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Computational techniques for singularly perturbed reaction-diffusion delay differential equations: a second-order approach

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Abstract

For the analysis of singularly perturbed delay differential equations exhibiting layer or oscillatory behaviour and a slight negative shift in the reaction term, this study introduces a second order numerical approach via Stormer's method. To approximate the term with negative shift, we use Taylor series, which in turn changes the equation into a singular perturbation problem with the same asymptotic behaviour. Finally, we have a recurrence relation with five terms that can be resolved using the Gauss elimination method. The computational results are shown by solving some model problems for different delay and perturbation parameters. The rate of convergence, both theoretically and numerically, has been demonstrated and is compatible with the present approach. The findings acquired using the new approach are shown to be more accurate than those obtained using the earlier investigations.

Keywords: Singular perturbation problems, delay differential equations, Stormer's method, numerical methods.

2020 MSC: 65L06, 65L11, 65L20, 65L50.

1. Introduction

Singularly perturbed delay differential equation (SPDDE) is a developing field of mathematics with a considerable history and a bright future full of significant applications in science and engineering. SPDDE is one that restricts the class of delay differential equations to those in which the highest derivative is multiplied by a very small parameter. The SPDDEs model is widely used to describe most processes in bioscience, control theory, economics and engineering. Due to their applications in numerous scientific and technical fields, such as micro-scale heat transfer [7], control theory [21], hydrodynamics of liquid helium [14], the first exit-time problem [19], describing the human pupil-light reflex [22], models for different physiological processes or diseases [23], the theory of plates and shells [17], magneto-hydrodynamic flow [12], etc, there has been an increase in interest in the numerical study of singularly perturbed delay differential equations in recent years.

Lange and Miura [20] investigated a class of boundary-value problems and discussed an asymptotic method to approximatively solve this type of differential equation. Vaid and Arora [30] developed a numerical method utilizing trigonometric cubic B-spline functions, in which the derivatives are approximated as a sum of the basis functions. Chakravarthy and Kumar [3] approached these problems via

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Numerov's method and the study in [29] developed non-polynomial cubic spline approximations. The work in [1] proposed an exponentially fitted method by use of Taylor series expansion to derive the scheme. Ranjan and Prasad [26] developed an exponentially fitted three term finite difference scheme by using Taylor series approach. Swamy et al. [28] employed numerical integration and the linear interpolation technique after converting the singularly perturbed delay differential equation to a neutral delay differential equation. Kumar and Kumari [18] suggested B-spline functions on a piecewise-uniform mesh to solve singularly perturbed problems with an integral boundary condition. Sharma et al. [27] developed Liouville-Green transformation technique to solve singularly perturbed delay differential equations by the use of Taylor series expansion. Erdogan and Sakar [8] suggested a numerical technique based on a piecewise uniform Shishkin mesh with an exponentially fitted difference scheme for each time subinterval. Phaneendra et al. [25] suggested a numerical integration approach for singularly perturbed delay differential equations with layer or oscillatory behaviour. Amiraliyev and Cimen [2] carried out an exponentially fitted difference scheme on a uniform mesh. This is achieved by the use of the method of integral identities, which involves the utilization of exponential basis functions and interpolating quadrature rules that are formulated with the weight and remainder term in integral form. Kadalbajoo and Sharma [15] developed a standard upwind finite difference scheme on a special type of mesh to tackle the delay argument. The researchers developed a piecewise uniform mesh of Shishkin type to solve singularly perturbed problems of linear [5] and non-linear [4] second order delay differential equations. Cimen and Cakir [6] introduced an exponentially fitted difference scheme on a uniformly spaced grid, constructed through integral identities employing exponential basis functions and interpolating quadrature rules incorporating weight and remainder terms in integral form.

All of these works deal with second-order singularly perturbed delay differential equations, and they show the effectiveness of the methods by analysing them from various angles and producing good, accurate numerical solutions that correspond to a variety of rates of convergence. However, the obtained approximate solution and the corresponding order of convergence are not more satisfactory, indicating that other numerical methods must be developed in order to solve singularly perturbed delay reaction-diffusion problems and produce a more precise numerical solution. Traditional numerical methods often struggle to provide accurate and efficient solutions for such equations, especially when the perturbation parameter is small, leading to numerical instability or excessive computational costs. The motivation for this work is to develop advanced computational technique that specifically target SPDDEs, with a focus on achieving second-order accuracy. By improving the accuracy and efficiency of numerical solutions for SPDDEs, significant implications across various scientific and engineering disciplines can be developed. Furthermore, it enables researchers to model and analyze complex systems more effectively. In this work, we take into account reaction diffusion problems, which fall under the category of singularly perturbed delay differential equations. In Section 2, we define the problem and assumptions on the parameter. In Section 3, we employ Stormer's method to arrive at its numerical solution. Using Taylor series, an equivalent equation is developed to approximate the given problem to get a recurrence relation, which is then solved by Thomas Algorithm. The convergence analysis of the proposed method is also discussed in Section 4. In Section 5, four model examples are computed with varying delay and perturbation parameters, and their results are shown. Discussion and conclusion follow in Section 6.

2. Statement of the problem

We consider the following singularly perturbed delay differential equation of reaction diffusion type:

$$\mu\omega''(\vartheta) + c(\vartheta)\omega(\vartheta-\eta) + d(\vartheta)\omega(\vartheta) = g(\vartheta),\ 0 < \vartheta < 1, \tag{2.1}$$

together with interval and boundary conditions,

$$\omega(\vartheta) = \psi(\vartheta),\quad -\eta \leqslant \vartheta \leqslant 0,\quad \omega(1) = \gamma, \tag{2.2}$$

where μ $(0 < \mu << 1)$ is the perturbation parameter and η is the delay parameter $(0 < \eta < 1)$. Here $c(\vartheta), d(\vartheta), g(\vartheta)$, and $\psi(\vartheta)$ are bounded smooth functions in $(0, 1)$ and γ is a fixed constant. The above assumptions guarantee the existence of unique solution (refer to [10]) and it is also discussed in [11].

The above problem depends on a small positive parameter in a way that causes the solution to vary slowly in certain regions of the domain and fast in others. As a result, the solution usually behaves regularly and changes slowly outside of narrow transition layers, where it jumps or fluctuates quickly. For $\eta \neq 0$, the sign of $c(\vartheta) + d(\vartheta)$ acts as a discriminant for the solution of the problem (2.1)-(2.2). If the sign is positive, then the solution is oscillatory and if the sign is negative, the solution behaves in layers. Consequently, if the solution is having layers, it will be at $\vartheta = 0$ and $\vartheta = 1$. We have discussed the solution of the problems having layer behaviour as well as oscillatory behaviour. Applying the expansion of Taylor series in the region around ϑ, we obtain

$$\omega(\vartheta - \eta) = \omega(\vartheta) - \eta\omega'(\vartheta) + O(\eta^2). \tag{2.3}$$

By substituting (2.3) into (2.1), we develop the following problem, containing small perturbation parameter μ,

$$\mu\omega''(\vartheta) + p(\vartheta)\omega'(\vartheta) + q(\vartheta)\omega(\vartheta) = f(\vartheta), \tag{2.4}$$

under the boundary conditions,

$$\omega(0) = \psi(0) \text{ and } \omega(1) = \gamma, \tag{2.5}$$

where

$$p(\vartheta) = -\eta c(\vartheta), \quad q(\vartheta) = c(\vartheta) + d(\vartheta), \quad \text{and} \quad f(\vartheta) = g(\vartheta).$$

3. Derivation of the method

The objective of this section is to introduce a special kind of mesh such that the terms containing those parameters will be located on the nodal points and the numerical approach that is being examined in this article. Let $\bar{\Gamma}^N = \{\vartheta_i\}_{i=0}^N$ be the discretized domain with N mesh -intervals on the domain $\bar{\Gamma} = [0, 1]$, where N is as an even positive integer. Let $0 = \vartheta_0, \vartheta_1, \ldots, \vartheta_N = 1$ be the mesh points obtained while dividing $[0, 1]$, such that the step size $h_i = \vartheta_i - \vartheta_{i-1}$ for $i = 1, 2, \ldots, N-1$.

3.1. *Numerical algorithm*

The subsequent procedure is suggested for acquiring the numerical solution of the problem.

Step 1: Introduce the uniform mesh by partitioning the domain $[0, 1]$ into N mesh intervals.

Step 2: Stormer's technique is applied to the statement problem to arrive at a five term recurrence relation.

Step 3: We make use of Taylor series expansion of first order derivatives in the system obtained in Step 2 to obtain a scheme.

Step 4: Using reduced problem and the given history function, an initial value problem is formed to find the solution at $\vartheta = N + 1$.

Step 5: We employ the scheme obtained in Step 3 and we use the value $\omega(N + 1)$ obtained in Step 4 to find the solution of the problem using Gauss elimination method.

3.2. *The proposed numerical scheme*

Consider the equation

$$\mu\omega''(\vartheta) + p(\vartheta)\omega'(\vartheta) + q(\vartheta)\omega(\vartheta) = f(\vartheta), \vartheta \in [0, 1]. \tag{3.1}$$

We rearrange the differential equation (3.1) as

$$\mu\omega''(\vartheta) = r(\vartheta, \omega(\vartheta), \omega'(\vartheta)), \quad \vartheta \in [0, 1], \tag{3.2}$$

where $r(\vartheta, \omega(\vartheta), \omega'(\vartheta)) = f(\vartheta) - q(\vartheta)\omega(\vartheta) - p(\vartheta)\omega'(\vartheta)$.

Now, we consider the Stormer's method [13] to solve the equation (3.2) and this equation is approximated by the finite difference scheme:

$$\begin{aligned} \omega_{i+1}-2\omega_i+\omega_{i-1} &= \frac{h^2}{12}\left[13\omega_i''-2\omega_{i-1}''+\omega_{i-2}''\right], \\ \frac{\mu}{h^2}\left(\omega_{i+1}-2\omega_i+\omega_{i-1}\right) &= \frac{1}{12}\left[13r_i-2r_{i-1}+r_{i-2}\right]. \end{aligned} \tag{3.3}$$

Using the definition of r_i in equation (3.3), we get

$$\begin{aligned} \frac{\mu}{h^2}\left(\omega_{i+1}-2\omega_i+\omega_{i-1}\right) = &\frac{1}{12}\left[13(f_i-q_i\omega_i-p_i\omega_i')\right] \\ &-\frac{1}{12}\left[2(f_{i-1}-q_{i-1}\omega_{i-1}-p_{i-1}\omega_{i-1}')+(f_{i-2}-q_{i-2}\omega_{i-2}-p_{i-2}\omega_{i-2}')\right]. \end{aligned} \tag{3.4}$$

Now, the finite difference approximation for ω_i', ω_{i-1}', and ω_{i+1}' is derived with the help of Taylor's series of ω_{i+1} and ω_{i-1} upto $O(h^5)$,

$$\begin{aligned} \omega_{i+1} &= \omega_i+h\omega_i'+\frac{h^2}{2!}\omega_i''+\frac{h^3}{3!}\omega_i^{(3)}+\frac{h^4}{4!}\omega_i^{(4)}+O(h^5), \\ \omega_{i-1} &= \omega_i-h\omega_i'+\frac{h^2}{2!}\omega_i''-\frac{h^3}{3!}\omega_i^{(3)}+\frac{h^4}{4!}\omega_i^{(4)}+O(h^5). \end{aligned}$$

The following approximations for the first derivative of ω are used to substitute in (3.4),

$$\begin{aligned} \omega_i' &\simeq \frac{\omega_{i+1}-\omega_{i-1}}{2h}-\frac{h^2}{6}\omega_i^{(3)}+O(h^4), \\ \omega_{i-1}' &\simeq \frac{-\omega_{i+1}+4\omega_i-3\omega_{i-1}}{2h}+\frac{h^2}{3}\omega_i^{(3)}-\frac{h^3}{12}\omega_i^{(4)}+O(h^4), \\ \omega_{i-2}' &\simeq \frac{-\omega_{i+2}+4\omega_i-3\omega_{i-2}}{4h}+\frac{4h^2}{3}\omega_i^{(3)}-\frac{2h^3}{3}\omega_i^{(4)}+O(h^4). \end{aligned}$$

We obtain the following scheme:

$$\begin{aligned} &\left(\frac{h^2q_{i-2}}{12}-\frac{hp_{i-2}}{16}\right)\omega_{i-2}+\left(\mu-\frac{h^2q_{i-1}}{6}-\frac{13hp_i}{24}+\frac{hp_{i-1}}{4}\right)\omega_{i-1} \\ &+\left(-2\mu+\frac{13h^2q_i}{12}-\frac{hp_{i-1}}{3}+\frac{hp_{i-2}}{12}\right)\omega_i+\left(\mu+\frac{hp_{i-1}}{12}+\frac{13hp_i}{24}\right)\omega_{i+1} \\ &+\left(-\frac{hp_{i-2}}{48}\right)\omega_{i+2}=\frac{h^2}{12}(13f_i-2f_{i-1}+f_{i-2}). \end{aligned}$$

Finally we obtain a five term recurrence relation,

$$\mathcal{L}^N \equiv A_i\omega_{i-2}+B_i\omega_{i-1}+C_i\omega_i+D_i\omega_{i+1}+E_i\omega_{i+2}=F_i, \tag{3.5}$$

where

$$\begin{aligned} A_i &= \frac{h^2q_{i-2}}{12}-\frac{hp_{i-2}}{16}, & B_i &= \mu-\frac{h^2q_{i-1}}{6}-\frac{13hp_i}{24}+\frac{hp_{i-1}}{4}, \\ C_i &= -2\mu+\frac{13h^2q_i}{12}-\frac{hp_{i-1}}{3}+\frac{hp_{i-2}}{12}, & D_i &= \mu+\frac{hp_{i-1}}{12}+\frac{13hp_i}{24}, \\ E_i &= -\frac{hp_{i-2}}{48}, & F_i &= \frac{h^2}{12}(13f_i-2f_{i-1}+f_{i-2}). \end{aligned}$$

We have the following reduced problem by putting the perturbation parameter $\mu = 0$ in (3.1),

$$p(\vartheta)\omega_0' + q(\vartheta)\omega_0 = f(\vartheta), \ \vartheta \in [0,1],$$

subject to $\omega(\vartheta) = \psi(\vartheta)$, $\vartheta \in [-1,0]$. $\omega_0(\vartheta-\eta) = \psi(\vartheta-\eta)$, since $\omega(\vartheta) = \psi(\vartheta)$ in the interval $[-1,0]$. So, by using the above condition, we get $p(\vartheta)\omega_0' + q(\vartheta)\omega_0 = f(\vartheta)$, which gives

$$\omega_0' = \frac{1}{p(\vartheta)}\left[f(\vartheta) - q(\vartheta)\omega_0\right],$$

with $\omega_0(0) = \psi(0)$. We make use of Runge-Kutta method to get the solution at $\vartheta = N+1$, say $\bar{\gamma}$ (i.e., $\omega_0(N+1) = \bar{\gamma}$). The scheme (3.5) gives a system of $(N-1)$ equations, which is solved using the numerical algorithm mentioned in Section 3.1, with the help of MATLAB R2022a mathematical software.

4. Stability and convergence analysis

Lemma 4.1. *For the case of $\beta(\bar{\vartheta}) = c(\bar{\vartheta}) + d(\bar{\vartheta}) < 0$, for all $\bar{\vartheta} \in (0,1)$, the operator $\mathcal{L}^N$ has the discrete minimum principle, if $\omega_0 \geqslant 0$ and $\mathcal{L}\omega(\bar{\vartheta}) \leqslant 0$, for all $\bar{\vartheta} \in (0,1)$, then $\omega(\bar{\vartheta})$ is non-negative, for all $\bar{\vartheta} \in (0,1)$.*

Proof. Suppose $x \in (0,1)$, such that $\omega(x) = \min_{\vartheta \in (0,1)} \omega(\vartheta)$ and $\omega(x) < 0$. Since x is not contained in $\{0,1\}$ and is a point of minima, the first derivative of ω attains zero at the point x and $\omega''(x)$ is non-negative. Since by assumption $\omega(x)$ is strictly negative and $q(x) < 0$, we obtain

$$\mathcal{L}\omega(\vartheta) = \mu\omega''(x) + p(x)\omega(x) + q(x)\omega(x) > 0.$$

But this gives a contradiction to our assumption. Therefore, it follows that $\omega(\vartheta) \geqslant 0$ for $\vartheta \in (0,1)$. □

Theorem 4.2. *For the case of layer behaviour, i.e., $\beta(\bar{\vartheta}) < 0$, the operator in (3.5) is stable and satisfies*

$$\mid \omega(\bar{\vartheta}) \mid \leqslant K_1 \max\left\{\mid \omega_0 \mid, \max_{\bar{\vartheta}\in(0,1)} \mid \mathcal{L}\omega(\bar{\vartheta}) \mid\right\}$$

for some constant $K_1 \geqslant 1$.

Proof. Let

$$\tau^{\pm}(\bar{\vartheta}) = K_1 \max\left\{\mid \omega_0 \mid, \max_{\bar{\vartheta}\in(0,1)} \mid \mathcal{L}\omega(\bar{\vartheta}) \mid\right\} \pm \omega(\bar{\vartheta}).$$

Hence, $\tau^{\pm}(0)$ is non-negative and

$$\mathcal{L}\tau^{\pm}(\bar{\vartheta}) = K_1\beta(\bar{\vartheta}) \max\left\{\mid \omega_0 \mid, \max_{\bar{\vartheta}\in(0,1)} \mid \mathcal{L}\omega(\bar{\vartheta}) \mid\right\} \pm \mathcal{L}\omega(\vartheta).$$

Note that $\mathcal{L}\tau^{\pm}(\bar{\vartheta})$ is non-positive, since $\beta(\bar{\vartheta}) < 0$ and for appropriate value of K_1. Then by Lemma 4.1, we obtain $\tau^{\pm}(\bar{\vartheta}) \geqslant 0$, for all $\bar{\vartheta} \in (0,1)$. Thus

$$\mid \omega(\bar{\vartheta}) \mid \leqslant K_1 \max\left\{\mid \omega_0 \mid, \max_{\bar{\vartheta}\in(0,1)} \mid \mathcal{L}\omega(\bar{\vartheta}) \mid\right\}.$$

So for the case of layer behaviour, solution's stability is proved. □

Lemma 4.3. *For the case of $\beta(\bar{\vartheta}) = c(\bar{\vartheta}) + d(\bar{\vartheta}) > 0$, for all $\bar{\vartheta} \in (0,1)$, the operator $\mathcal{L}^N$ has the discrete maximum principle, if ω_0 and $\mathcal{L}\omega(\bar{\vartheta})$ are non-negative, for all $\bar{\vartheta} \in (0,1)$, then $\omega(\bar{\vartheta}) \geqslant 0$ for all $\bar{\vartheta} \in (0,1)$.*

Proof. Suppose $x \in (0,1)$, such that $\omega(x) = \max_{\vartheta \in (0,1)} \omega(\vartheta)$ and $\omega(x) < 0$. Since x is not contained in $\{0,1\}$ and is a point of maxima, the first derivative of ω attains zero at the point x and $\omega''(x)$ is non-positive. Since by assumption $\omega(x)$ is strictly negative and $q(x) > 0$, we obtain

$$\mathcal{L}\omega(\vartheta) = \mu\omega''(x) + p(x)\omega(x) + q(x)\omega(x) < 0.$$

But this gives a contradiction to our assumption. Therefore, it follows that $\omega(\vartheta) \geqslant 0$ for $\vartheta \in (0,1)$. □

Theorem 4.4. *For the case of oscillatory behaviour, i.e.,* $\beta(\bar{\vartheta}) > 0$*, the operator in* (3.5) *is stable and satisfies*

$$|\ \omega(\bar{\vartheta})\ | \leqslant K_1 \max\left\{|\ \omega_0\ |, \max_{\bar{\vartheta}\in(0,1)} |\ \mathcal{L}\omega(\bar{\vartheta})\ |\right\}$$

for some constant $K_1 \geqslant 1$.

Proof. Let

$$\tau^{\pm}(\bar{\vartheta}) = K_1 \max\left\{|\ \omega_0\ |, \max_{\bar{\vartheta}\in(0,1)} |\ \mathcal{L}\omega(\vartheta)\ |\right\} \pm \omega(\bar{\vartheta}).$$

Hence, $\tau^{\pm}(0)$ is non-negative and

$$\mathcal{L}\tau^{\pm}(\bar{\vartheta}) = K_1\beta(\bar{\vartheta}) \max\left\{|\ \omega_0\ |, \max_{\bar{\vartheta}\in(0,1)} |\ \mathcal{L}\omega(\bar{\vartheta})\ |\right\} \pm \mathcal{L}\omega(\vartheta).$$

Note that $\mathcal{L}\tau^{\pm}(\bar{\vartheta})$ is non-positive, since $\beta(\bar{\vartheta}) > 0$ and for appropriate value of K_1. Then by Lemma 4.3, we obtain $\tau^{\pm}(\bar{\vartheta}) \geqslant 0$ for all $\bar{\vartheta} \in (0, 1)$. Thus,

$$|\ \omega(\bar{\vartheta})\ | \leqslant K_1 \max\left\{|\ \omega_0\ |, \max_{\bar{\vartheta}\in(0,1)} |\ \mathcal{L}\omega(\bar{\vartheta})\ |\right\}.$$

So for the case of oscillatory behaviour, solution's stability is proved. □

Lemma 4.5. *The bound for derivative of the solution* $\omega(\vartheta)$ *of the problem* (2.1)-(2.2) *when* $\vartheta \in (0, 1)$ *is given by*

$$\begin{cases} |\ \omega^k(\vartheta)\ | \leqslant C\left(1 + \mu^{-k}\exp\left(\frac{-\alpha\vartheta}{\mu}\right)\right), & \textit{for left layer},\\ |\ \omega^k(\vartheta)\ | \leqslant C\left(1 + \mu^{-k}\exp\left(\frac{-\alpha(1-\vartheta)}{\mu}\right)\right), & \textit{for right layer},\end{cases}$$

for $0 \leqslant k \leqslant 4$*, where* $p(\vartheta) \geqslant \alpha > 0$ *for right boundary layer case and* $p(\vartheta) \leqslant \alpha < 0$ *for left boundary layer case.*

Proof. For the proof, refer to [16, 24]. □

Theorem 4.6. *Let* $\omega(\vartheta_i)$ *be the analytical solution of the problem in* (2.4) *and* (2.5) *and* W^N *be the computational solution of the discretized problem in* (3.5)*. Then, for sufficiently large* N*, the following parameter uniform error estimate holds:*

$$\sup_{0\leqslant\mu\leqslant1} |\ \omega(\vartheta_i) - W^N\ | \leqslant C_1 N^{-2}.$$

Proof. We write (3.5) in matrix form as

$$MW = Y, \tag{4.1}$$

where $M = (m_{ij}), i = 1(1)N - 1$, is a square matrix of order $N - 1$. So from (3.5), we can write as

$$m_{ii-2} = \frac{h^2 q_{i-2}}{12} - \frac{hp_{i-2}}{16}, \qquad m_{ii-1} = \mu - \frac{h^2 q_{i-1}}{6} - \frac{13hp_i}{24} + \frac{hp_{i-1}}{4},$$

$$m_{ii} = -2\mu + \frac{13h^2 q_i}{12} - \frac{hp_{i-1}}{3} + \frac{hp_{i-2}}{12}, \qquad m_{ii+1} = \mu + \frac{hp_{i-1}}{12} + \frac{13hp_i}{24}, \qquad m_{ii+2} = -\frac{hp_{i-2}}{48},$$

and the column vector $Y = (y_i)$ is given by

$$y_1 = F_1 - A_1\psi_{-1} - B_1\psi(0), \qquad y_2 = F_2 - A_2\psi(0), \qquad y_i = F_i,\ i = 3, 4, \ldots, N-3,$$

$$y_{N-2} = F_{N-2} - E_{N-2}\gamma, \qquad y_{N-1} = F_{N-1} - D_{N-1}\gamma - E_{N-1}\bar{\gamma}.$$

Truncation error $T_e(h)$ obtained is:

$$T_e(h) = h^4\left[\frac{13}{72}p_i\omega_i^{(3)} + \frac{1}{18}p_{i-1}\omega_i^{(3)} - \frac{1}{9}p_{i-2}\omega_i^{(3)}\right] + O(h^5).$$

Equation (4.1) can also be written in error form as $M\overline{W} - T_e(h) = Y$, where $\overline{W} = \left(\overline{w}_1 \quad \overline{w}_2 \quad \cdots \quad \overline{w}_{N-1}\right)^t$

is the exact solution and $T_e(h) = \begin{pmatrix} T_{e_1}(h) & T_{e_2}(h) & \cdots & T_{e_{N-1}}(h) \end{pmatrix}^t$ is the truncation error. Then we obtain

$$ME^* = T_e(h), \tag{4.2}$$

where

$$E^* = \overline{W} - W = \begin{pmatrix} e_1^* & e_2^* & \cdots & e_{N-1}^* \end{pmatrix}^t.$$

Now, let $\bar{S_i}$ be the sum of elements of i^{th} row of the matrix M. Then

$$S_1 = -\mu + h^2\left[\frac{13q_i}{12}\right] + h\left[-\frac{p_{i-1}}{4} + \frac{13p_i}{24} + \frac{p_{i-2}}{16}\right], \quad S_2 = h^2\left[\frac{13q_i}{12} - \frac{q_{i-1}}{6}\right] + h\left[\frac{p_{i-2}}{16}\right].$$

For $i = 3$ to $N-3$,

$$S_i = h^2 Q_i^*,$$

where $Q_i^* = \left[\frac{q_{i-2}}{12} + \frac{13q_i}{12} - \frac{q_{i-1}}{6}\right]$,

$$S_{N-2} = h^2\left[\frac{q_{i-2}}{12} - \frac{q_{i-1}}{6} + \frac{13q_i}{12}\right] + h\left[\frac{p_{i-2}}{48}\right],$$
$$S_{N-1} = -\mu + h^2\left[\frac{q_{i-2}}{12} - \frac{q_{i-1}}{6} + \frac{13q_i}{12}\right] + h\left[\frac{p_{i-2}}{48} - \frac{13p_i}{24} - \frac{p_{i-1}}{12}\right].$$

From (4.2),

$$E^* = M^{-1}T_e(h), \tag{4.3}$$

which implies,

$$\|E^*\| \leqslant \|M^{-1}\|\|T_e(h)\|, \tag{4.4}$$

where the norm $\|.\|$ denotes the maximum norm. Let the (i,k)-th element of M^{-1} be denoted by $\overline{m}_{i,k}$, which are non-negative. Then,

$$\sum_{k=1}^{N-1} \overline{m}_{i,k}\bar{S_i} = 1, i = 1, 2, \ldots, N-1.$$

Hence,

$$\sum_{k=1}^{N-1} \overline{m}_{i,k} \leqslant \frac{1}{\min_{1\leqslant k\leqslant N-1} S_k} \leqslant \frac{1}{h^2 \mid Q_{k0}^* \mid}, \tag{4.5}$$

for some k_0 between 1 and $N-1$. From Equations (4.1), (4.3), (4.4), and (4.5), we have,

$$e_k = \sum_{i=1}^{N-1} \overline{m}_{i,j} T_{e_k}(h), k = 1, 2, \ldots, N-1,$$

which gives

$$e_i \leqslant \left(\sum_{i=1}^{N-1} \overline{m}_{i,j}\right) \max_{1\leqslant k\leqslant N-1} \mid T_{e_k}(h) \mid \leqslant \frac{1}{h^2 \mid Q_{k0}^* \mid} \times O(h^4) = O(h^2).$$

As a result, $\|E\| = O(h^2)$ and hence our method is of second order convergent. □

5. Numerical Experiments

We took four numerical experiments into consideration to demonstrate the applicability of the suggested technique. Tables are given to show the computed solution for different values of μ as well as different values of η. Maximum absolute errors are determined by applying the double mesh principle [9] to the presented examples

$$E_{\mu}^{N} = \max_{0 \leqslant i \leqslant N} \mid \omega_i^N - \omega_{2i}^{2N} \mid,$$

where ω_i^N represents the numerical solution of the problem on N number of mesh points and ω_{2i}^{2N} represents the numerical solution of the problem on 2N number of mesh points. For a value of N, the μ -uniform maximum absolute error is calculated by the formula

$$E^N = \max_{\mu} E_{\mu}^N.$$

The computational rate of convergence ρ is also obtained by using the double mesh principle defined as

$$\rho = \frac{\log(E_{\mu}^N) - \log\left(E_{\mu}^{2N}\right)}{\log 2}.$$

Example 5.1. Consider

$$\mu\omega''(\vartheta) - 2\omega(\vartheta - \eta) - \omega(\vartheta) = 1,$$

together with the boundary conditions

$$\omega(\vartheta) = 1, \ -\eta \leqslant \vartheta \leqslant 0, \ \omega(1) = 0.$$

Table 1 gives the maximum absolute error for different values of delay parameter with $\mu = 0.1$ and Table 2 presents the maximum absolute error for different values of perturbation parameter with $\eta = 0.03$. The results are compared with the existing method in [25, 28] and it is found that results by our method are better as compared to the discussed method. Also, we present Figure 1, which gives the computed solution of the problem for different values of μ with $\eta = 0.05$ and the graphs of point-wise absolute errors for different values of N with $\mu = 0.1$ and $\eta = 0.05$. Figure 2 represents the graphs of maximum absolute error for different values of η with $\mu = 0.1$ and loglog plot of the maximum pointwise error.

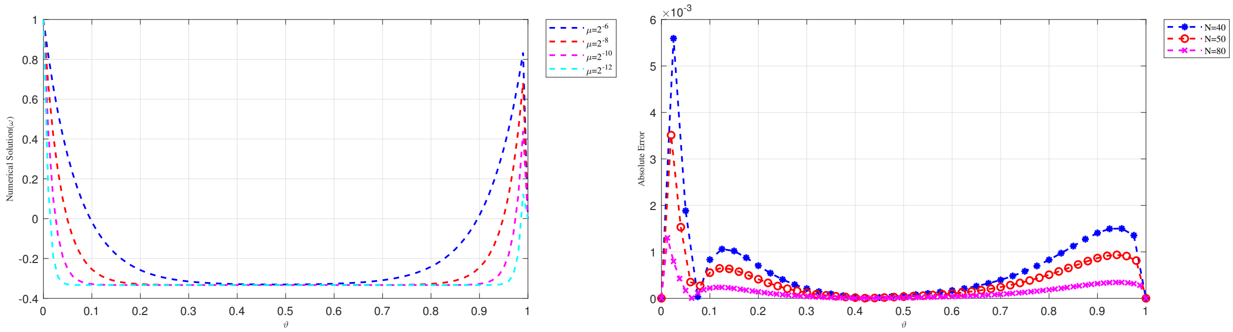

Figure 1: The numerical solution and point-wise absolute errors of Example 5.1.

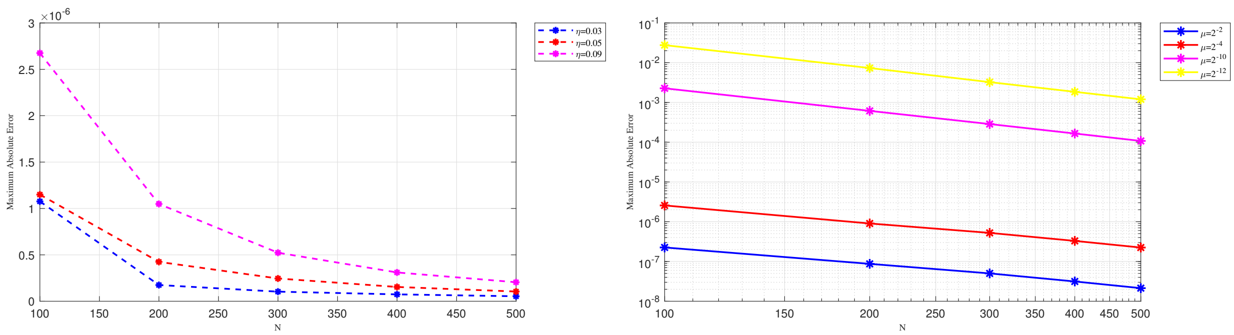

Figure 2: The maximum absolute error and loglog plot of maximum point-wise errors of Example 5.1.

Table 1: The maximum absolute error of Example 5.1 for different values of η with $\mu = 0.1$.

η	N				
	100	200	300	400	500
0.03	3.5633e-05	7.1880e-06	2.9299e-06	1.5725e-06	9.7719e-07
0.05	5.0425e-05	1.0984e-05	4.6314e-06	2.5337e-06	1.5939e-06
0.09	8.2623e-05	1.9300e-05	8.3673e-06	4.6463e-06	2.9503e-06
Results in [28]					
0.03	3.1674e-03	1.6058e-03	1.0754e-03	8.0837e-04	6.4760e-04
0.05	3.1437e-03	1.5949e-03	1.0685e-03	8.0338e-04	6.4367e-04
0.09	3.0784e-03	1.5660e-03	1.0502e-03	7.9000e-04	6.3310e-04
Results in [25]					
0.03	9.3352e-03	4.9360e-03	3.3540e-03	2.5398e-03	2.0438e-03
0.05	8.7514e-03	4.7344e-03	3.2355e-03	2.4561e-03	1.9803e-03
0.09	7.2037e-03	4.1449e-03	2.8840e-03	2.2111e-03	1.7913e-03

Table 2: The maximum absolute error of Example 5.1 for different values of μ and $\eta = 0.03$.

μ	N				
	2^4	2^5	2^6	2^7	2^8
2^{-4}	6.2767e-03	1.1691e-03	2.1426e-04	4.1713e-05	8.8041e-06
2^{-5}	1.3232e-02	2.5312e-03	4.3864e-04	7.7501e-05	1.4801e-05
2^{-6}	2.6939e-02	5.6764e-03	9.7927e-04	1.6134e-04	2.7788e-05
2^{-7}	5.0289e-02	1.2574e-02	2.2910e-03	3.6818e-04	5.8595e-05
2^{-8}	8.2700e-02	2.6293e-02	5.3857e-03	8.8697e-04	1.3579e-04
Results in [28]					
2^{-4}	2.1118e-02	1.1692e-02	6.1941e-03	3.1887e-03	1.6178e-03
2^{-5}	2.7872e-02	1.6023e-02	8.6367e-03	4.4957e-03	2.2948e-03
2^{-6}	3.5711e-02	2.1293e-02	1.1869e-02	6.2731e-03	3.2240e-03
2^{-7}	4.6679e-02	2.8350e-02	1.6107e-02	8.6728e-03	4.5120e-03
2^{-8}	5.4895e-02	3.6018e-02	2.1373e-02	1.1929e-02	6.2847e-03

Table 3: Rate of convergence ρ of Example 5.1 for $\mu = 0.1$ and $\eta = 0.05$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	5.0425e-05	1/400	1.0984e-05	2.1987
1/200	1/400	1.0984e-05	1/800	2.5337e-06	2.1161
1/300	1/600	4.6314e-06	1/1200	1.0940e-06	2.0819

Example 5.2. Consider

$$\mu\omega''(\vartheta) + 0.25\omega(\vartheta - \eta) - \omega(\vartheta) = 1,$$

together with the boundary conditions

$$\omega(\vartheta) = 1, \ -\eta \leqslant \vartheta \leqslant 0, \ \omega(1) = 0.$$

Table 4 gives the maximum absolute error for different values of delay parameter with $\mu = 0.1$ and Table 5 presents the maximum absolute error for different values of perturbation parameter with $\eta = 0.03$. The results are compared with the existing method in [25, 28] and it is found that results by our method are better as compared to the discussed method. Also, we present Figure 3, which gives the computed solution of the problem for different values of μ with $\eta = 0.05$ and the graphs of point-wise absolute errors for different values of N with $\mu = 0.1$ and $\eta = 0.05$. Figure 4 represents the graphs of maximum absolute error for different values of η with $\mu = 0.1$ and loglog plot of the maximum pointwise error.

Table 4: The maximum absolute error of Example 5.2 for different values of η with $\mu = 0.1$.

η	N				
	100	200	300	400	500
0.03	1.0763e-06	1.7460e-07	1.0331e-07	7.4030e-08	5.3509e-08
0.05	1.1497e-06	4.2418e-07	2.4509e-07	1.5386e-07	1.0463e-07
0.09	2.6753e-06	1.0485e-06	5.2303e-07	3.1032e-07	2.0481e-07
Results in [28]					
0.03	2.1999e-03	1.1041e-03	7.3705e-04	5.5315e-04	4.4269e-04
0.05	2.2012e-03	1.1049e-03	7.3749e-04	5.5345e-04	4.4293e-04
0.09	2.1999e-03	1.1038e-03	7.3676e-04	5.5289e-04	4.4247e-04
Results in [25]					
0.03	8.9194e-03	4.5468e-03	3.0511e-03	2.2959e-03	1.8404e-03
0.05	8.9177e-03	4.5440e-03	3.0482e-03	2.2934e-03	1.8382e-03
0.09	8.8966e-03	4.5252e-03	3.0345e-03	2.2825e-03	1.8292e-03

Table 5: The maximum absolute error of Example 5.2 for different values of μ and $\eta = 0.03$.

μ	N				
	100	200	300	400	500
2^{-4}	1.1995e-03	1.4607e-04	1.4217e-05	8.9545e-07	2.5132e-07
2^{-5}	3.2019e-03	4.1123e-04	4.1149e-05	2.7127e-06	7.9697e-07
2^{-6}	7.9709e-03	1.0995e-03	1.1406e-04	7.9924e-06	2.3986e-06
2^{-7}	1.8647e-02	2.8360e-03	3.0884e-04	2.3396e-05	7.0218e-06
2^{-8}	4.0268e-02	7.0258e-03	8.1909e-04	6.9265e-05	2.0561e-05
Results in [28]					
2^{-4}	1.8632e-02	9.6189e-03	4.8865e-03	2.4643e-03	1.2376e-03
2^{-5}	2.8161e-02	1.4818e-02	7.6255e-03	3.8713e-03	1.9509e-03
2^{-6}	3.7958e-02	2.0967e-02	1.0977e-02	5.6273e-03	2.8498e-03
2^{-7}	5.0640e-02	2.8316e-02	1.5267e-02	7.9105e-03	4.0287e-03
2^{-8}	6.3580e-02	3.7706e-02	2.0984e-02	1.1012e-02	5.6555e-03

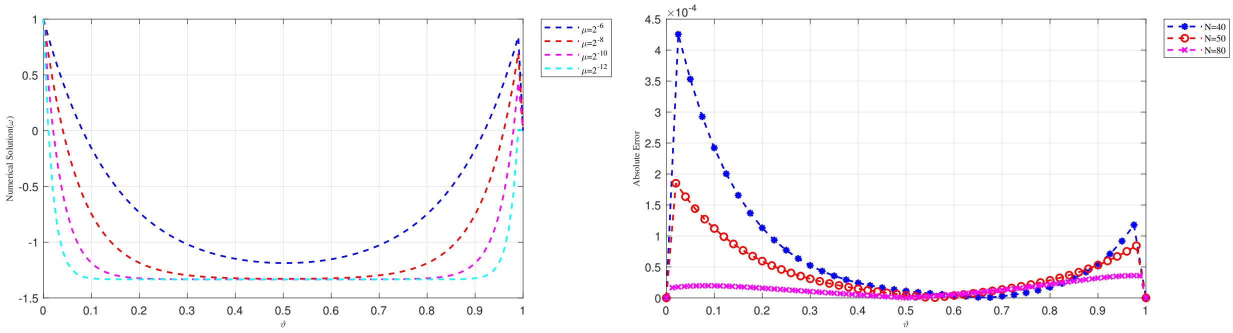

Figure 3: The numerical solution and point-wise absolute errors of Example 5.2.

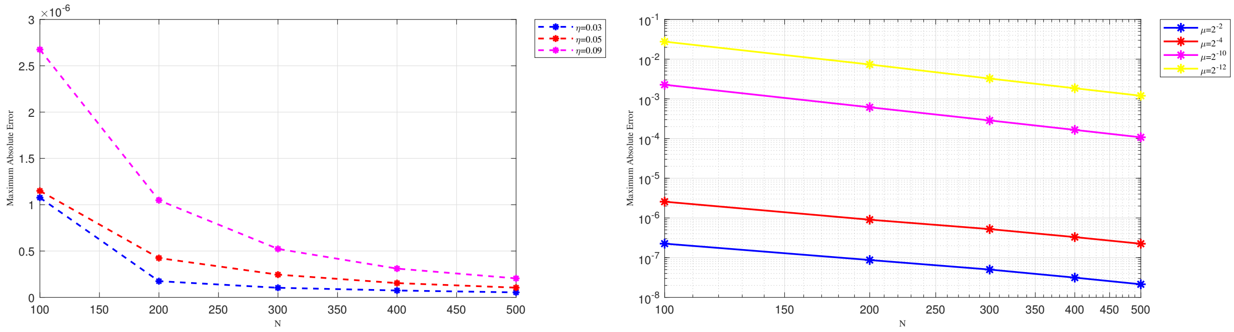

Figure 4: The maximum absolute error and loglog plot of maximum point-wise errors of Example 5.2.

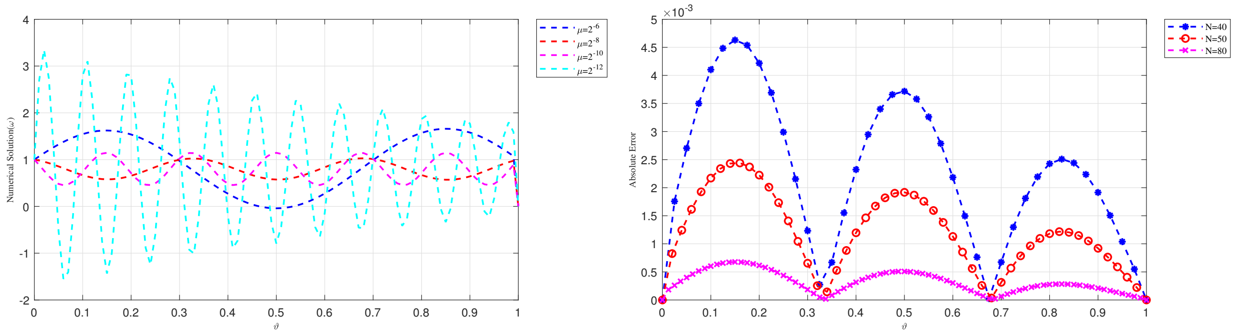

Figure 5: The numerical solution and point-wise absolute errors of Example 5.3.

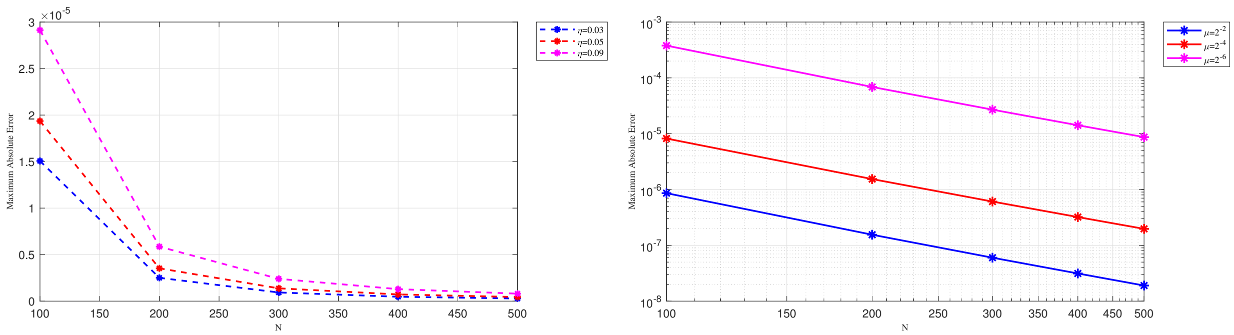

Figure 6: The maximum absolute error and loglog plot of maximum point-wise errors of Example 5.3.

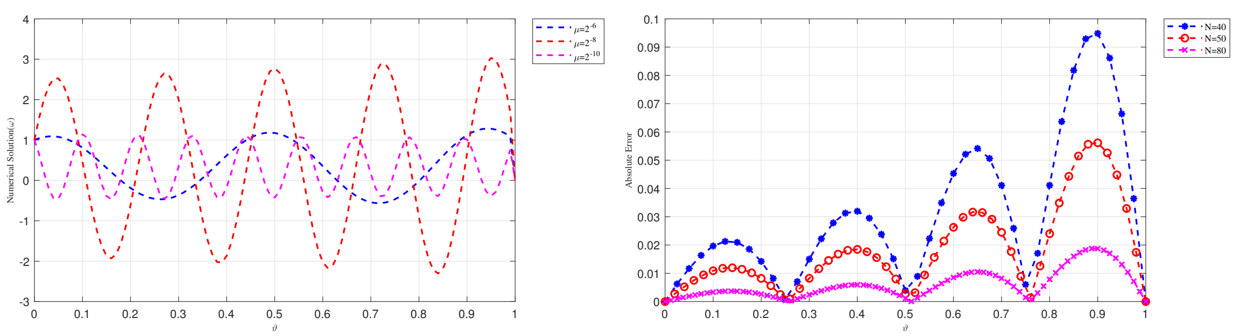

Figure 7: The numerical solution and point-wise absolute errors of Example 5.4.

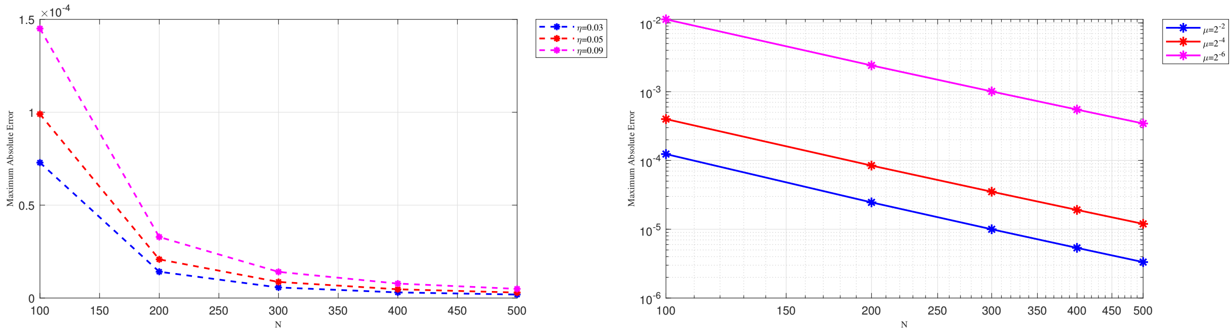

Figure 8: The maximum absolute error and loglog plot of maximum point-wise errors of Example 5.4.

Table 6: Rate of convergence ρ of Example 5.2 for $\mu = 0.1$ and $\eta = 0.05$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	2.6753e-06	1/400	1.0485e-06	1.7566
1/200	1/400	1.0485e-06	1/800	3.1032e-07	1.8498
1/300	1/600	5.2303e-07	1/1200	1.4511e-07	1.8913

Table 7: Rate of convergence ρ of Example 5.3 for $\mu = 0.1$ and $\eta = 0.05$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/100	1/200	1.9359e-05	1/400	3.5266e-06	2.2917
1/200	1/400	3.5266e-06	1/800	7.2028e-07	2.2147
1/300	1/600	1.3758e-06	1/1200	2.9640e-07	2.1698

Example 5.3. Consider the following singularly perturbed delay differential equation with oscillatory behaviour

$$\mu\omega''(\vartheta) + 0.25\omega(\vartheta - \eta) + \omega(\vartheta) = 1,$$

together with the boundary conditions

$$\omega(\vartheta) = 1, \ -\eta \leqslant \vartheta \leqslant 0, \ \omega(1) = 0.$$

Table 8: The maximum absolute error of Example 5.3 for different values of η with $\mu = 0.1$.

η	N				
	100	200	300	400	500
0.03	1.5067e-05	2.5073e-06	9.3074e-07	4.7221e-07	2.8256e-07
0.05	1.9359e-05	3.5266e-06	1.3758e-06	7.2028e-07	4.4045e-07
0.09	2.9131e-05	5.8582e-06	2.3953e-06	1.2891e-06	8.0268e-07
Results in [28]					
0.03	2.5991e-03	1.2872e-03	8.5528e-04	6.4039e-04	5.1179e-04
0.05	2.6270e-03	1.3013e-03	8.6474e-04	6.4750e-04	5.1749e-04
0.09	2.6813e-03	1.3289e-03	8.8320e-04	6.6139e-04	5.2863e-04
Results in [25]					
0.03	7.1024e-02	3.5558e-02	2.3661e-02	1.7721e-02	1.4163e-02
0.05	6.9203e-02	3.4790e-02	2.3181e-02	1.7373e-02	1.3890e-02
0.09	6.6055e-02	3.3490e-02	2.2377e-02	1.6794e-02	1.3439e-02

Table 9: The maximum absolute error of Example 5.4 for different values of η with $\mu = 0.1$.

η	N				
	100	200	300	400	500
0.03	7.2895e-05	1.4158e-05	5.6838e-06	3.0253e-06	1.8702e-06
0.05	9.9020e-05	2.0821e-05	8.6635e-06	4.7061e-06	2.9476e-06
0.09	1.4505e-04	3.2864e-05	1.4090e-05	7.7787e-06	4.9217e-06
Results in [28]					
0.03	1.5929e-02	7.4850e-03	4.8816e-03	3.6202e-03	2.8764e-03
0.05	1.5470e-02	7.2782e-03	4.7473e-03	3.5209e-03	2.7975e-03
0.09	2.1396e-02	1.0097e-02	6.5922e-03	4.8916e-03	3.8879e-03
Results in [25]					
0.03	1.9740e-01	1.0467e-01	7.0844e-02	5.3521e-02	4.2985e-02
0.05	2.5749e-01	1.3585e-01	9.2035e-02	6.9554e-02	5.5884e-02
0.09	1.5004e-00	7.1504e-01	4.6444e-01	3.4319e-01	2.7196e-01

Example 5.4. Consider the following singularly perturbed delay differential equation with oscillatory behaviour

$$\mu\omega''(\vartheta) + \omega(\vartheta - \eta) + 2\omega(\vartheta) = 1,$$

together with the boundary conditions

$$\omega(\vartheta) = 1, \ -\eta \leqslant \vartheta \leqslant 0, \ \omega(1) = 0.$$

The maximum absolute error obtained for Examples 5.3 and 5.4 for different values of delay parameter with $\mu = 0.1$ are presented in Tables 8 and 9, respectively. The results are compared with the existing method in [25, 28] and it is found that results by our method are better as compared to the discussed method. Also, Figures 5 and 7 represent the graphs of the computed solution of Examples 5.3 and 5.4 for different values of μ with $\eta = 0.05$ and the graphs of point-wise absolute errors for different values of N with $\mu = 0.1$ and $\eta = 0.05$. Figures 6 and 8 represents the graphs of maximum absolute error for different values of η with $\mu = 0.1$ and loglog plot of the maximum pointwise error of Examples 5.3 and 5.4, respectively.

Table 10: Rate of convergence ρ of Example 5.4 for $\mu = 0.1$ and $\eta = 0.05$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
$1/100$	$1/200$	9.9020e-05	$1/400$	2.0821e-05	2.2497
$1/200$	$1/400$	2.0821e-05	$1/800$	4.7061e-06	2.1454
$1/300$	$1/600$	8.6635e-06	$1/1200$	2.0172e-06	2.1026

Example 5.5. Consider the following singularly perturbed delay differential equation with variable coefficients

$$\mu\omega''(\vartheta) + \vartheta\omega(\vartheta - \eta) + (1 - \vartheta)\omega(\vartheta) = \vartheta,$$

together with the boundary conditions

$$\omega(\vartheta) = 1, \ -\eta \leqslant \vartheta \leqslant 0, \ \omega(1) = 0.$$

Table 11 gives the maximum absolute error for different values of delay parameter with $\mu = 0.1$. Also, we present Figure 9, which gives the computed solution of the problem for different values of μ with $\eta = 0.05$ and the graphs of point-wise absolute errors for different values of N with $\mu = 0.1$ and $\eta = 0.05$.

Table 11: The maximum absolute error of Example 5.5 for different values of η with $\mu = 0.1$.

η	N				
	100	200	300	400	500
0.03	8.5640e-04	4.4057e-04	2.9649e-04	2.2341e-04	1.7923e-04
0.05	1.4282e-03	7.3453e-04	4.9427e-04	3.7242e-04	2.9876e-04
0.09	2.5740e-03	1.3230e-03	8.9006e-04	6.7057e-04	5.3792e-04

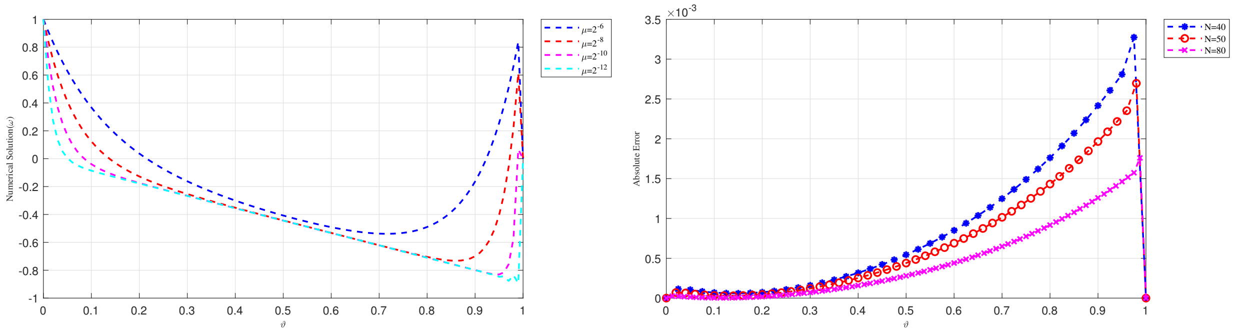

Figure 9: The numerical solution and point-wise absolute errors of Example 5.5.

6. Discussion and conclusion

In this study, a second order numerical approach for solving singularly perturbed delay differential equations of reaction diffusion type is explored. The findings, as shown in the tables, indicate that the current method provides a reasonable approximation of the solution and is superior to other numerical strategies described in the literature. The results in Tables 3, 6, 7, 10 suggest that the current technique is of second order convergence based on theoretical error estimates and numerical rate of convergence. Graphs have been used to show the numerical solution of the problems so that the effect of delay term on the solution profile can be studied. The effect of delay on the layer or oscillatory behaviour of the solution is demonstrated by a few numerical examples. To see the impact on layer behaviour, we look at Examples 5.1 and 5.2, whereas Examples 5.3 and 5.4 demonstrate the impact on oscillatory behaviour. Also, we can observe fro graphs that the absolute error goes down as N goes up. As a result, our current technique is demonstrated to be accurate, stable, and convergent.

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