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Certain classes of analytic functions defined by polylogarithm functions

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Abstract

Polylogarithm functions are special functions defined in terms of the polylogarithm, which is a generalization of the logarithm function. These functions appear in various physical systems and are essential for understanding the behavior of these systems at both classical and quantum levels. In this paper, we introduce and study a new subclass of analytic functions which are defined by means of a new differential operator. Some results connected to coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity, close-to-convexity, extreme points, Hadamard product and closure property related to the subclass are obtained.

Keywords: Polylogarithm function, analytic, starlike, convexity, coefficient estimate, distortion property, extreme points.

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1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order ξ, $0 \leqslant \xi < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \xi, \quad z \in \mathbb{U}.$$

We denote this class with $S^*(\xi)$. A function $u \in A$ is a convex function of the order ξ, $0 \leqslant \xi < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \xi, \quad z \in \mathbb{U}.$$

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We denote this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$, respectively. For $u \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \mathbb{U}).$$

Note that $u * g \in A$. Let T denotes the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geqslant 0 \ \ (z \in \mathbb{U}), \tag{1.2}$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [31]. Let $u \in A$. Denote by $D^\lambda : A \to A$ the operator defined by

$$D^\lambda = \frac{z}{(1-z)^{\lambda+1}} * u(z) \ \ (\lambda > -1).$$

It is obvious that $D^0 u(z) = u(z), D^1 u(z) = zu'(z)$ and

$$D^\delta u(z) = \frac{z(z^{\delta-1}u(z))^\delta}{\delta!}, \quad (\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Note that $D^\delta u(z) = z + \sum_{k=2}^{\infty} C(\delta, k) a_k z^k$, where $C(\delta, k) = \binom{k+\delta-1}{\delta}$ and $\delta \in \mathbb{N}_0$. The operator $D^\delta u(z)$ is called the Ruscheweyh derivative operator (see [28]). The evolution of polylogarithm function, also known as Jonquiere's function, was started in 1696 by two eminent mathematicians, Leibniz and Bernoulli [14]. In their work, the polylogarithm function was defined using an absolute convergent series. The development of this function was so significant that it was utilized in the research work of other prominent mathematicians such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan, etc, allowing them to discover various functional identities of great importance as a result [22]. It should come as no surprise that the increased utilization of the polylogarithm function appears to be related to its importance in a number of key areas of mathematics and physics such as topology, algebra, geometry, complex analysis quantum field theory, and mathematical physics [15, 23, 26].

Polylogarithm functions and the analytic functions defined by them have applications in various areas of mathematics and physics, including: Number theory, Quantum field theory, Statistical mechanics, String theory. These functions appear in various physical systems and are essential for understanding the behavior of these systems at both classical and quantum levels. Recntly Al-Shaqsi and Darus [35], Soybas et al. [32], Al-Shaqsi and Darus [9], Stalin et al. [34], and Thirucheran et al. [35] generalized Ruscheweyh and Salagean operators using polylogarithm functions on class A of analytic functions (see also [2, 4–8, 10, 17–20, 23, 27, 30, 33, 36]).

We recall here the definition of the well-known generalization of the Riemann Zeta and polylogarithm function, or simply the n^{th} order polylogarithm function $G(n; z)$ given by

$$\Phi_n(b; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+b)^n} \ \ (n, b \in \mathbb{C}, \ z \in \mathbb{U}),$$

where any term with $k + b$ =0 is excluded (see Lerch [21] and also [12, Sections 1.10 and 1.12]). Using the the definition of Gamma function [12, p.27] a simply transformation produces the integral formula

$$\Phi_n(b; z) = \frac{1}{\Gamma(n)} \int_0^1 z(\log\frac{1}{t})^{n-1} \frac{t^b}{1-tz} dt, \ \ \text{Re}\ b > -1, \ \text{and} \ \text{Re}\ n > 1.$$

We note that $\Phi_{-1}(0;z) = \frac{z}{(1-z)^2}$ is Koebe function. For more about polylogarithm in the theory of univalent functions see [26]. Now, for $u \in A, n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, and $z \in \mathbb{U}$, we define the function $G(n, b; z)$ by

$$G(n,b;z) = (1+b)^n \Phi_n(b;z) = \sum_{k=1}^{\infty} \left(\frac{1+b}{k+b}\right)^n z^k. \tag{1.3}$$

Also we introduce a function $(G(n, b; z))^{-1}$ given by

$$G(n,b;z) * (G(n,b;z))^{-1} = \frac{z}{(1-z)^{\lambda+1}} \quad (n \in \mathbb{C},\ b \in \mathbb{C} \setminus \mathbb{Z}^-,\ \lambda > -1;\ z \in \mathbb{U}), \tag{1.4}$$

and obtain the following linear operator

$$\mathfrak{D}^n_{b,\lambda} u(z) = (G(n,b;z))^{-1} * u(z).$$

Now we find the explicit form of the function $(G(n, b; z))^{(-1)}$. It is well known that $\lambda > -1$,

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1} \quad (z \in \mathbb{U}). \tag{1.5}$$

Putting (1.3) and (1.5) in (1.4), we get

$$\sum_{k=1}^{\infty} \left(\frac{1+b}{k+b}\right)^n z^k * (G(n,b;z))^{(-1)} = \sum_{k=1}^{\infty} \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k.$$

Therefore the function $(G(n, b; z))^{(-1)}$ has the following form

$$(G(n,b;z))^{(-1)} = \sum_{k=1}^{\infty} \left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k \quad (z \in \mathbb{U}).$$

Now we note that

$$\mathfrak{D}^n_{b,} u(z) = z + \sum_{k=2}^{\infty} \Theta(k,b,\lambda,n) a_k z^k \quad (n \in \mathbb{C},\ b \in \mathbb{C} \setminus \mathbb{Z}^-,\ \lambda > -1;\ z \in \mathbb{U}), \tag{1.6}$$

where

$$\Theta(k,b,\lambda,n) = \left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!}.$$

It is clear that $\mathfrak{D}^n_{b,}$ are multiplier transformations. For $n \in \mathbb{Z}, b = 1$, and $\lambda = 0$ the operators $\mathfrak{D}^n_{b,}$ were studied by Uralegaddi and Somanatha [37], and for $n \in \mathbb{Z}, \lambda = 0$ the operators $\mathfrak{D}^n_{b,}$ are closely related to the multiplier transformations studied by Flett [13], also, for $n = -1, \lambda = 0$, the operators $\mathfrak{D}^n_{b,}$ is the integral operator studied by Owa and Srivastava [25]. And for any negative real number n and $b = 1, \lambda = 0$ the operators $\mathfrak{D}^n_{b,}$ is the multiplier transformation studied by Jung et al. [16], and for any nonnegative integer n and $b = \lambda = 0$, the operators $\mathfrak{D}^n_{b,}$ is the differential operator defined by Salagean [29]. Furthermore, for $n = 0$ and $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup 0$, the operator $\mathfrak{D}^n_{b,}$ is the differential operator $\mathfrak{D}^n$ defined by Ruscheweyh [28]. For $n, \lambda \in \mathbb{N}_0$, and $b = 0$ the operator $\mathfrak{D}^n$ is the differential operator defined in [9]. Finally, for different choices of n, b, and λ we obtain several operator investigated earlier by other authors, see, for example, [3, 11, 24].

Motivated by the aforementioned work, we introduce the new subclass involving differential operator $\mathfrak{D}^n_{b,} u(z)$, as below.

Definition 1.1. For $0 \leqslant \omega < 1, 0 \leqslant \sigma < 1, 0 < \sigma < 1,$ and $0 \leqslant \vartheta < 1$, we let $TS^n_{b,\lambda}(\omega, \sigma, \sigma)$ be the subclass of u consisting of functions of the form (1.2) and its geometrical condition satisfies

$$\left| \frac{\omega\left((\mathfrak{D}^n_{b,}u(z))' - \frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)}{\sigma(\mathfrak{D}^n_{b,}u(z))' + (1-\omega)\frac{\mathfrak{D}^n_{b,}u(z)}{z}} \right| < \sigma, \quad z \in \mathbb{U},$$

where $\mathfrak{D}^n_{b,}u(z)$ is given by (1.6).

The aims of studying polylogarithm functions include understanding their analytical properties, establishing their connections to other special functions, and applying them in various areas of mathematics, physics, and engineering. In this paper, we introduce and study a new subclass of analytic functions which are defined by means of a new differential operator. We obtain coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity, close-to-convexity, extreme points Hadamard product, and closure property .

2. Coefficient inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS^n_{b,\lambda}(\omega, \sigma, \sigma)$.

Theorem 2.1. *Let the function* u *be defined by* (1.2)*. Then* $u \in TS^n_{b,\lambda}(\omega, \sigma, \sigma)$ *if and only if*

$$\sum_{k=2}^{\infty} [\omega(k-1) + \sigma(k\sigma + 1 - \omega)]\Theta(k, b, \lambda, n)a_k \leqslant \sigma(\sigma + (1-\omega)), \tag{2.1}$$

where $0 < \sigma < 1, 0 \leqslant \omega < 1, 0 \leqslant \sigma < 1,$ *and* $0 \leqslant \vartheta < 1$*. The result* (2.1) *is sharp for the function*

$$u(z) = z - \frac{\sigma(\sigma + (1-\omega))}{[\omega(k-1) + \sigma(k\sigma + 1 - \omega)]\Theta(k, b, \lambda, n)} z^k, \quad k \geqslant 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\omega\left((\mathfrak{D}^n_{b,}u(z))' - \frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)\right| - \sigma\left|\sigma\left(\mathfrak{D}^n_{b,}u(z))' + (1-\omega)\frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)\right| \\
&= \left|-\omega\sum_{k=2}^{\infty}(k-1)\Theta(k,b,\lambda,n)a_k z^{k-1}\right| - \sigma\left|\sigma + (1-\omega) - \sum_{k=2}^{\infty}(k\sigma + 1 - \omega)\Theta(k,b,\lambda,n)a_k z^{k-1}\right| \\
&\leqslant \sum_{k=2}^{\infty}[\omega(k-1) + \sigma(k\sigma + 1 - \omega)]\Theta(k,b,\lambda,n)a_k - \sigma(\sigma + (1-\omega)) \leqslant 0.
\end{aligned}$$

Hence, by maximum modulus principle, $u \in TS^n_{b,\lambda}(\omega, \sigma, \sigma)$. Now assume that $u \in TS^n_{b,\lambda}(\omega, \sigma, \sigma)$ so that

$$\left| \frac{\omega\left((\mathfrak{D}^n_{b,}u(z))' - \frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)}{\sigma(\mathfrak{D}^n_{b,}u(z))' + (1-\omega)\frac{\mathfrak{D}^n_{b,}u(z)}{z}} \right| < \sigma, \quad z \in \mathbb{U}.$$

Hence

$$\left|\omega\left((\mathfrak{D}^n_{b,}u(z))' - \frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)\right| < \sigma\left|\sigma\left(\mathfrak{D}^n_{b,}u(z))' + (1-\omega)\frac{\mathfrak{D}^n_{b,}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{k=2}^{\infty}\omega(k-1)\Theta(k,b,\lambda,n)a_n z^{k-1}\right| < \sigma\left|\sigma + (1-\omega) - \sum_{k=2}^{\infty}(k\sigma + 1 - \omega)\Theta(k,b,\lambda,n)a_k z^{k-1}\right|.$$

Thus

$$\sum_{k=2}^{\infty}[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)a_k \leqslant \sigma(\sigma+(1-\omega))$$

and this completes the proof. □

Corollary 2.2. *Let the function* $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. *Then*

$$a_k \leqslant \frac{\sigma(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}z^k, \quad k \geqslant 2.$$

3. Distortion and covering theorem

We introduce the growth and distortion theorems for the functions in the class $TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$.

Theorem 3.1. *Let the function* $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. *Then*

$$|z| - \frac{\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z|^2 \leqslant |u(z)| \leqslant |z| + \frac{\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z|^2.$$

The result is sharp and attained

$$u(z) = z - \frac{\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}z^2.$$

Proof.

$$|u(z)| = \left|z - \sum_{k=2}^{\infty} a_k z^k\right| \leqslant |z| + \sum_{k=2}^{\infty} a_k|z|^k \leqslant |z| + |z|^2\sum_{k=2}^{\infty} a_k.$$

By Theorem 2.1, we get

$$\sum_{k=2}^{\infty} a_k \leqslant \frac{\sigma(\sigma+(1-\omega))}{[\omega+\sigma(2\sigma+1-\omega)]\Theta(k,b,\lambda,n)}.$$

Thus

$$|u(z)| \leqslant |z| + \frac{\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z|^2.$$

Also

$$|u(z)| \geqslant |z| - \sum_{k=2}^{\infty} a_k|z|^k \geqslant |z| - |z|^2\sum_{k=2}^{\infty} a_k \geqslant |z| - \frac{\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z|^2.$$

□

Theorem 3.2. *Let* $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. *Then*

$$1 - \frac{2\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z| \leqslant |u'(z)| \leqslant 1 + \frac{2\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}|z|$$

with equality for

$$u(z) = z - \frac{2\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}z^2.$$

Proof. Notice that

$$\begin{aligned}\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]\sum_{k=2}^{\infty}ka_k\\ \leqslant \sum_{k=2}^{\infty}n[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)a_k \leqslant \sigma(\sigma+(1-\omega)),\end{aligned}$$

from Theorem 2.1. Thus

$$\begin{aligned}|u'(z)| = \left|1-\sum_{k=2}^{\infty}ka_kz^{k-1}\right| &\leqslant 1+\sum_{k=2}^{\infty}ka_k|z|^{k-1}\\ &\leqslant 1+|z|\sum_{k=2}^{\infty}ka_k \leqslant 1+|z|\frac{2\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}.\end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned}|u'(z)| = \left|1-\sum_{k=2}^{\infty}ka_kz^{k-1}\right| &\geqslant 1-\sum_{k=2}^{\infty}ka_k|z|^{k-1}\\ &\geqslant 1-|z|\sum_{k=2}^{\infty}ka_k \geqslant 1-|z|\frac{2\sigma(\sigma+(1-\omega))}{\Theta(2,b,\lambda,n)[\omega+\sigma(2\sigma+1-\omega)]}.\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we get the result. □

4. Radii of starlikeness, convexity, and close-to-convexity

In the following theorems, we obtain the radii of starlikeness, convexity, and close-to-convexity for the class $TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$.

Theorem 4.1. *Let $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. Then u is starlike in $|z| < R_1$ of order $\wp$, $0 \leqslant \wp < 1$, where*

$$R_1 = \inf_k\left\{\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{(k-\wp)\sigma(\sigma+(1-\omega))}\right\}^{\frac{1}{k-1}}, \quad k \geqslant 2.$$

Proof. u is starlike of order $\wp$, $0 \leqslant \wp < 1$, if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \wp.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)a_kz^{k-1}}{1-\sum_{k=2}^{\infty}a_kz^{k-1}}\right| \leqslant \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leqslant 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty}\frac{(k-\wp)}{(1-\wp)}a_k|z|^{k-1} \leqslant 1. \tag{4.1}$$

Hence by Theorem 2.1, (4.1) will be true if

$$\frac{k-\wp}{1-\wp}|z|^{k-1} \leqslant \frac{(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega)}$$

or if

$$|z| \leqslant \left[\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{(k-\wp)\sigma(\sigma+(1-\omega))}\right]^{\frac{1}{k-1}}, \quad k \geqslant 2. \tag{4.2}$$

Theorem follows easily from (4.2). □

Theorem 4.2. *Let* $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. *Then* u *is convex in* $|z| < R_2$ *of order* $\wp$, $0 \leqslant \wp < 1$, *where*

$$R_2 = \inf_{k}\left\{\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{k(k-\wp)\sigma(\sigma+(1-\omega))}\right\}^{\frac{1}{k-1}}, \quad k \geqslant 2.$$

Proof. u is convex of order $\wp$, $0 \leqslant \wp < 1$, if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \wp.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} k a_k z^{k-1}}\right| \leqslant \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty} k a_k |z|^{k-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \leqslant 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty}\frac{k(k-\wp)}{(1-\wp)}a_k|z|^{k-1} \leqslant 1. \tag{4.3}$$

Hence by Theorem 2.1, (4.3) will be true if

$$\frac{k(k-\wp)}{1-\wp}|z|^{k-1} \leqslant \frac{(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega)}$$

or if

$$|z| \leqslant \left[\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{k(k-\wp)\sigma(\sigma+(1-\omega))}\right]^{\frac{1}{k-1}}, \quad k \geqslant 2. \tag{4.4}$$

Theorem follows easily from (4.4). □

Theorem 4.3. *Let* $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. *Then* u *is close-to-convex in* $|z| < R_3$ *of order* $\wp$, $0 \leqslant \wp < 1$, *where*

$$R_3 = \inf_{n}\left\{\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{k\sigma(\sigma+(1-\omega))}\right\}^{\frac{1}{k-1}}, \quad k \geqslant 2.$$

Proof. u is close-to-convex of order $\wp$, $0 \leqslant \wp < 1$, if

$$\Re\left\{u'(z)\right\} > \wp.$$

Thus it is enough to show that

$$|u'(z)-1| = \left|-\sum_{k=2}^{\infty} k a_k z^{n-1}\right| \leqslant \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|u'(z)-1| \leqslant 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty}\frac{k}{(1-\wp)}a_k|z|^{k-1} \leqslant 1. \tag{4.5}$$

Hence by Theorem 2.1, (4.5) will be true if

$$\frac{k}{1-\wp}|z|^{k-1} \leqslant \frac{(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega)}$$

or if

$$|z| \leqslant \left[\frac{(1-\wp)(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{k\sigma(\sigma+(1-\omega))}\right]^{\frac{1}{k-1}}, \quad n \geqslant 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

5. Extreme points

In the following theorem, we obtain extreme points for the class $TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$.

Theorem 5.1. *Let* $u_1(z) = z$ *and*

$$u_k(z) = z - \frac{\sigma(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}z^k, \quad \text{for } n = 2,3,\ldots.$$

Then $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$ *if and only if it can be expressed in the form*

$$u(z) = \sum_{k=1}^{\infty} \theta_k u_k(z), \quad \text{where } \theta_k \geqslant 0 \text{ and } \sum_{k=1}^{\infty} \theta_k = 1.$$

Proof. Assume that $u(z) = \sum_{k=1}^{\infty} \theta_k u_k(z)$, hence we get

$$u(z) = z - \sum_{k=2}^{\infty} \frac{\sigma(\sigma+(1-\omega))\theta_n}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}z^k.$$

Now, $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$, since

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))} \frac{\sigma(\sigma+(1-\omega))\theta_n}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)} = \sum_{k=2}^{\infty} \theta_k = 1-\theta_1 \leqslant 1.$$

Conversely, suppose $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$. Then we show that u can be written in the form $\sum_{k=1}^{\infty} \theta_k u_k(z)$. Now $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$ implies from Theorem 2.1,

$$a_k \leqslant \frac{\sigma(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}.$$

Setting $\theta_n = \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}a_k$, $k = 2,3,\ldots$, and $\theta_1 = 1 - \sum_{k=2}^{\infty} \theta_k$, we obtain $u(z) = \sum_{k=1}^{\infty} \theta_k u_k(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belonging to the class $TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$.

Theorem 6.1. *Let* $u, g \in TS(\omega,\sigma,\sigma,\vartheta)$. *Then* $u * g \in TS(\omega,\sigma,\zeta,\vartheta)$ *for*

$$u(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad \text{and} \quad (u*g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k,$$

where

$$\zeta \geqslant \frac{\sigma^2(\sigma+(1-\omega))\omega(k-1)}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]^2\Theta(k,b,\lambda,n) - \sigma^2(\sigma+(1-\omega))(k\sigma+1-\omega)}.$$

Proof. $u \in TS^{n}_{b,\lambda}(\omega,\sigma,\sigma)$ and so

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))} a_k \leqslant 1,$$

and

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))} b_k \leqslant 1.$$

We have to find the smallest number ζ such that

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\zeta(\sigma+(1-\omega))} a_k b_k \leqslant 1.$$

By Cauchy-Schwarz inequality

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))} \sqrt{a_k b_k} \leqslant 1, \tag{6.1}$$

therefore it is enough to show that

$$\frac{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\zeta(\sigma+(1-\omega))} a_k b_k \leqslant \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))} \sqrt{a_k b_k}.$$

That is

$$\sqrt{a_n b_n} \leqslant \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\zeta}{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\sigma}. \tag{6.2}$$

From (6.1),

$$\sqrt{a_k b_k} \leqslant \frac{\sigma(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}.$$

Thus it is enough to show that

$$\frac{\sigma(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)} \leqslant \frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\zeta}{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\sigma},$$

which simplifies to

$$\zeta \geqslant \frac{\sigma^2(\sigma+(1-\omega))\omega(k-1)}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]^2\Theta(k,b,\lambda,n) - \sigma^2(\sigma+(1-\omega))(k\sigma+1-\omega)}.$$

□

7. Closure theorems

We shall prove the following closure theorems for the class $TS^n_{b,\lambda}(\omega,\sigma,\sigma)$.

Theorem 7.1. *Let* $u_j \in TS^n_{b,\lambda}(\omega,\sigma,\sigma),\ j=1,2,\dots,s$. *Then*

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS^n_{b,\lambda}(\omega,\sigma,\sigma)$$

for $u_j(z)=z-\sum_{k=2}^{\infty} a_{k,j}z^k$, *where* $\sum_{j=1}^{s} c_j=1$.

Proof.

$$g(z)=\sum_{j=1}^{s} c_j u_j(z)=z-\sum_{k=2}^{\infty}\sum_{j=1}^{s} c_j a_{k,j} z^k = z-\sum_{k=2}^{\infty} e_k z^k,$$

where $e_k=\sum_{j=1}^{s} c_j a_{k,j}$. Thus $g(z)\in TS^n_{b,\lambda}(\omega,\sigma,\sigma)$ if

$$\sum_{k=2}^{\infty}\frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}e_k \leqslant 1,$$

that is, if

$$\begin{aligned}&\sum_{k=2}^{\infty}\sum_{j=1}^{s}\frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}c_j a_{k,j}\\ &=\sum_{j=1}^{s} c_j \sum_{k=2}^{\infty}\frac{[\omega(k-1)+\sigma(k\sigma+1-\omega)]\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}a_{k,j} \leqslant \sum_{j=1}^{s} c_j = 1.\end{aligned}$$

□

Theorem 7.2. *Let* $u, g \in TS^n_{b,\lambda}(\omega,\sigma,\sigma)$. *Then*

$$h(z)=z-\sum_{k=2}^{\infty}(a_k^2+b_k^2)z^k \in TS^n_{b,\lambda}(\omega,\sigma,\sigma),$$

where

$$\zeta \geqslant \frac{2\omega(k-1)\sigma^2(\sigma+(1-\omega))}{[\omega(k-1)+\sigma(k\sigma+1-\omega)]^2\Theta(k,b,\lambda,n)-2\sigma^2(\sigma+(1-\omega))(k\sigma+1-\omega)}.$$

Proof. Since $u, g \in TS^n_{b,\lambda}(\omega,\sigma,\sigma)$, so Theorem 2.1 yields

$$\sum_{k=2}^{\infty}\left[\frac{(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}a_k\right]^2 \leqslant 1$$

and

$$\sum_{k=2}^{\infty}\left[\frac{(\omega(k-1)+\sigma(k\sigma+1-\omega))\Theta(k,b,\lambda,n)}{\sigma(\sigma+(1-\omega))}b_k\right]^2 \leqslant 1.$$

We obtain from the last two inequalities,

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{(\omega(k-1) + \sigma(k\sigma + 1 - \omega))\Theta(k, b, \lambda, n)}{\sigma(\sigma + (1-\omega))} \right]^2 (a_k^2 + b_k^2) \leqslant 1. \tag{7.1}$$

But $h(z) \in TS(\omega, \sigma, \zeta, q, m)$, if and only if

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1) + \zeta(k\sigma + 1 - \omega)]\Theta(k, b, \lambda, n)}{\zeta(\sigma + (1-\omega))} (a_k^2 + b_k^2) \leqslant 1, \tag{7.2}$$

where $0 < \zeta < 1$, however (7.1) implies (7.2) if

$$\frac{[\omega(k-1) + \zeta(k\sigma + 1 - \omega)]\Theta(k, b, \lambda, n)}{\zeta(\sigma + (1-\omega))} \leqslant \frac{1}{2} \left[\frac{(\omega(k-1) + \sigma(k\sigma + 1 - \omega))\Theta(k, b, \lambda, n)}{\sigma(\sigma + (1-\omega))} \right]^2.$$

Simplifying, we get

$$\zeta \geqslant \frac{2\omega(k-1)\sigma^2(\sigma + (1-\omega))}{[\omega(k-1) + \sigma(k\sigma + 1 - \omega)]^2\Theta(k, b, \lambda, n) - 2\sigma^2(\sigma + (1-\omega))(k\sigma + 1 - \omega)}.$$

□

8. Conclusion

The polylogarithm functions and the classes of analytic functions defined by them provide a rich framework for studying complex functions and their applications in mathematics and physics. In this paper, by making use of the well-known polylogarithm functions, a new class of analytic functions was systematically defined. Then, for this newly defined functions class, we studied well-known results, such as coefficient estimates, growth and distortion properties, radii of starlike and convexity, extreme points, Hadamard product, and closure properties. Furthermore, we believe that this study will motivate a number of researchers to extend this idea to meromorphic functions, bi-univalent functions, harmonic functions, q-calculus, and (p, q)-calculus. One may also apply this idea to the use of sine domain, cosine domain, and petal shaped domain. We hope that this distribution series play a significant role in several branches of mathematics, science, and technology.

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References

[1] S. I. Ahmed, A. K. Radhi, *Estimating coefficients for subclasses of meromorphic bi-univalent functions involving the polylogarithm function*, Int. J. Nonlinear Anal. Appl., **13** (2022), 3085–3092. 1

[2] S. I. Ahmed, A. K. Radhi, *Estimating coefficients for subclasses of meromorphic bi-univalent functions involving the polylogarithm function*, Int. J. Nonlinear Anal. Appl., **13** (2022), 3085–3092. 1

[3] O. P. Ahuja, *Integral operators of certain univalent functions*, Int. J. Math. Math. Sci., **8** (1985), 653–662. 1

[4] A. Akgül, *New subclasses of analytic and Bi-Univalent functions involving a new integral operator defined by Polylogarithm function*, Theory Appl. Math. Comput. Sci., **7** (2017), 31–40. 1

[5] A. Akgül, *Coefficient estimates for certain subclass of Bi-Univalent functions obtained with Polylogarithms*, Math. Sci. Appl. E-Notes, **6** (2018), 70–76.

[6] F. I. Akinwale, J. O. Hamzat, A. T. Oladipo, *Coefficient bounds for certain subclasses of analytic functions associated with generalized distribution defined by polylogarithm*, Int. J. Math. Sci. Optim.: Theory Appl., **9** (2023), 148–155.

[7] K. R. Alhindi, *Convex Families of q-Derivative Meromorphic Functions Involving the Polylogarithm Function*, Symmetry, **15** (2023), 10 pages.

[8] K. R. Alhindi, M. Darus, *Fekete-Szegö inequalities for the polylogarithm function and fractional derivative operator.*, Southeast Asian Bull. Math., **42** (2018), 307–314. 1

[9] K. Al-Shaqsi, M. Darus, *An operator defined by convolution involving the polylogarithms functions*, J. Math. Stat., **4** (2008), 46–50. 1, 1

[10] M. K. Aouf, R. M. El-Ashwah, S. M. El-Deeb, *Certain subclasses of uniformly starlike and convex functions defined by convolution*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), **26** (2010), 55–70. 1

[11] N. E. Cho, T. H. Kim, *Multiplier transformations and strongly close to-convex functions*, Bull. Korean Math. Soc., **40** (2003), 399–410. 1

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions. Vols. I, II*, McGraw-Hill Book Co., New York-Toronto-London, (1953). 1

[13] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38** (1972), 746–765. 1

[14] C. I. Gerhardt, G. W. Leibniz, *Leibnizens Mathematische Schriften III/1*, Georg Olms Verlag, Hildesheim, (1962), 1–420. 1

[15] A. B. Goncharov, *Polylogarithms in Arithmetic and Geometry* In: Proceedings of the International Congress ofMathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, **3** (1995), 374-387. 1

[16] I. B. Jung, Y. C. Kim, H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176** (1993), 138–147. 1

[17] M. G. Khan, B. Ahmad , N. E. Cho, *Properties of harmonic mappings associated with polylogarithm functions*, Appl. Math. E-Notes, **21** (2021), 599–609. 1

[18] B. Khan, I. Aldawish, S. Araci, M. G. Khan, *Third Hankel determinant for the logarthmic coefficients of starlike functions associated with Sine function*, Fractal Fract., **6** (2022), 11 pages.

[19] M. G. Khan, B. Khan, F. M. O. Tawfiq, J. S. Ro, *Zalcman functional and majorization results for certain subfamilies of holomorphic functions*, Axioms, **12** (2023), 13 pages.

[20] V. A. S. J. Lavanya, M. P. Jeyaraman, H. A. Farzana, *On certain geometric properties of generalized polylogarithm function*, Malaya J. Mat., **8** (2020), 183–188. 1

[21] E. Lerch, *Note sur la fonction* $\mathfrak{K}(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi i x}}{(w+k)^s}$, Acta Math., **11** (1887), 19–24. 1

[22] L. Lewin, *Polylogarithms and associated functions*, North-Holland Publishing Co., New York-Amsterdam, (1981). 1

[23] L. Lewin, *Structural properties of polylogarithms*, American Mathematical Society, Providence, RI, (1991). 1

[24] L. J. Lin, S. Owa, *Properties of the Salagean operator*, Georgian Math. J., **5** (1998), 361–366. 1

[25] S. Owa, H. M. Srivastava, *Some applications of the generalized Libera integral operator*, Proc. Japan Acad. Ser. A Math. Sci., **62** (1986), 125–128. 1

[26] S. Ponnusamy, S. Sabapathy, *Polylogarithms in the theory of univalent functions*, Results Math., **30** (1996), 136–150. 1, 1

[27] S. M. Popade, R. N. Ingle, P. T. Reddy, B. Venkateswarlu, *Integral operator defined by polylogarithm function for certain subclass of analytic functions*, Ganita, **70** (2020), 105–112. 1

[28] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109–115. 1, 1

[29] G. Ş. Sălăgean, *Subclasses of univalent functions*, Springer, Berlin, **1013** (1983), 362–372. 1

[30] L. Shi, B. Ahmad, N. Khan, M. G. Khan, S. Araci, W. K. Mahhwani, B. Khan, *Coefficient estimates for a subclass of meromorphic multivalent q close-to-convex functions*, Symmetry, **13** (2021), 12 pages. 1

[31] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51** (1975), 109–116. 1

[32] D. Soybaş, B. Joshi, H. Pawar, *On a certain subclass of analytic functions involving integral operator defined by polylogarithm function*, Mathematics, **7** (2019), 9 pages. 1

[33] H. M. Srivastava, K. R. Alhindi, M. Darus, *An investigation into the polylogarithm function and its associated class of meromorphic functions*, Maejo Int. J. Sci. Technol., **10** (2016), 166–174. 1

[34] T. Stalin, M. Thirucheran, A. Anand, *Obtain subclass of analytic functions connected with convolution of polylogarithm functions*, Adv. Math.: Sci. J., **9** (2020), 9639–9645. 1

[35] M. Thirucheran, M. Vinoth kumar, T. Stalin, *New subclass analytic functions associated with polylogarithm function defined by a linear differential operator*, Stoch. Model. Appl., **25** (2021), 35–40. 1

[36] T. Thulasiram, T. V. Sudharshan, K. Suchitra, *On certain subclass of analytic functions associated with polylogarithm function*, Adv. Appl. Math. Sci., **22** (2023), 2099–2108. 1

[37] B. A. Uralegaddi, C. Somanatha, *Certain classes of univalent functions*, World Sci. Publ., River Edge, NJ, (1992), 371–374. 1