}(\mathbb{R}\times\mathbb{R})}\leqslant\left|\operatorname{supp}\left(\mathcal{A}_{f,g}^{A}(t,\omega)\right)\right|^{\frac{s_1-s_2}{s_1s_2}}2c^{2}(r_1)\frac{(2\pi)^{-\frac{1}{2}}}{|b|^{\frac{1}{2}-\frac{s_1}{2}}}\|f\|_{L^{r_1}(\mathbb{R})}\|g\|_{L^{r_1}(\mathbb{R})}. \tag{4.9}$$

Notice that

$$\|g\|_{L^{r_1}(\mathbb{R})} = \left(\int_{\mathbb{R}} |X_G g|^{r_1} dx\right)^{\frac{1}{r_1}} \leqslant \left(\int_{\mathbb{R}} |X_G|^{\frac{r_1 r_2}{r_2 - r_1}} dx\right)^{\frac{r_2-r_1}{r_1 r_2}} \left(\int_{\mathbb{R}} |g|^{\frac{r_1 r_2}{r_1}} dx\right)^{\frac{r_1}{r_1 r_2}} = \left|\text{supp}\,(g)\right|^{\frac{r_2-r_1}{r_1 r_2}} \|g\|_{L^{r_2}(\mathbb{R})}, \tag{4.10}$$

where X_G is the indicator function of $G = \text{supp}(g)$. Substituting (4.10) into (4.9), we immediately obtain

$$\|\mathcal{A}^A_{f,g}(t,\omega)\|_{L^{s_2}(\mathbb{R}\times\mathbb{R})} \leqslant \left|\text{supp}\,(\mathcal{A}^A_{f,g}(t,\omega))\right|^{\frac{s_1-s_2}{s_1 s_2}} 2c^2(r_1)\frac{(2\pi)^{-\frac{1}{2}}}{|b|^{\frac{1}{2}-\frac{s_1}{2}}}\left|\text{supp}\,(f)\right|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})} \times \left|\text{supp}\,(g)\right|^{\frac{r_2-r_1}{r_1 r_2}} \|g\|_{L^{r_2}(\mathbb{R})}.$$

This is the required result. □

5. Heisenberg inequality for LCAF

In the following, we present Heisenberg-type inequality in the context of the LCAF.

Theorem 5.1 (Heisenberg inequality)**.** *Given* $f, g \in L^2(\mathbb{R})$, *then one has*

$$b^2\pi\|f\|^4_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})} \leqslant \left(\int_{\mathbb{R}}\left(x+\frac{t}{2}\right)^2 |f(t)|^2\, dt\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \omega^2 \left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega dt\right).$$

Proof. By Heisenberg-type uncertainty principle related to the Fourier transform, it follows that

$$\left(\int_{\mathbb{R}} x^2\left|R_{f,g}(x,t)e^{\frac{ia}{2b}x^2}\right|^2 dx\right)\left(\int_{\mathbb{R}} \omega^2\left|\mathcal{F}\{R_{f,g}(x,t)e^{\frac{ia}{2b}x^2}\}(\omega)\right|^2 d\omega\right) \geqslant \frac{\pi}{2}\|R_{f,g}(x,t)\|^4_{L^2(\mathbb{R})}. \tag{5.1}$$

Applying relation (3.2), equation (5.1) changes to

$$\left(\int_{\mathbb{R}} x^2\left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\right)\left(\frac{2\pi}{b^2}\int_{\mathbb{R}}\omega^2\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega\right) \geqslant \frac{\pi}{2}\left(\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\right)^2.$$

Integrating both sides of this equation with respect to t, we obtain

$$\frac{\pi}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|f\left(x+\frac{t}{2}\right)\right|^2\left|g\left(x-\frac{t}{2}\right)\right|^2 dxdt \leqslant \left[\left(\int_{\mathbb{R}}\int_{\mathbb{R}} x^2\left|f\left(x+\frac{t}{2}\right)\right|^2\left|g\left(x-\frac{t}{2}\right)\right|^2 dtdx\right)\left(\frac{2\pi}{b^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\omega^2\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega dt\right)\right]^{\frac{1}{2}}.$$

An application of Fubini's theorem will lead to

$$\pi\int_{\mathbb{R}}|f(t)|^2dt\int_{\mathbb{R}}|g(x)|^2dx \leqslant \left[\left(\int_{\mathbb{R}}\left(x+\frac{t}{2}\right)^2|f(t)|^2\, dt\int_{\mathbb{R}}|g(x)|^2\, dx\right)\left(\frac{\pi}{b^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\omega^2\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega dt\right)\right]^{\frac{1}{2}}.$$

Therefore,

$$b^2\pi\|f\|^4_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})} \leqslant \left(\int_{\mathbb{R}}\left(x+\frac{t}{2}\right)^2 |f(t)|^2\, dt\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \omega^2 \left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2 d\omega dt\right),$$

which completes the proof. □

6. Generalized version of Pitt's inequality for LCAF

A useful generalization of the Pitt's inequality for the Fourier transform is given by the following result.

Lemma 6.1. *For* $f \in \mathcal{S}(\mathbb{R})$, *we have*

$$\int_{\mathbb{R}} |x|^{\alpha}|f(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \, |\mathcal{F}\{f(x)\}(\omega)|^2 \, d\omega \geqslant \frac{1}{\pi^{\alpha-2}} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right] \|f\|^2_{L^2(\mathbb{R})},$$

with $1 < \alpha < 3$.

Proof. From the classical Pitt's inequality for the Fourier transform defined by equation (4.4) we get

$$\int_{\mathbb{R}} \left|\omega\right|^{-\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega \leqslant c_{\alpha} \int_{\mathbb{R}} \left|x\right|^{\alpha} \left|f(x)\right|^2 dx,$$

where c_α is defined by equation (4.3). Putting $g(x) = xf(x)$, we immediately obtain

$$\int_{\mathbb{R}} \left|x\right|^{\alpha} |f(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega = \int_{\mathbb{R}} |x|^{\alpha-2}|g(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega.$$

Applying Pitt's inequality in equation (4.4), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |x|^{\alpha-2}|g(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega \\
&\geqslant \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}} |\omega|^{2-\alpha} \left|\mathcal{F}\{g(x)\}(\omega)\right|^2 d\omega \int_{\mathbb{R}} |\omega|^{\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega \\
&= \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}} \left|\omega^{1-\frac{\alpha}{2}} \mathcal{F}\{g(x)\}(\omega)\right|^2 d\omega \int_{\mathbb{R}} |\omega^{\frac{\alpha}{2}} \mathcal{F}\{f(x)\}(\omega)|^2 d\omega.
\end{aligned}$$

By Cauchy-schwarz inequality, we see that

$$\begin{aligned}
\int_{\mathbb{R}} |x|^{\alpha}|f(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha}|\mathcal{F}\{f(x)\}(\omega)|^2 d\omega &\geqslant \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \left|\int_{\mathbb{R}} \omega^{1-\frac{\alpha}{2}} \mathcal{F}\{g(x)\}(\omega) \overline{\omega^{\frac{\alpha}{2}} \mathcal{F}\{f(x)\}(\omega)} d\omega\right|^2 \\
&= \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \left|\int_{\mathbb{R}} \mathcal{F}\{g(x)\}(\omega) \overline{\omega \mathcal{F}\{f(x)\}(\omega)} d\omega\right|^2.
\end{aligned}$$

Using the derivative property of the FT, that is,

$$\mathcal{F}\left\{\frac{df}{dx}\right\}(\omega) = i\omega\mathcal{F}\{f\}(\omega),$$

we further obtain

$$\int_{\mathbb{R}} \left|x\right|^{\alpha}|f(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \left|\mathcal{F}\{f(x)\}(\omega)\right|^2 d\omega \geqslant \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \left|\int_{\mathbb{R}} \mathcal{F}\{g(x)\}(\omega) \overline{\mathcal{F}\left\{\frac{df}{dx}\right\}(\omega)} d\omega\right|^2.$$

Using Parseval identity for the FT, this equation will lead to

$$\int_{\mathbb{R}} |x|^{\alpha}|f(x)|^2 dx \int_{\mathbb{R}} |\omega|^{\alpha} \, |\mathcal{F}\{f\}(\omega)|^2 \, d\omega \geqslant \pi^{\alpha-2} \left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 2\pi \left|\int_{\mathbb{R}} g(x) \overline{\frac{df}{dx}} dx\right|^2$$

$$= 2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \left|\int_{\mathbb{R}} xf(x)\overline{\frac{df}{dx}}dx\right|^2.$$

Hence,

$$\int_{\mathbb{R}} |x|^{\alpha}|f(x)|^2dx \int_{\mathbb{R}} |\omega|^{\alpha}\, |\mathcal{F}\{f\}(\omega)|^2\, d\omega \geqslant 2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]\int_{\mathbb{R}} |xf(x)|^2dx = 2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]\|xf\|^2_{L^2(\mathbb{R})},$$

and the proof is complete. □

Let us now establish a generalized version of Pitt's inequality related to the LCAF, as expressed as follows.

Theorem 6.2. *For* $f \in L^2(\mathbb{R})$*, the following inequality is valid:*

$$4|b|^{\alpha}\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}}\left|x-\frac{t}{2}\right|^2 |f(t)|^2\, dt \leqslant \int_{\mathbb{R}}\left|x-\frac{t}{2}\right|^{\alpha} |f(t)|^2\, dt \int_{\mathbb{R}}\int_{\mathbb{R}} |\omega|^{\alpha}\left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega dt$$

Proof. It follows from Lemma 6.1 that

$$2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \|xf\|^2_{L^2(\mathbb{R})} \leqslant \int_{\mathbb{R}} |x|^{\alpha}|f(x)|^2dx \int_{\mathbb{R}} |\omega|^{\alpha}\, |\mathcal{F}\{f(x)\}(\omega)|^2\, d\omega. \tag{6.1}$$

By replacing $f(x)$ with $R_{f,g}(x,t)$ in equation (6.1) mentioned before, we infer that

$$2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \|xR_{f,g}(x,t)\|^2_{L^2(\mathbb{R})} \leqslant \int_{\mathbb{R}} |x|^{\alpha}|R_{f,g}(x,t)|^2dx \int_{\mathbb{R}} |\omega|^{\alpha}\left|\mathcal{F}\{R_{f,g}(x,t)\}(\omega)\right|^2 d\omega.$$

Hence,

$$\begin{aligned}2\pi^{\alpha-1}&\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}} x^2\left|f\left(x+\frac{t}{2}\right)\right|^2\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\\ &\leqslant \int_{\mathbb{R}} |x|^{\alpha}\left|f\left(x+\frac{t}{2}\right)\right|^2\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\int_{\mathbb{R}} |\omega|^{\alpha}\left|\mathcal{F}\{R_{f,g}(x,t)\}(\omega)\right|^2 d\omega.\end{aligned}$$

We further obtain

$$\begin{aligned}2\pi^{\alpha-1}&\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}} x^2\left|f\left(x+\frac{t}{2}\right)\right|^2\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\\ &\leqslant \int_{\mathbb{R}} |x|^{\alpha}\left|f\left(x+\frac{t}{2}\right)\right|^2\left|\overline{g\left(x-\frac{t}{2}\right)}\right|^2 dx\int_{\mathbb{R}} \left|\frac{\omega}{b}\right|^{\alpha}\left|\mathcal{F}\{R_{f,g}(x,t)\}\left(\frac{\omega}{b}\right)\right|^2 d\left(\frac{\omega}{b}\right).\end{aligned}$$

This equation is the same as

$$\begin{aligned}2\pi^{\alpha-1}&\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2 \int_{\mathbb{R}} x^2\left|f\left(x+\frac{t}{2}\right)\right|^2\left|g\left(x-\frac{t}{2}\right)\right|^2 dx\\ &\leqslant \int_{\mathbb{R}} |x|^{\alpha}\left|f\left(x+\frac{t}{2}\right)\right|^2\left|g\left(x-\frac{t}{2}\right)\right|^2 dx\,\frac{2\pi}{|b|^{\alpha}}\int_{\mathbb{R}} |\omega|^{\alpha}\left|\mathcal{A}^{A}_{f,g}(t,\omega)\right|^2 d\omega.\end{aligned} \tag{6.2}$$

Integrating both sides of equation (6.2) with respect to t, we immediately get

$$2\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2\int_{\mathbb{R}}x^2\left|f\left(x+\frac{t}{2}\right)\right|^2dx\int_{\mathbb{R}}|g(t)|^2dt$$
$$\leqslant\int_{\mathbb{R}}|x|^\alpha\left|f\left(x+\frac{t}{2}\right)\right|^2dx\int_{\mathbb{R}}|g(x)|^2dx\frac{2\pi}{|b|^\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^\alpha\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2d\omega dt.$$

Therefore,

$$4|b|^\alpha\pi^{\alpha-2}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2\int_{\mathbb{R}}x^2\left|f\left(x+\frac{t}{2}\right)\right|^2dx\|f\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}$$
$$\leqslant\int_{\mathbb{R}}\left|x-\frac{t}{2}\right|^\alpha|f(t)|^2\,dt\|g\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^\alpha\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2d\omega dt.$$

This is equal to

$$4|b|^\alpha\pi^{\alpha-1}\left[\frac{\Gamma\left(\frac{3-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha-1}{4}\right)}\right]^2\int_{\mathbb{R}}\left|x-\frac{t}{2}\right|^2|f(t)|^2\,dt\leqslant\int_{\mathbb{R}}\left|x-\frac{t}{2}\right|^\alpha|f(t)|^2\,dt\int_{\mathbb{R}}\int_{\mathbb{R}}|\omega|^\alpha\left|\mathcal{A}^A_{f,g}(t,\omega)\right|^2d\omega dt,$$

which completes the proof. □

7. Conclusion

In this paper, we have introduced the linear canonical ambiguity function and collected its properties. We have made a direct relation between the linear canonical ambiguity function and classical ambiguity function. We developed the relation to obtain some inequalities related to the proposed linear canonical ambiguity function.

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