J. Math. Computer Sci., 35 (2024), 348–361

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

K-Humbert confluent hypergeometric functions $\Phi_{1,k}$, $\Phi_{2,k}$, and $\Phi_{3,k}$

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Abstract

The main aim of this work is to introduce the k-Humbert confluent hypergeometric functions $\Phi_{1,k}$, $\Phi_{2,k}$, $\Phi_{3,k}$ and developed some formulae by using the idea of k-calculus. k-Humbert confluent hypergeometric functions are an extension of the classical confluent hypergeometric functions, incorporating an additional parameter, k, which enriches their analytical properties and applicability in diverse mathematical contexts. We introduce new results by applying q-derivative operator on k-Humbert confluent hypergeometric functions $\Phi_{1,k}$, $\Phi_{2,k}$, and $\Phi_{3,k}$. Contiguous function relations of various types, (q, k)-recurrence relations, and q-derivatives formulae are constructed.

Keywords: Humbert confluent hypergeometric functions, q-derivative, (q, k)-recurrence relations and contiguous function.

2020 MSC: 33C15, 33C20.

1. Introduction

The Humbert series [10] introduced by Pierre Humbert in the 1920s was a set of seven hypergeometric series of two variables that generalize Kummer's confluent hypergeometric series of one variable and the confluent hypergeometric limit function of one variable. They are regarded as the confluent hypergeometric differential equation's solutions

$$h(y)'' + \left(\frac{a-b}{c}\right) h(y)' - ch(y) = 0,$$

where a, b, and c are constants and $h(y)$ is a function of y. The solutions to this equation are given by the Humbert confluent hypergeometric functions, which are denoted by $U(a, b.c; y)$ and $V(a, b.c.y)$ is one of the most important functions in the theory of special functions since in terms of the shifting factorials (rising factorial), all hypergeometric series are defined. The Humbert confluent hypergeometric functions can be continued for the whole complex plane excluding non-positive integers. The Humbert confluent

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doi: 10.22436/jmcs.035.03.07

Received: 2024-03-23 *Revised:* 2024-04-10 *Accepted:* 2024-04-29

hypergeometric functions are of great importance in pure and applied mathematics, for example, number theory, mathematical analysis, probability theory, mathematical statistics, and physics [6]. Numerous authors have examined the Humbert function [4, 5, 15, 17] and shown that it is possible to convert a Humbert confluent hypergeometric series with two bases into an expression with just one base further provided the new expansion formula for Humbert hypergeometric functions. Recently Shehata et al. [16] introduced the q-Humbert confluent hypergeometric functions

$$\Phi_1 = \Phi_1\left[(t),(u),(v);q,r,s\right] = \sum_{l,m=0}^{\infty} \frac{(t;q)_{l+m}(u;q)_l}{(v,q)_{l+m}} \frac{r^l s^m}{(q;q)_l\,(q,q)_m},$$

$$\Phi_2 = \Phi_2\left[(t),(u),(v);q,r,s\right] = \sum_{l,m=0}^{\infty} \frac{(t;q)_{l,k}(u;q)_m}{(v,q)_{l+m}} \frac{r^l s^m}{(q;q)_l\,(q,q)_m},$$

$$\Phi_3 = \Phi_3\left[(t),(u);q,r,s\right] = \sum_{l,m=0}^{\infty} \frac{(t)_l}{(u,q)_{l+m}} \frac{r^l s^m}{(q;q)_l\,(q,q)_m}.$$

Furthermore, it offers a variety of characteristics including new q-contiguous function relations and q-recursion formulae [1] for the q-Humbert confluent hypergeometric functions.

2. Preliminaries

In this section, we briefly review some basic definitions. The k-analog of gamma, beta and hypergeometric functions was first introduced in [7, 8, 18] by Daiz and also demonstrated several of these characteristics. Numerous researchers investigated a variety of findings on these functions [2, 11–14].

Now, we discussed the new extension of Pochhammer (q,k)-shifted symbol. Let $v \geqslant 0$ and $0 < |q| < 1$, $q \in C$, then Pochhammer (q,k)-shifted symbol is defined as

$$(v;q)_{m,k} = \begin{cases} \prod_{r=0}^{m-1} \left(1 - vq^{kr}\right), & \textit{if } m \geqslant 0, \\ 1, & \textit{if } m = 0, \end{cases}$$
$$= \begin{cases} (1-v)\left(1-v^k\right)\left(1-vq^{2k}\right)\left(1-vq^{3k}\right)\cdots\left(1-vq^{k(m-1)}\right), & m \in N, \\ 1, & m = 0. \end{cases}$$

We will use the relations

$$\begin{aligned} \left(vq^k;q\right)_{m,k} &= (v;q)_{m,k}\left(\frac{1-vq^{km}}{1-v}\right) = (v;q)_{m,k}\left[1+\frac{v(1-q^{km})}{1-v}\right], \\ \left(vq^{-k};q\right)_{m,k} &= (v;q)_{m,k}\left(\frac{1-vq^{-k}}{1-vq^{k(m-1)}}\right), \\ \frac{1}{\left(vq^{-k};q\right)_{m,k}} &= \left[1+\frac{v\left(1-q^{km}\right)}{q^k-v}\right]\frac{1}{(v;q)_{m,k}}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \left(vq^k;q\right)_{l+m,k} &= (v;q)_{l+m,k}\left[1+\frac{v\left(1-q^{lk}\right)}{1-v}+\frac{vq^{lk}\left(1-q^{mk}\right)}{1-v}\right] \\ &= (v;q)_{l+m,k}\left[1+\frac{v\left(1-q^{mk}\right)}{1-v}+\frac{vq^{mk}\left(1-q^{lk}\right)}{1-v}\right], \\ \frac{1}{\left(vq^{-k};q\right)_{l+m,k}} &= \left[1+\frac{v\left(1-q^{lk}\right)}{q^k-v}+\frac{vq^{lk}\left(1-q^{mk}\right)}{q^k-v}\right]\frac{1}{(v;q)_{l+m,k}} \\ &= \left[1+\frac{v\left(1-q^{mk}\right)}{q^k-v}+\frac{vq^{mk}\left(1-q^{lk}\right)}{q^k-v}\right]\frac{1}{(v;q)_{l+m,k}}, \end{aligned} \tag{2.2}$$

and

$$\frac{1}{(vq^{-k};q)_{l+m,k}}=\left[\frac{vq^{k(l+m)}}{v-q^{k}}-\frac{q^{k}}{v-q^{k}}\right]\frac{1}{(v;q)_{l+m,k}}.$$

Suppose h is a function that is defined on a subset of the real or complex plane. The q-derivative operator [9] is defined as

$$D_{q}h(y)=\frac{h(y)-h(qy)}{(1-q)\,y}.$$

For $k\geqslant 0$ and $s\geqslant 0$, we have the relation ([3])

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}V\,(k,s)=\sum_{s=0}^{\infty}\sum_{k=0}^{s}V\,(k,s-k)\,.$$

3. (q,k)-contiguous function relations and (q,k)-recursion formulas for K-Humbert confluent hypergeometric functions $\Phi_{1,k}$

Let

$$\begin{aligned}U_{l,m,k}&=\frac{(t;q)_{l+m,k}(u;q)_{l,k}}{(v;q)_{l+m,k}},\\ \Phi_{1,k}=\Phi_{1,k}\,[(t,k),(u,k),(v,k);q,r,s]&=\sum_{l,m=0}^{\infty}U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}.\end{aligned}\tag{3.1}$$

If we multiplying or dividing by q^{k} one of the K-Humbert confluent hypergeometric series parameters, the resultant function is said contiguous to $\Phi_{1,k}$. For (q,k)-contiguous functions, there are following relationships:

$$\begin{aligned}\Phi_{1,k}\left(tq^{k}\right)&=\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),(v,k);q,r,s\right]\\ &=\sum_{l,m=0}^{\infty}\left(\frac{1-tq^{k(l+m)}}{1-t}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &=\sum_{l,m=0}^{\infty}\left[1+\frac{t(1-q^{k(l+m)})}{1-t}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &=\sum_{l,m=0}^{\infty}\left[1+\frac{t\left(1-q^{lk}\right)}{1-t}+\frac{tq^{lk}\left(1-q^{mk}\right)}{1-t}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &=\sum_{l,m=0}^{\infty}\left[1+\frac{t\left(1-q^{mk}\right)}{1-t}+\frac{tq^{mk}\left(1-q^{lk}\right)}{1-t}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned}\tag{3.2}$$

$$\begin{aligned}\Phi_{1,k}\left(tq^{-k}\right)&=\Phi_{1,k}\left[\left(tq^{-k},k\right),(u,k),(v,k);q,r,s\right]\\ &=\sum_{l,m=0}^{\infty}\left(\frac{1-tq^{-k}}{1-tq^{k(l+m-1)}}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &=\sum_{l,m=0}^{\infty}\left(1-\frac{tq^{-k}(1-q^{k(l+m)})}{1-tq^{k(l+m-1)}}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned}\tag{3.3}$$

$$\begin{aligned}
\Phi_{1,k}\left(uq^{k}\right) &= \Phi_{1,k}\left[(t,k),\left(uq^{k},k\right),(v,k);q,r,s\right] \\
&= \sum_{l,m=0}^{\infty}\left(\frac{1-uq^{lk}}{1-u}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\
&= \sum_{l,m=0}^{\infty}\left[1+\frac{u(1-q^{lk})}{1-u}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}, \\
\Phi_{1,k}\left(uq^{-k}\right) &= \Phi_{1,k}\left[(t,k),(uq^{-k},k),(v,k);q,r,s\right] \\
&= \sum_{l,m=0}^{\infty}\left(\frac{1-uq^{-k}}{1-uq^{k(l-1)}}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\
&= \sum_{l,m=0}^{\infty}\left[1-\frac{uq^{-k}\left(1-q^{lk}\right)}{1-uq^{k(l-1)}}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}, \\
\Phi_{1,k}\left(vq^{k}\right) &= \Phi_{1,k}\left[(t,k),(u,k),\left(vq^{k},k\right);q,r,s\right] \\
&= \sum_{l,m=0}^{\infty}\left(\frac{1-v}{1-vq^{k(l+m)}}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\
&= \sum_{l,m=0}^{\infty}\left[1-\frac{v\left(1-q^{k(l+m)}\right)}{1-vq^{k(l+m)}}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}, \\
\Phi_{1,k}\left(vq^{-k}\right) &= \Phi_{1,k}\left[(t,k),(u,k),\left(vq^{-k},k\right);q,r,s\right] \\
&= \sum_{l,m=0}^{\infty}\left(\frac{1-vq^{k(l+m-1)}}{1-vq^{-k}}\right)U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\
&= \sum_{l,m=0}^{\infty}\left[1+\frac{v\left(1-q^{lk}\right)}{q^{k}-v}+\frac{vq^{lk}\left(1-q^{mk}\right)}{q^{k}-v}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\
&= \sum_{l,m=0}^{\infty}\left[1+\frac{v\left(1-q^{mk}\right)}{q^{k}-v}+\frac{vq^{mk}\left(1-q^{lk}\right)}{q^{k}-v}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}.
\end{aligned}$$

Now we consider the operators $\theta_{r,q}=r\frac{\partial}{\partial r}=rD_{r,q}$ and $\theta_{s,q}=s\frac{\partial}{\partial s}=sD_{s,q}$ in (3.1), we get

$$\theta_{r,q}\Phi_{1,k}=\sum_{l,m=0}^{\infty}\frac{\left(1-q^{lk}\right)}{(1-q)}U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}=r\frac{(1-t)\,(1-u)}{(1-v)\,(1-q)}\Phi_{1,k}\left(tq^{k},uq^{k},vq^{k}\right) \tag{3.4}$$

and

$$\theta_{s,q}\Phi_{1,k}=\sum_{l,m=0}^{\infty}\frac{\left(1-q^{mk}\right)}{(1-q)}U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}=s\frac{(1-t)}{(1-v)\,(1-q)}\Phi_{1,k}\left(tq^{k},vq^{k}\right). \tag{3.5}$$

From (3.4) and (3.5) we get the q-derivatives of $\Phi_{1,k}$ with respect to r and s,

$$\begin{aligned}
D^{n}_{r,q}\Phi_{1,k} &= \frac{(t;q)_{n,k}\,(u;q)_{n,k}}{(v;q)_{n,k}\,(1-q)^{n}}\Phi_{1,k}\left[\left(tq^{nk},k\right),\left(uq^{nk},k\right),\left(vq^{nk},k\right);q,r,s\right], \\
D^{n}_{s,q}\Phi_{1,k} &= \frac{(t;q)_{n,k}}{(v;q)_{n,k}\,(1-q)^{n}}\Phi_{1,k}\left[\left(tq^{nk},k\right),\left(vq^{nk},k\right);q,r,s\right].
\end{aligned}$$

From (3.2) and (3.4) we get the relations

$$\left[t\theta_{r,q}+\frac{1-t}{1-q}\right]\Phi_{1,k}+t\theta_{s,q}\Phi_{1,k}\,(rq)=\frac{1-t}{1-q}\Phi_{1,k}\left(tq^{k}\right), \tag{3.6}$$

$$\left[t\theta_{s,q}+\frac{1-t}{1-q}\right]\Phi_{1,k}+t\theta_{r,q}\Phi_{1,k}\left(sq\right)=\frac{1-t}{1-q}\Phi_{1,k}\left(tq^{k}\right), \tag{3.7}$$
$$\left[u\theta_{r,q}+\frac{1-u}{1-q}\right]\Phi_{1,k}=\frac{1-u}{1-q}\Phi_{1,k}\left(uq^{k}\right).$$

From (3.6) and (3.7) we obtain

$$\theta_{r,q}\Phi_{1,k}+\theta_{s,q}\Phi_{1,k}\left(rq\right)=\theta_{s,q}\Phi_{1,k}+\theta_{r,q}\Phi_{1,k}\left(sq\right).$$

Using (3.3), (3.4), and (3.5) for $\Phi_{1,k}$, we obtain

$$\begin{aligned}
\left[tq^{-k}\theta_{r,q}+\frac{1-tq^{-k}}{1-q}\right]\Phi_{1,k}\left(tq^{-k}\right)+tq^{-k}\theta_{s,q}\Phi_{1,k}\left(tq^{-k},rq\right)&=\left(\frac{1-tq^{-k}}{1-q}\right)\Phi_{1,k},\\
\left[tq^{-k}\theta_{s,q}+\frac{1-tq^{-k}}{1-q}\right]\Phi_{1,k}\left(tq^{-k}\right)+tq^{-k}\theta_{r,q}\Phi_{1,k}\left(tq^{-k},sq\right)&=\left(\frac{1-tq^{-k}}{1-q}\right)\Phi_{1,k},\\
\left[uq^{-k}\theta_{r,q}+\frac{1-uq^{-k}}{1-q}\right]\Phi_{1,k}\left(uq^{-k}\right)&=\left(\frac{1-uq^{-k}}{1-q}\right)\Phi_{1,k},\\
\left[vq^{-k}\theta_{r,q}+\frac{1-vq^{-k}}{1-q}\right]\Phi_{1,k}+vq^{-k}\theta_{s,q}\Phi_{1,k}\left(rq\right)&=\left(\frac{1-vq^{-k}}{1-q}\right)\Phi_{1,k}\left(vq^{-k}\right),\\
\left[vq^{-k}\theta_{s,q}+\frac{1-vq^{-k}}{1-q}\right]\Phi_{1,k}+vq^{-k}\theta_{r,q}\Phi_{1,k}\left(sq\right)&=\left(\frac{1-vq^{-k}}{1-q}\right)\Phi_{1,k}\left(vq^{-k}\right).
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we have the relation

$$\left(t-vq^{-k}\right)\Phi_{1,k}=\left(1-vq^{-k}\right)t\Phi_{1,k}\left(vq^{-k}\right)-\left(1-t\right)vq^{-k}\Phi_{1,k}\left(tq^{k}\right).$$

Theorem 3.1. *The following* q*-recursion formulae of* $\Phi_{1,k}$ *with numerator parameter* t *exists for* $v\neq 1$ *and* $n\in\mathbb{N}$*:*

$$\begin{aligned}
\Phi_{1,k}\left(tq^{nk}\right)&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\
&\quad+\frac{tr(1-u)}{1-v}\sum_{h=1}^{n}q^{h-1}\Phi_{1,k}\left[\left(tq^{hk},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right]\\
&\quad+\frac{ts}{1-v}\sum_{h=1}^{n}q^{h-1}\Phi_{1,k}\left[\left(tq^{hk},k\right),\left(vq^{k},k\right);q,qr,s\right],
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\Phi_{1,k}\left(tq^{nk}\right)&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\
&\quad+\frac{ts}{1-v}\sum_{h=1}^{n}q^{h-1}\Phi_{1,k}\left[\left(tq^{hk},k\right),\left(vq^{k},k\right);q,r,s\right]\\
&\quad+\frac{tr(1-u)}{1-v}\sum_{h=1}^{n}q^{h-1}\Phi_{1,k}\left[\left(tq^{hk},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,qs\right],
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\Phi_{1,k}\left(tq^{-nk}\right)&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\
&\quad-\frac{tr(1-u)}{1-v}\sum_{h=1}^{n}q^{-h}\Phi_{1,k}\left[\left(tq^{(-h+1)k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right]\\
&\quad-\frac{ts}{1-v}\sum_{h=1}^{n}q^{-h}\Phi_{1,k}\left[\left(tq^{(-h+1)k},k\right),\left(vq^{k},k\right);q,qr,s\right],
\end{aligned}$$

$$\begin{aligned}\Phi_{1,k}\left(tq^{-nk}\right) &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right] \\ &\quad -\frac{ts}{1-v}\sum_{h=1}^{n} q^{-h}\Phi_{1,k}\left[\left(tq^{(-h+1)k},k\right),\left(vq^{k},k\right);q,r,s\right] \\ &\quad -\frac{tr\left(1-u\right)}{1-v}\sum_{h=1}^{n} q^{-h}\Phi_{1,k}\left[\left(tq^{(-h+1)k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,qs\right],\end{aligned} \tag{3.11}$$

Proof. From (3.2) and (2.2) we get the contiguous relation

$$\begin{aligned}\Phi_{1,k}\left(tq^{k}\right) &= \sum_{l,m=0}^{\infty}\left[1+\frac{t\left(1-q^{lk}\right)}{1-t}+\frac{tq^{lk}\left(1-q^{mk}\right)}{1-t}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\ &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+\frac{tr\left(1-u\right)}{1-v}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right] \\ &\quad +\frac{ts}{1-v}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(vq^{k},k\right);q,qr,s\right].\end{aligned} \tag{3.12}$$

Iterating in this computation $\Phi_{1,k}$ for n-times, we get the recursion formula of (3.9) with parameter tq^{nk}. From (2.2) and (3.1) we get the q-contiguous relation

$$\begin{aligned}\Phi_{1,k}\left(tq^{k}\right) &= \sum_{l,m=0}^{\infty}\left[1+\frac{t\left(1-q^{mk}\right)}{1-t}+\frac{tq^{mk}\left(1-q^{lk}\right)}{1-t}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} \\ &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+\frac{ts}{1-v}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(vq^{k},k\right);q,r,s\right] \\ &\quad +\frac{tr\left(1-u\right)}{1-v}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,qs\right].\end{aligned} \tag{3.13}$$

Iterating the computation on $\Phi_{1,k}$ for n-times, we get the recursion formula of (3.10) with the parametor tq^{nk}. Performing the replacement $t \to tq^{(-1)k}$ in the contiguous relation (3.12), we have

$$\begin{aligned}\Phi_{1,k}\left(tq^{-k}\right) &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-\frac{tr\left(1-u\right)}{q\left(1-v\right)}\Phi_{1,k}\left[(t,k),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right] \\ &\quad -\frac{ts}{q\left(1-v\right)}\Phi_{1,k}\left[(t,k),\left(vq^{k},k\right);q,qr,s\right].\end{aligned}$$

Using the function $\Phi_{1,k}$ for n iterations while using the parameter tq^{-nk}, we get (3.14). Performing the replacement $t \to tq^{(-1)k}$ in the contiguous relation (3.13), we have

$$\begin{aligned}\Phi_{1,k}\left(tq^{-k}\right) &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-\frac{ts}{q\left(1-v\right)}\Phi_{1,k}\left[(t,k),\left(vq^{k},k\right);q,r,s\right] \\ &\quad -\frac{tr\left(1-u\right)}{q\left(1-v\right)}\Phi_{1,k}\left[(t,k),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,qs\right].\end{aligned} \tag{3.14}$$

Using the function $\Phi_{1,k}$ for n iterations while using the parameter tq^{-nk}, we get (3.11). □

Theorem 3.2. *The following q-recursion formulae of $\Phi_{1,k}$ with numerator parameter u exists for $v \neq 1$ and $n \in \mathbb{N}$:*

$$\begin{aligned}\Phi_{1,k}\left(uq^{nk}\right) &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right] \\ &\quad +\frac{ur\left(1-t\right)}{1-v}\sum_{h=1}^{n} q^{h-1}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{hk},k\right),\left(vq^{k},k\right);q,r,s\right],\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Phi_{1,k}\left(uq^{-nk}\right) &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right] \\ &\quad -\frac{ur\left(1-t\right)}{1-v}\sum_{h=1}^{n} q^{-h}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{(-h+1)k},k\right),\left(vq^{k},k\right);q,r,s\right].\end{aligned} \tag{3.16}$$

Proof. From (3.1) and (2.1) we get the q-contiguous relation

$$\begin{aligned}\Phi_{1,k}\left(uq^{k}\right) &= \Phi_{1,k}\left[(t,k),\left(uq^{k},k\right),(v,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{u\left(1-q^{lk}\right)}{1-u}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+\frac{ur(1-t)}{1-v}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right].\end{aligned} \tag{3.17}$$

Iterating the computation on $\Phi_{1,k}$ for n-times, we get the recursion formula of (3.15) with the parameter uq^{nk}. Performing the replacement $u \to uk^{(-1)k}$ in the contiguous relation (3.17), we have

$$\Phi_{1,k}\left(uq^{-k}\right) = \Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-\frac{ur(1-t)}{q(1-v)}\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{k},k\right);q,r,s\right].$$

Using the function $\Phi_{1,k}$ for n iterations while using the parameter uq^{-nk}, we get (3.16). □

Theorem 3.3. *The* q*-recursion formulae for* $\Phi_{1,k}$ *with the lower or denominator parameter* v *are as follows:*

$$\begin{aligned}\Phi_{1,k}\left(vq^{-nk}\right) =&\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+vr(1-t)(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{(2-h)k},k\right);q,r,s\right]\\ &+vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(2-h)k},k\right);q,qr,s\right], v\neq q^{kh},q^{(h-1)k},k\in N,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\Phi_{1,k}\left(vq^{-nk}\right) =&\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{(2-h)k},k\right);q,r,s\right]\\ &+vr(1-t)(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(2-h)k},k\right);q,r,qs\right], v\neq q^{kh},q^{(h-1)k},k\in N,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\Phi_{1,k}\left(vq^{nk}\right) =&\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-vr(1-t)(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{(h+1)k},k\right);q,r,s\right]\\ &-vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(h+1)k},k\right);q,qr,s\right], v\neq q^{-kh},q^{(1-h)k},k\in N,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\Phi_{1,k}\left(vq^{nk}\right) =&\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ &\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{(h+1)k},k\right);q,r,s\right]\\ &-vs(1-t)(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\end{aligned} \tag{3.21}$$

$$\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(h+1)k},k\right);q,r,qs\right],v\neq q^{-kh},q^{(1-h)k},k\in\mathbb{N}.$$

Proof. From (3.2) and (2.2) we get the contiguous relation

$$\begin{aligned}\Phi_{1,k}\left(vq^{-k}\right)&=\sum_{l,m=0}^{\infty}\left[1+\frac{v\left(1-q^{lk}\right)}{q^{k}-v}+\frac{vq^{lk}\left(1-q^{mk}\right)}{q^{k}-v}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\&\quad+\frac{vr(1-t)(1-u)}{\left(q^{k}-v\right)(1-v)}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,s\right]\\&\quad+\frac{vs(1-t)}{\left(q^{k}-v\right)(1-v)}\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{k},k\right);q,qr,s\right].\end{aligned}\tag{3.22}$$

Iterating in this computation $\Phi_{1,k}$ for n-times, we get the recursion formula of (3.18) with parameter vq^{-nk}. From (3.2) and (2.2) we get the contiguous relation

$$\begin{aligned}\Phi_{1,k}\left(vq^{-k}\right)&=\sum_{l,m=0}^{\infty}\left[1+\frac{v\left(1-q^{mk}\right)}{q^{k}-v}+\frac{vq^{mk}\left(1-q^{lk}\right)}{q^{k}-v}\right]U_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\&\quad+\frac{vs(1-t)}{\left(q^{k}-v\right)(1-v)}\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{k},k\right);q,r,s\right]\\&\quad+\frac{vr(1-t)(1-u)}{\left(q^{k}-v\right)(1-v)}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{k},k\right);q,r,qs\right].\end{aligned}\tag{3.23}$$

Iterating in this computation $\Phi_{1,k}$ for n-times, we get the recursion formula of (3.19) with parameter vq^{-nk}. Performing the replacement $v\to vq^{k}$ in the contiguous relation (3.22), we have

$$\begin{aligned}\Phi_{1,k}\left(vq^{k}\right)&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\&\quad-\frac{vr(1-t)(1-u)}{(1-v)\left(1-vq^{k}\right)}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{2k},k\right);q,r,s\right]\\&\quad-\frac{vs(1-t)}{(1-v)\left(1-vq^{k}\right)}\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{2k},k\right);q,qr,s\right].\end{aligned}$$

Using the function $\Phi_{1,k}$ for n iterations while using the parameter vq^{nk}, we get (3.20). Performing the replacement $v\to vq^{k}$ in the contiguous relation (3.23), we have

$$\begin{aligned}\Phi_{1,k}\left(vq^{k}\right)&=\Phi_{1,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\&\quad-\frac{vs(1-t)}{(1-v)\left(1-vq^{k}\right)}\Phi_{1,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{2k},k\right);q,r,s\right]\\&\quad-\frac{vr(1-t)(1-u)}{(1-v)\left(1-vq^{k}\right)}\Phi_{1,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{2k},k\right);q,r,qs\right].\end{aligned}$$

Using the function $\Phi_{1,k}$ for n iterations while using the parameter vq^{nk}, we get (3.21). □

4. (q,k)-contiguous function relations and (q,k)-recursion formulas for K-Humbert confluent hypergeometric functions $\Phi_{2,k}$

Let

$$V_{l,m,k}=\frac{(t;q)_{l,k}(u;q)_{m,k}}{(v;q)_{l+m,k}},$$

$$\Phi_{2,k} = \Phi_{2,k}\left[(t,k),(u,k),(\nu,k);q,r,s\right] = \sum_{l,m=0}^{\infty} V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}. \tag{4.1}$$

If we multiplying or dividing by q^k one of the K-Humbert confluent hypergeometric series parameters, the resultant function is said contiguous to $\Phi_{2,k}$. For (q,k)-contiguous functions, there are following relationships:

$$\begin{aligned}\Phi_{2,k}\left(tq^k\right) &= \Phi_{2,k}\left[\left(tq^k,k\right),(u,k),(\nu,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-tq^{lk}}{1-t}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{t(1-q^{lk})}{1-t}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned} \tag{4.2}$$

$$\begin{aligned}\Phi_{2,k}\left(tq^{-k}\right) &= \Phi_{2,k}\left[\left(tq^{-k},k\right),(u,k),(\nu,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-tq^{-k}}{1-tq^{k(l-1)}}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left(1-\frac{tq^{-k}(1-q^{lk})}{1-tq^{k(l-1)}}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Phi_{2,k}\left(uq^k\right) &= \Phi_{2,k}\left[(t,k),\left(uq^k,k\right),(\nu,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-uq^{mk}}{1-u}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{u(1-q^{mk})}{1-u}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\\ \Phi_{2,k}\left(uq^{-k}\right) &= \Phi_{2,k}\left[(t,k),(uq^{-k},k),(\nu,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-uq^{-k}}{1-uq^{k(m-1)}}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1-\frac{uq^{-k}\left(1-q^{mk}\right)}{1-uq^{k(m-1)}}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\\ \Phi_{2,k}\left(\nu q^k\right) &= \Phi_{2,k}\left[(t,k),(u,k),\left(\nu q^k,k\right);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-\nu}{1-\nu q^{k(l+m)}}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1-\frac{\nu\left(1-q^{k(l+m)}\right)}{1-\nu q^{k(l+m)}}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\\ \Phi_{2,k}\left(\nu q^{-k}\right) &= \Phi_{2,k}\left[(t,k),(u,k),\left(\nu q^{-k},k\right);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-\nu q^{k(l+m-1)}}{1-\nu q^{-k}}\right)V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{\nu\left(1-q^{lk}\right)}{q^k-\nu}+\frac{\nu q^{lk}\left(1-q^{mk}\right)}{q^k-\nu}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\end{aligned}$$

$$= \sum_{l,m=0}^{\infty}\left[1+\frac{v\left(1-q^{mk}\right)}{q^{k}-v}+\frac{vq^{mk}\left(1-q^{lk}\right)}{q^{k}-v}\right]V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}.$$

Now we consider the operators $\theta_{r,q}=r\frac{\partial}{\partial r}=rD_{r,q}$ and $\theta_{s,q}=s\frac{\partial}{\partial s}=sD_{s,q}$ in (4.1), we get

$$\theta_{r,q}\Phi_{2,k}=\sum_{l,m=0}^{\infty}\frac{\left(1-q^{lk}\right)}{(1-q)}V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}=r\frac{(1-t)}{(1-v)(1-q)}\Phi_{2,k}\left(tq^{k},u,vq^{k}\right) \tag{4.4}$$

and

$$\theta_{s,q}\Phi_{2,k}=\sum_{l,m=0}^{\infty}\frac{\left(1-q^{mk}\right)}{(1-q)}V_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}=s\frac{(1-u)}{(1-v)(1-q)}\Phi_{2,k}\left(uq^{k},vq^{k}\right). \tag{4.5}$$

From (4.4) and (4.5) we get the q-derivatives of $\Phi_{2,k}$ with respect to r and s:

$$D^{n}_{r,q}\Phi_{2,k}=\frac{(t;q)_{n,k}}{(v;q)_{n,k}(1-q)^{n}}\Phi_{2,k}\left[\left(tq^{nk},k\right),\left(vq^{nk},k\right);q,r,s\right],$$
$$D^{n}_{s,q}\Phi_{2,k}=\frac{(u;q)_{n,k}}{(v;q)_{n,k}(1-q)^{n}}\Phi_{2,k}\left[\left(uq^{nk},k\right),\left(vq^{nk},k\right);q,r,s\right].$$

From (4.2) and (4.4) we get the relations

$$\left[t\theta_{r,q}+\frac{1-t}{1-q}\right]\Phi_{2,k}=\frac{1-t}{1-q}\Phi_{2,k}\left(tq^{k}\right)$$

and

$$\left[u\theta_{r,q}+\frac{1-u}{1-q}\right]\Phi_{2,k}=\frac{1-u}{1-q}\Phi_{2,k}\left(uq^{k}\right).$$

Using (4.3), (4.4), and (4.5) for $\Phi_{2,k}$, we obtain

$$\left[tq^{-k}\theta_{r,q}+\frac{1-tq^{-k}}{1-q}\right]\Phi_{2,k}\left(tq^{-k}\right)=\left(\frac{1-tq^{-k}}{1-q}\right)\Phi_{2,k},$$
$$\left[uq^{-k}\theta_{r,q}+\frac{1-uq^{-k}}{1-q}\right]\Phi_{2,k}\left(uq^{-k}\right)=\left(\frac{1-uq^{-k}}{1-q}\right)\Phi_{2,k},$$
$$\left[vq^{-k}\theta_{r,q}+\frac{1-vq^{-k}}{1-q}\right]\Phi_{2,k}+vq^{-k}\theta_{s,q}\Phi_{2,k}(rq)=\left(\frac{1-vq^{-k}}{1-q}\right)\Phi_{2,k}\left(vq^{-k}\right),$$
$$\left[vq^{-k}\theta_{s,q}+\frac{1-vq^{-k}}{1-q}\right]\Phi_{2,k}+vq^{-k}\theta_{r,q}\Phi_{2,k}(sq)=\left(\frac{1-vq^{-k}}{1-q}\right)\Phi_{2,k}\left(vq^{-k}\right).$$

Theorem 4.1. *The following* q*-recursion formulae of* $\Phi_{2,k}$ *with numerator parameter* t *exist for* $v\neq 1$ *and* $n\in\mathbb{N}$*:*

$$\Phi_{2,k}\left(tq^{nk}\right)=\Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right]+\frac{tr}{1-v}\sum_{h=1}^{n}q^{h-1}\Phi_{2,k}\left[\left(tq^{hk},k\right),(u,k),\left(vq^{k},k\right);q,r,s\right],$$
$$\Phi_{2,k}\left(tq^{-nk}\right)=\Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right]-\frac{tr}{1-v}\sum_{h=1}^{n}q^{-h}\Phi_{2,k}\left[\left(tq^{(-h+1)k},k\right),(u,k),\left(vq^{k},k\right);q,r,s\right].$$

Theorem 4.2. *The following* q*-recursion formulae of* $\Phi_{2,k}$ *with numerator parameter* u *exist for* $v \neq 1$ *and* $n \in \mathbb{N}$*:*

$$\Phi_{2,k}\left(uq^{nk}\right) = \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right] + \frac{ur}{1-v}\sum_{h=1}^{n} q^{h-1}\Phi_{2,k}\left[(t,k),\left(uq^{hk},k\right),\left(vq^{k},k\right);q,r,s\right],$$

$$\begin{aligned}\Phi_{2,k}\left(uq^{-nk}\right) &= \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right]\\ &\quad - \frac{ur}{1-v}\sum_{h=1}^{n} q^{-h}\Phi_{2,k}\left[(t,k),\left(uq^{(-h+1)k},k\right),\left(vq^{k},k\right);q,r,s\right].\end{aligned}$$

Theorem 4.3. *The* q*-recursion formulas for* $\Phi_{2,k}$ *with the lower or denominator parameter* v *are as follows:*

$$\begin{aligned}\Phi_{2,k}\left(vq^{-nk}\right) =& \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right] + vr(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ & \Phi_{2,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(2-h)k},k\right);q,r,s\right] + vs(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ & \Phi_{2,k}\left[(t,k),\left(uq^{k},k\right),\left(vq^{(2-h)k},k\right);q,qr,s\right], v \neq q^{kh}, q^{(h-1)k}, k \in \mathbb{N},\\ \Phi_{2,k}\left(vq^{-nk}\right) =& \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right] + vs(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ & \Phi_{2,k}\left[(t,k),\left(uq^{k},k\right),\left(vq^{(2-h)k},k\right);q,r,s\right] + vr(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-v\right)\left(q^{(h-1)k}-v\right)}\\ & \Phi_{2,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(2-h)k},k\right);q,r,qs\right], v \neq q^{kh}, q^{(h-1)k}, k \in \mathbb{N},\\ \Phi_{2,k}\left(vq^{nk}\right) =& \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right] - vr(1-t)(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h+1)k}\right)\left(1-vq^{kh}\right)}\\ & \Phi_{2,k}\left[\left(tq^{k},k\right),\left(uq^{k},k\right),\left(vq^{(h+1)k},k\right);q,r,s\right]\\ & - vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ & \Phi_{2,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(h+1)k},k\right);q,qr,s\right], v \neq q^{-kh}, q^{(1-h)k}, k \in \mathbb{N},\\ \Phi_{2,k}\left(vq^{nk}\right) =& \Phi_{2,k}\left[(t,k),(u,k),(v,k);q,r,s\right] - vs(1-u)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ & \Phi_{2,k}\left[(t,k),\left(uq^{k},k\right),\left(vq^{(h+1)k},k\right);q,qr,s\right]\\ & - vs(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-vq^{(h-1)k}\right)\left(1-vq^{kh}\right)}\\ & \Phi_{2,k}\left[\left(tq^{k},k\right),(u,k),\left(vq^{(h+1)k},k\right);q,r,qs\right], v \neq q^{-kh}, q^{(1-h)k}, k \in \mathbb{N}.\end{aligned}$$

5. (q,k)-contiguous function relations and (q,k)-recursion formulas for K-Humbert confluent hypergeometric functions $\Phi_{3,k}$

Let

$$W_{l,m,k} = \frac{(t;q)_{l,k}}{(u;q)_{l+m,k}},$$

$$\Phi_{3,k} = \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] = \sum_{l,m=0}^{\infty} W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}. \tag{5.1}$$

If we multiply or divide by q^k one of the K-Humbert confluent hypergeometric series parameters, the resultant function is said contiguous to $\Phi_{3,k}$. For (q,k)-contiguous functions, there are following relationships:

$$\begin{aligned}\Phi_{3,k}\left(tq^k\right) &= \Phi_{3,k}\left[\left(tq^k,k\right),(u,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-tq^{lk}}{1-t}\right)W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{t(1-q^{lk})}{1-t}\right]W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\Phi_{3,k}\left(tq^{-k}\right) &= \Phi_{3,k}\left[\left(tq^{-k},k\right),(u,k);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-tq^{-k}}{1-tq^{k(l-1)}}\right)W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left(1-\frac{tq^{-k}(1-q^{lk})}{1-tq^{k(l-1)}}\right)W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\Phi_{3,k}\left(uq^k\right) &= \Phi_{3,k}\left[(t,k),\left(vq^k,k\right);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-u}{1-uq^{k(l+m)}}\right)W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1-\frac{u\left(1-q^{k(l+m)}\right)}{1-vq^{k(l+m)}}\right]W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}},\\ \Phi_{3,k}\left(vq^{-k}\right) &= \Phi_{3,k}\left[(t,k),\left(uq^{-k},k\right);q,r,s\right]\\ &= \sum_{l,m=0}^{\infty}\left(\frac{1-uq^{k(l+m-1)}}{1-uq^{-k}}\right)W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{u\left(1-q^{lk}\right)}{q^k-u}+\frac{uq^{lk}\left(1-q^{mk}\right)}{q^k-u}\right]W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}\\ &= \sum_{l,m=0}^{\infty}\left[1+\frac{u\left(1-q^{mk}\right)}{q^k-u}+\frac{uq^{mk}\left(1-q^{lk}\right)}{q^k-u}\right]W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}}.\end{aligned} \tag{5.4}$$

Now we consider the operators $\theta_{r,q} = r\frac{\partial}{\partial r} = rD_{r,q}$ and $\theta_{s,q} = s\frac{\partial}{\partial s} = sD_{s,q}$ in (5.1), we get

$$\theta_{r,q}\Phi_{3,k} = \sum_{l,m=0}^{\infty}\frac{(1-q^{lk})}{(1-q)}W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} = r\frac{(1-t)}{(1-v)(1-q)}\Phi_{3,k}\left(tq^k,uq^k\right) \tag{5.5}$$

and

$$\theta_{s,q}\Phi_{3,k} = \sum_{l,m=0}^{\infty}\frac{(1-q^{mk})}{(1-q)}W_{l,m,k}\frac{r^{lk}s^{mk}}{(q;q)_{l,k}(q,q)_{m,k}} = s\frac{1}{(1-u)(1-q)}\Phi_{3,k}\left(t,uq^k\right). \tag{5.6}$$

From (5.5) and (5.6) we get the q-derivatives of $\Phi_{3,k}$ with respect to r and s:

$$D^n_{r,q}\Phi_{3,k} = \frac{(t;q)_{n,k}}{(u;q)_{n,k}(1-q)^n}\Phi_{3,k}\left[\left(tq^{nk},k\right),\left(uq^{nk},k\right);q,r,s\right],$$
$$D^n_{s,q}\Phi_{3,k} = \frac{1}{(u;q)_{n,k}(1-q)^n}\Phi_{3,k}\left[(t,k),\left(uq^{nk},k\right);q,r,s\right].$$

From (5.2) and (5.5) we get the relation

$$\left[t\theta_{r,q} + \frac{1-t}{1-q}\right]\Phi_{3,k} = \frac{1-t}{1-q}\Phi_{3,k}\left(tq^k\right).$$

Using (5.3), (5.4), and (5.5) for $\Phi_{3,k}$, we obtain

$$\left[tq^{-k}\theta_{r,q} + \frac{1-tq^{-k}}{1-q}\right]\Phi_{3,k}\left(tq^{-k}\right) = \left(\frac{1-tq^{-k}}{1-q}\right)\Phi_{3,k},$$
$$\left[uq^{-k}\theta_{r,q} + \frac{1-uq^{-k}}{1-q}\right]\Phi_{3,k} + uq^{-k}\theta_{s,q}\Phi_{3,k}(rq) = \left(\frac{1-uq^{-k}}{1-q}\right)\Phi_{3,k},$$
$$\left[uq^{-k}\theta_{s,q} + \frac{1-uq^{-k}}{1-q}\right]\Phi_{3,k} + uq^{-k}\theta_{r,q}\Phi_{3,k}(sq) = \left(\frac{1-uq^{-k}}{1-q}\right)\Phi_{3,k}.$$

Theorem 5.1. *The following* q*-recursion formulae of* $\Phi_{3,k}$ *with numerator parameter* t *exists for* $v \neq 1$ *and* $n \in N$*:*

$$\Phi_{3,k}\left(tq^{nk}\right) = \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] + \frac{tr}{1-u}\sum_{h=1}^{n} q^{h-1}\Phi_{3,k}\left[\left(tq^{hk},k\right),\left(uq^{k},k\right);q,r,s\right],$$
$$\Phi_{3,k}\left(tq^{-nk}\right) = \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] - \frac{tr}{1-u}\sum_{h=1}^{n} q^{-h}\Phi_{3,k}\left[\left(tq^{(-h+1)k},k\right),\left(uq^{k},k\right);q,r,s\right]$$
$$-\frac{ts}{1-u}\sum_{h=1}^{n} q^{-h}\Phi_{3,k}\left[\left(tq^{(-h+1)k},k\right),\left(uq^{k},k\right);q,qr,s\right].$$

Theorem 5.2. *The* q*-recursion formulas for* $\Phi_{3,k}$ *with the lower or denominator parameter* u *are as follows:*

$$\begin{aligned}\Phi_{3,k}\left(uq^{-nk}\right) =& \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] + ur(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-u\right)\left(q^{(h-1)k}-u\right)}\\ &\Phi_{3,k}\left[\left(tq^{k},k\right),\left(uq^{(2-h)k},k\right);q,r,s\right] + us\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-u\right)\left(q^{(h-1)k}-u\right)}\\ &\Phi_{3,k}\left[(t,k),\left(uq^{(2-h)k},k\right);q,qr,s\right], u \neq q^{kh}, q^{(h-1)k}, k \in N,\\ \Phi_{3,k}\left(uq^{-nk}\right) =& \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] + us\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-u\right)\left(q^{(h-1)k}-u\right)}\\ &\Phi_{3,k}\left[(t,k),\left(uq^{(2-h)k},k\right);q,r,s\right] + vr(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(q^{hk}-u\right)\left(q^{(h-1)k}-u\right)}\\ &\Phi_{3,k}\left[\left(tq^{k},k\right),\left(uq^{(2-h)k},k\right);q,r,qs\right], u \neq q^{kh}, q^{(h-1)k}, k \in N,\\ \Phi_{3,k}\left(uq^{nk}\right) =& \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] - ur(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-uq^{(h+1)k}\right)\left(1-uq^{kh}\right)}\\ &\Phi_{3,k}\left[\left(tq^{k},k\right),\left(uq^{(h+1)k},k\right);q,r,s\right] - us\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-uq^{(h-1)k}\right)\left(1-uq^{kh}\right)}\\ &\Phi_{3,k}\left[(t,k),\left(uq^{(h+1)k},k\right);q,qr,s\right], u \neq q^{-kh}, q^{(1-h)k}, k \in N,\end{aligned}$$

$$\Phi_{3,k}\left(uq^{nk}\right) = \Phi_{3,k}\left[(t,k),(u,k);q,r,s\right] - us\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-uq^{(h-1)k}\right)\left(1-uq^{kh}\right)}$$
$$\Phi_{3,k}\left[(t,k),\left(uq^{(h+1)k},k\right);q,r,s\right] - vr(1-t)\sum_{h=1}^{n}\frac{q^{(h-1)k}}{\left(1-uq^{(h-1)k}\right)\left(1-uq^{kh}\right)}$$
$$\Phi_{3,k}\left[\left(tq^{k},k\right),\left(vq^{(h+1)k},k\right);q,r,qs\right], u \neq q^{-kh}, q^{(1-h)k}, k \in \mathbb{N}.$$

6. Conclusions

Studying K-Humbert confluent hypergeometric functions offers insights into complex mathematical phenomena, enabling advancements in various scientific fields likes theoretical physics, engineering, and statistics, facilitating the solution of intricate problems. In future research we will be employed further to introduce the k-Humbert confluent hypergeometric functions $\Psi_{1,k}$, $\Psi_{2,k}$, $\Xi_{1,k}$, and $\Xi_{2,k}$ with the help of Pochhammer (q,k)-shifted symbol. And also derive the (q,k)-contiguous function relations, (q,k)-recurrence relations, q-derivatives formulas, and use of q-derivative operator on these bibasic k-Humbert confluent hypergeometric functions $\Psi_{1,k}$, $\Psi_{2,k}$, $\Xi_{1,k}$, and $\Xi_{2,k}$ on one independent bases q.

Acknowledgment

The authors are thankful to the editor and anonymous reviewers for their valuable comments and suggestions. One of the authors (M. Usman) dedicates his contribution in this work to his beloved father (Riaz Ahmad), who passed away but whose spirit continues to inspire him every day.

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