J. Math. Computer Sci., 35 (2024), 431–456

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Novel H_∞ performance and delay-dependent exponential passivity for neural networks in response to leakage delay

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Abstract

This article studies the problem of exponential passivity and H_∞ performance for neural networks (NNs) under the effect of leakage and distributed delays. A novel criterion for achieving exponential passivity in these neural networks is derived. Moreover, we establish new criteria for analyzing the exponential stability and H_∞ performance of the system. Utilizing the Lyapunov-Krasovskii stability theory, we employ an integral inequality to assess the derivative of the Lyapunov-Krasovskii functionals, often referred to as LKFs. This estimation involves constructing novel LKFs that incorporate triple and quadruple integral terms. Furthermore, we obtain results contingent upon the leakage delay and the upper bound of the time-varying delays. To provide context, we conduct comparisons to contrast with existing results. In order to demonstrate the usefulness of the findings, a few numerical examples are provided together with computer simulations.

Keywords: Exponential passivity and H_∞ performance, neural networks, leakage delay.

2020 MSC: 34A34, 26D10, 34D20, 92B20.

1. Introduction

Numerous fields have seen the successful application of neural networks (NNs), such as signal processing, pattern recognition, optimization problems and associative memory [5]. For the majority of these applications, the planned network's equilibrium states must be stable. In applications involving control systems, stability is critical. Stable NNs can be integrated into control loops to make accurate and reliable real-time decisions, helping maintain the overall system's stability. So, it becomes crucial to investigate the stability of neural networks. In practice, neural networks frequently encounter time-delay systems, and these time delays often serve as triggers for oscillations and instability. NNs with delays are suitable for modeling dynamic systems where delays play a significant role, such as chemical reactions, biological processes, and control systems. Control systems often involve time delays between input commands and system responses [1, 2, 40, 41]. Delayed NNs can model and predict these delays for better control system

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doi: 10.22436/jmcs.035.04.04

Received: 2024-02-09 *Revised:* 2024-04-02 *Accepted:* 2024-05-10

design. To confirm the exponential or asymptotic stability of delay neural networks, delay-independent and delay-dependent conditions have been proposed, as mentioned in pertinent references. Consequently, NNs with time delays have become a significant concern within control theory and have been extensively studied [3–5, 16, 19, 29–33, 36, 43, 46, 47, 52].

Neural network implementations and applications frequently experience abrupt changes in their parameters (such as connection weights and biases) as a result of unanticipated failures or intentional switching [22]. In this scenario, a switching model that is essentially a collection of parametric settings that transition between them by a particular Markov chain can be used to describe neural networks. However, when neural networks are implemented in hardware, the tolerances of the electrical components used in the design may cause certain modifications to the neural system's network parameters.

The essence of passivity theory lies in the concept that a system's passive attributes contribute directly to its internal stability, prompting significant research exploration [3, 7, 8, 16, 19, 21, 23, 36, 43, 46, 47, 51]. Exponential stability ensures that perturbed trajectories converge exponentially towards equilibrium points, providing controlled behavior. Exponential passivity seamlessly integrates stability with a system's energy behavior, which is crucial in maintaining overall stability and energy efficiency. Recent works have addressed the study of exponential passivity for NNs with time-varying delay [28, 46, 47, 52]. Furthermore, H_∞ performance analysis specifically focuses on evaluating and enhancing the ability of NNs with time delay to maintain stability and desired performance levels despite uncertainties. The challenge of the H_∞ control problem for NNs has garnered remarkable attention among researchers. Moreover, [34] examined the joint H_∞ and passivity state estimate of memristive neural networks. Our objective is to address the lack of attention given to the combined passive and H_∞ analysis problem for neural networks (NNs) under the influence of leakage and distributed delays.

Motivated by the discussions above, this letter presents the derivation of the exponential passivity, exponential stability and H_∞ performance for NNs with leakage and mixed time-varying delays. Moreover, we construct novel augments LKFs using various techniques, including inequalities, descriptor model transformation and Leibniz-Newton formula. We show the new sufficient conditions for the problem in the form of linear matrix inequalities (LMIs). Besides, we got some results dependent on the leakage delay and upper bound of mixed time-varying delays. In the numerical part, we provide some examples to demonstrate the effectiveness of the proposed criteria. The major contributions of this article can be concluded as follows.

- We firstly investigate the exponential passivity, exponential stability, and H_∞ performance for NNs incorporating three types of delays: leakage, discrete, and distributed.

- Working in this field, the system incorporates leakage and mixed time-varying delays. Specifically, we focus on the scenario where the distributed delay is non-differentiable. In real-world systems, delay behaviors may not conform to smooth and differentiable patterns. Non-differentiable distributed time-varying delay can more effectively capture a specific process's irregular and discontinuous nature.

- We propose new sufficient conditions to exponential passivity, exponential stability and H_∞ performance for NNs with leakage and mixed time-varying delays, encompassing a broad range of performances. And there have never been any previous works that have combined all of these aspects.

- To estimate the LKFs, we have independently established Lemmas 2.6-2.8. Furthermore, integral inequalities are applied and new LKFs with single, double, triple, and quadruple integral terms are presented. This approach contrasts with the techniques employed in the cited works [9–12, 14, 15, 18, 25–27, 39, 45, 49, 50]. The utilization of the Leibniz-Newton formula and the concept of the zero equation is employed in our methodology. Notably, the approach adopted in this study leads to less conservative results compared to previous findings [9–12, 14, 15, 18, 25–27, 39, 45, 49, 50].

- To demonstrate the theory's validity and use of the theory, we provide five numerical examples. These examples serve to illustrate how the theory effectively addresses various aspects of the subject under study.

This article is structured into five sections as follows. We provide the neural network model and preliminaries in Section 2. In Section 3, we present the analysis of exponential passivity in the NNs and investigates the exponential stability analysis of a particular case of NNs. Section 4 shows the numerical examples. Finally, Section 5 addresses the conclusions.

Notations: The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, while the set of all $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. The maximum and minimum eigenvalues of A are denoted by the expressions $\lambda_{max}(P)$ and $\lambda_{min}(P)$, respectively. A symmetric positive definite matrix is shown by A, and block diagonal matrix by diag$\{\cdots\}$. Matrix transposition is indicated by the superscript "T" and Sym$\{A\} = A + A^T$.

2. Network model and preliminaries

The following are the continuous neural networks (NNs) with mixed time-varying delays and leakage delay:

$$\begin{aligned}
\dot{\xi}(t) &= -A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds + u(t),\\
z(t) &= C_1(\xi(t)) + C_2(\xi(t-h(t))) + C_3u(t),\\
\xi(t) &= \phi(t), \quad t \in [-\tau_{max}, 0], \quad \tau_{max} = \max\{h_M, \rho_M\}.
\end{aligned} \tag{2.1}$$

The neural state vector is denoted by $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_n(t)] \in \mathbb{R}^n$. The self-feedback connection weight matrix is indicated by the diagonal matrix A. The connection weight matrices linking neurons with the proper dimensions are denoted by the letters B, C, and D. C_1, C_2, and C_3 represent the real matrices. The functions $f(\xi(t)), k(\xi(t)), h(\xi(t)) \in \mathbb{R}^n$ denote the activation functions. $z(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^n$ are the output and input vectors, respectively. There is an initial condition $\phi(t)$, where the constant leakage delay is denoted by $\delta \geqslant 0$. The variables $\rho(t)$ and $h(t)$ denote the distributed and discrete time-varying delays that satisfy

$$0 \leqslant h(t) \leqslant h_M, \quad 0 \leqslant \dot{h}(t) \leqslant h_d, \quad 0 \leqslant \rho(t) \leqslant \rho_M, \tag{2.2}$$

where h_M, h_d and ρ_M are positive real constants. The activation functions $f(\xi(t)), k(\xi(t))$, and $h(\xi(t))$ meet the following requirements.

(A1) f is a continuous function that meets the requirements listed below:

$$F_i^- \leqslant \frac{f_i(\alpha_1) - f_i(\alpha_2)}{\alpha_1 - \alpha_2} \leqslant F_i^+,$$

for all $\alpha_1 \neq \alpha_2$ and $F_i^-, F_i^+ \in \mathbb{R}$, $f_i(0) = 0$.

(A2) k is a continuous function that meets the requirements listed below:

$$K_i^- \leqslant \frac{k_i(\alpha_1) - k_i(\alpha_2)}{\alpha_1 - \alpha_2} \leqslant K_i^+,$$

for all $\alpha_1 \neq \alpha_2$ and $K_i^-, K_i^+ \in \mathbb{R}$, $k_i(0) = 0$.

(A3) h is a continuous function that meets the requirements listed below:

$$H_i^- \leqslant \frac{h_i(\alpha_1) - h_i(\alpha_2)}{\alpha_1 - \alpha_2} \leqslant H_i^+,$$

for all $\alpha_1 \neq \alpha_2$ and $H_i^-, H_i^+ \in \mathbb{R}$, $h_i(0) = 0$.

Definition 2.1 ([52]). If there is an exponential Lyapunov function (sometimes called an exponential storage function) V defined on $\mathbb{R}^n$ and a positive constant $\beta > 0$, then the neural network is said to be exponentially passive from input $u(t)$ to output $z(t)$. The inequality holds for all $u(t)$, all initial conditions $\xi(0)$, and all $t \geqslant t_0$,

$$\dot{V}(t) + \beta V(t) \leqslant 2z^T(t)u(t), \quad t \geqslant 0,$$

where $\dot{V}(t)$ denotes the entire derivative of $V(t)$ along the system (2.1) state trajectories $\xi(t)$ for $t \geqslant 0$.

Definition 2.2 ([18]). If there exist κ and β are real positive scalars and satisfy

$$\|\xi(t,\phi)\| \leqslant \beta\|\phi\|e^{-\kappa t}, \quad \forall t \geqslant 0,$$

subsequently the delayed NNs (2.1) exhibit exponential stability.

Definition 2.3 ([6]). If the system is both exponentially stable and satisfies the condition $\|z(t)\|2 \leqslant \gamma\|w(t)\|2$ for all nonzero $w(t) \in L_2[0,\infty)$, assuming zero initial conditions, then we declare the system described in (2.1) is exponentially stable with a H_∞ performance level of γ.

Lemma 2.4 ([42]). *When a vector function $\omega : [-k_1, k_2] \to \mathbb{R}^n$ is well-defined and there is a constant positive definite symmetric matrix $W \in \mathbb{R}^{n\times n}$, along with functions $k(t)$ with $0 < k_1 < k(t) < k_2$, then*

$$\frac{-(k_2^2 - k_1^2)}{2}\int_{-k_2}^{-k_1}\int_{t+s}^{t} \omega^T(u)W\omega(u)duds \leqslant -\phi^T(t)W\phi(t), \qquad \phi(t) = \Big(\int_{-k_2}^{-k_1}\int_{t+s}^{t} \omega(u)duds\Big).$$

Lemma 2.5 ([20]). *For a positive matrix S, the following inequality holds:*

$$-\frac{(\alpha-\beta)^3}{6}\int_{\beta}^{\alpha}\int_{s}^{\alpha}\int_{u}^{\alpha} \xi^T(\lambda)S\xi(\lambda)d\lambda duds \leqslant -\Big(\int_{\beta}^{\alpha}\int_{s}^{\alpha}\int_{u}^{\alpha} \xi(\lambda)d\lambda duds\Big)^T S\Big(\int_{\beta}^{\alpha}\int_{s}^{\alpha}\int_{u}^{\alpha} \xi(\lambda)d\lambda duds\Big).$$

Lemma 2.6 ([38]). *If there exists a constant symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, a discrete time-varying delay $h(t)$ defined in (2.2), and a vector function $\omega : [-h_M, 0] \to \mathbb{R}^n$ such that the relevant integration is well-defined, then*

$$-h_M\int_{-h_M}^{0} \xi^T(s)Q\xi(s)ds \leqslant -\int_{-h(t)}^{0} \xi^T(s)dsQ\int_{-h(t)}^{0} \xi(s)ds - \int_{-h_M}^{-h(t)} \xi^T(s)dsQ\int_{-h_M}^{-h(t)} \xi(s)ds.$$

Lemma 2.7 ([38]). *If $Q_1, Q_2, Q_3 \in R^{n\times n}$ are constant matrices of any kind, $Q_1 \geqslant 0, Q_3 > 0$, $\begin{bmatrix} Q_1 & Q_2 \\ & Q_3 \end{bmatrix} \geqslant 0$, $h(t)$ is a discrete time-varying delay with (2.2) and a vector function $\dot{\xi} : [-h_M, 0] \to R^n$ such that the subsequent integration is well defined, then*

$$\begin{aligned}
-h_M\int_{t-h_M}^{t} &\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix}\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix} ds \\
&\leqslant \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \\ \xi(t-h_M) \\ \int_{t-h(t)}^{t}\xi(s)ds \\ \int_{t-h_M}^{t-h(t)}\xi(s)ds \end{bmatrix}^T
\begin{bmatrix} -Q_3 & Q_3 & 0 & -Q_2^T & 0 \\ * & -Q_3-Q_3^T & Q_3 & Q_2^T & -Q_2^T \\ * & * & -Q_3 & 0 & Q_2^T \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_1 \end{bmatrix}
\begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \\ \xi(t-h_M) \\ \int_{t-h(t)}^{t}\xi(s)ds \\ \int_{t-h_M}^{t-h(t)}\xi(s)ds \end{bmatrix}.
\end{aligned}$$

Lemma 2.8 ([38]). *Given a vector-valued function $\xi(t) \in \mathbb{R}^n$, let its entries be continuous derivatives of first order. When $X, M_i \in \mathbb{R}^{n\times n}, i = 1,2,\dots,5$, and $h(t)$ are constant matrices, the integral inequality holds for discrete time-varying delays with (2.2) as follows:*

$$-\int_{t-h_M}^{t} \dot{\xi}^T(s)X\dot{\xi}(s)ds \leqslant \begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}^T \begin{bmatrix} M_1+M_1^T & -M_1^T+M_2 & 0\\ * & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ * & * & -M_2-M_2^T\end{bmatrix} \begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}$$
$$+h_M \begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}^T \begin{bmatrix} M_3 & M_4 & 0\\ * & M_3+M_5 & M_4\\ * & * & M_5\end{bmatrix} \begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix},$$

where

$$\begin{bmatrix} X & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5\end{bmatrix} \geqslant 0.$$

3. Main results

3.1. Analyzing exponential passivity in neural networks

First, we shall define sufficient requirements in this section that ensure NNs' exponential passivity. This entails taking into account the subsequent model:

$$\begin{aligned} \dot{\xi}(t) &= -A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds + u(t),\\ z(t) &= C_1(\xi(t)) + C_2(\xi(t-h(t))) + C_3u(t),\\ \xi(t) &= \phi(t), \quad t\in[-\tau_{max},0], \quad \tau_{max} = \max\{h_M,\rho_M\}. \end{aligned} \tag{3.1}$$

The following definitions are given in this article, along with an introduction to the notations for future reference:

$$\sum = [\Omega_{(i,j)}]_{26\times 26},$$

where $\Omega_{(i,j)} = \Omega_{(i,j)}^T$, $i,j = 1,2,\dots,26$,

$$\begin{aligned}
\Omega_{(1,1)} &= -Q_2^T A - A^T Q_2 + Q_3^T + Q3 + \alpha Q_1 + Q_1^T\alpha - e^{2\alpha h_M}P_2A - e^{2\alpha h_M}A^TP_2^T + P_3 + \delta^2 P_4\\
&\quad + P_5 + R_1 + R_4 + h_m^2 P_6 + (M_1+M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M}M_3 + h_M^2 R_7 - e^{2\alpha h_M}R_9\\
&\quad + e^{2\alpha h_M}\frac{h_M{}^4}{4}P_9 - e^{-4\alpha h_M}h_M^2 P_{10} + e^{2\alpha h_M}\frac{h_M^6}{36}P_{11} - F_1Y_1 - H_1Y_3 + 2h_M E_1 + 2\delta E_7,\\
\Omega_{(1,2)} &= Q_1,\\
\Omega_{(1,3)} &= -A^TQ_5 - Q_3^T + Q_6 + Q_4^T + (M_2 - M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M}M_4^T + e^{2\alpha h_M}R_9 - h_M E_1 + h_M E_2^T,\\
\Omega_{(1,4)} &= -A^TQ_8 + Q_9 - e^{2\alpha h_M}R_8^T,\\
\Omega_{(1,5)} &= -A^TQ_{11} + Q_{12} - Q_4^T,\\
\Omega_{(1,6)} &= -Q_2^T - A^TQ_{14} + Q_{15} + h_M^2 R_5,\\
\Omega_{(1,7)} &= Q_2^TB + e^{2\alpha h_M}B + R_2 + R_5 + F_2Y_1,\\
\Omega_{(1,8)} &= Q_2^TC + e^{2\alpha h_M}C,\\
\Omega_{(1,9)} &= Q_2^TD + e^{2\alpha h_M}D,\\
\Omega_{(1,10)} &= -Q_3^T - h_M E_1 + h_M E_3^T,
\end{aligned}$$

$$
\begin{aligned}
\Omega_{(1,11)} &= -Q_4^T,\\
\Omega_{(1,12)} &= Q_2^T + e^{2\alpha h_M} - C_1^T,\\
\Omega_{(1,13)} &= e^{2\alpha h_M} A^T A^T - \delta E_7 + \delta E_9^T,\\
\Omega_{(1,15)} &= \delta E_7 + \delta E_8^T,\\
\Omega_{(1,21)} &= e^{-4\alpha h_M} h_M,\\
\Omega_{(1,26)} &= H_2 Y_3,\\
\Omega_{(2,2)} &= -Q_1^T - Q_1,\\
\Omega_{(2,6)} &= Q_1,\\
\Omega_{(2,9)} &= E_{10}^T,\\
\Omega_{(3,3)} &= -Q_6^T - Q_6 + Q_7^T + Q_7 + (M_1 + M_1^T - M_2 - M_2^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M}(M_3 + M_5)\\
&\quad - K_1 Y_2 - 2h_M E_2 + 2\rho_M E_4 + (-R_9 - R_9^T)e^{-2\alpha h_M},\\
\Omega_{(3,4)} &= -Q_9 + Q_{10} + e^{2\alpha h_M} R_8^T,\\
\Omega_{(3,5)} &= -Q_{12} - Q_7^T + Q_{13} + (-M_1^T + M_2)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M} M_4 + e^{2\alpha h_M} R_9 - \rho_M E_4 + \rho_M E_5^T,\\
\Omega_{(3,6)} &= -Q_5^T - Q_{15} + Q_{16},\\
\Omega_{(3,7)} &= Q_5^T B,\\
\Omega_{(3,8)} &= Q_5^T C + K_2 Y_2,\\
\Omega_{(3,9)} &= Q_5^T D,\\
\Omega_{(3,10)} &= -Q_6^T - h_M E_2 - h_M E_3^T,\\
\Omega_{(3,11)} &= -Q_7^T - \rho_M E_4 + \rho_M E_6^T,\\
\Omega_{(3,12)} &= Q_5^T - C_2^T,\\
\Omega_{(3,19)} &= -e^{2\alpha h_M} R_8^T,\\
\Omega_{(4,4)} &= -h_M e^{-2\alpha h_M} P_6 - e^{2\alpha h_M} R_7,\\
\Omega_{(4,5)} &= -Q_{10}^T,\\
\Omega_{(4,6)} &= -Q_8^T,\\
\Omega_{(4,7)} &= Q_8^T B,\\
\Omega_{(4,8)} &= Q_8^T C,\\
\Omega_{(4,9)} &= Q_8^T D,\\
\Omega_{(4,10)} &= -Q_9^T,\\
\Omega_{(4,11)} &= -Q_{10}^T,\\
\Omega_{(4,12)} &= Q_8^T,\\
\Omega_{(5,5)} &= -Q_{13} - Q_{13}^T - e^{-2\alpha h_M} P_5 - e^{-2\alpha h_M} R_4 + (-M_2 - M_2^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M} M_5\\
&\quad - e^{2\alpha h_M} R_9 - 2\rho_M E_5,\\
\Omega_{(5,6)} &= -Q_{11}^T - Q_{16},\\
\Omega_{(5,7)} &= Q_{11}^T B,\\
\Omega_{(5,8)} &= Q_{11}^T C,\\
\Omega_{(5,9)} &= Q_{11}^T D,\\
\Omega_{(5,10)} &= -Q_{12}^T,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{(5,11)} &= -Q_{13}^T - \rho_M E_5 - \rho_M E_6^T, \\
\Omega_{(5,12)} &= Q_{11}^T, \\
\Omega_{(5,16)} &= -e^{-2\alpha h_M} R_5, \\
\Omega_{(5,19)} &= e^{2\alpha h_M} R_8^T, \\
\Omega_{(6,6)} &= -Q_{14}^T - Q_{14} + h_M P_7 + h_M^2 R_9 + e^{2\alpha h_M} \frac{h_M^4}{4} P_{10} + e^{2\alpha h_M} \frac{h_M^6}{36} P_{12}, \\
\Omega_{(6,7)} &= Q_{14}^T B, \\
\Omega_{(6,8)} &= Q_{14}^T C, \\
\Omega_{(6,9)} &= Q_{14}^T D, \\
\Omega_{(6,10)} &= -Q_{15}^T, \\
\Omega_{(6,11)} &= -Q_{16}^T, \\
\Omega_{(6,12)} &= Q_{14}^T, \\
\Omega_{(7,7)} &= R_3 + R_6 + \rho_M^2 P_8 - Y_1, \\
\Omega_{(7,9)} &= -E_{10}^T B^T, \\
\Omega_{(7,13)} &= e^{2\alpha h_M} B^T A^T, \\
\Omega_{(8,8)} &= -Y_2, \\
\Omega_{(8,9)} &= -E_{10}^T C^T, \\
\Omega_{(8,13)} &= e^{2\alpha h_M} C^T A^T, \\
\Omega_{(9,9)} &= -E_{10} D - D^T E_{10}^T, \\
\Omega_{(9,12)} &= -E_{10}, \\
\Omega_{(9,13)} &= e^{2\alpha h_M} D^T A^T, \\
\Omega_{(9,15)} &= A E_{10}, \\
\Omega_{(10,10)} &= -2h_M E_3, \\
\Omega_{(11,11)} &= -2\rho_M E_6, \\
\Omega_{(12,12)} &= -2C_3, \\
\Omega_{(12,13)} &= e^{2\alpha h_M} A^T, \\
\Omega_{(13,13)} &= -2\delta E_9, \\
\Omega_{(13,15)} &= -\delta E_8^T + \delta E_9, \\
\Omega_{(14,14)} &= -\delta^2 e^{2\alpha\delta} P_4, \\
\Omega_{(15,15)} &= -e^{2\alpha\delta} P_3 - 2\delta E_8, \\
\Omega_{(16,16)} &= -e^{-2\alpha h_M} R_6, \\
\Omega_{(17,17)} &= -e^{-2\alpha h_M}(1 - h_d) R_1, \\
\Omega_{(17,18)} &= -e^{-2\alpha h_M}(1 - h_d) R_2, \\
\Omega_{(18,18)} &= -e^{-2\alpha h_M}(1 - h_d) R_3, \\
\Omega_{(19,19)} &= -h_M e^{-2\alpha h_M} - e^{2\alpha h_M} R_7, \\
\Omega_{(20,20)} &= -e^{2\alpha\rho_M} \rho_M^2 P_8, \\
\Omega_{(21,21)} &= -e^{-4\alpha h_M} P_{10}, \\
\Omega_{(22,22)} &= -e^{-4\alpha h_M} P_9,
\end{aligned}
$$

$\Omega_{(23,23)} = -e^{-4\alpha h_M} P_{12}$,
$\Omega_{(23,24)} = -e^{-4\alpha h_M} P_{12}$,
$\Omega_{(24,24)} = -e^{4\alpha h_M} P_{12}$,
$\Omega_{(25,25)} = -e^{4\alpha h_M} P_{11}$,
$\Omega_{(26,26)} = -Y_3$,

and the others are 0.

Theorem 3.1. *The delayed NNs (3.1) is exponential passive with a decay rate α for given positive real constants h_M, h_d, ρ_M, and δ. Any suitable dimensional matrices Q_m and $m = 1, 2, \ldots, 16$ such that the LMIs hold as follows, if there are positive definite matrices Q_1, R_7, R_9, Y_1, Y_2, Y_3, P_i, $i \in \{1, 2, \ldots, 12\}$,*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_7 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \quad \sum < 0. \tag{3.2}$$

Proof. We generate the system (3.1) in the following way by applying the model transformation method:

$$\dot{\xi}(t) = y(t), \quad 0 = -y(t) - A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds + u(t).$$

Create a possible LKF in the following format for the neural networks with delays shown in (3.1):

$$V(t) = \sum_{i=1}^{9} V_i(\xi(t), t), \tag{3.3}$$

where

$$\begin{aligned}
V_1(\xi(t), t) &= \varpi^T(t) E P_1 \varpi(t), \\
V_2(\xi(t), t) &= e^{-2\alpha t}[\big(\xi(t) - A\int_{t-\delta}^{t} \dot{\xi}(s)ds\big)^T P_2 \big(\xi(t) - A\int_{t-\delta}^{t} \xi(s)ds\big)], \\
V_3(\xi(t), t) &= \int_{t-\delta}^{t} e^{2\alpha(s-t)} \xi^T(s) P_3 \xi(s) ds + \delta \int_{-\delta}^{0} \int_{t+\delta}^{t} e^{2\alpha(s-t)} \xi^T(s) P_4 \xi(s) ds d\theta, \\
V_4(\xi(t), t) &= \int_{t-h_M}^{t} e^{2\alpha(s-t)} \xi^T(s) P_5 \xi(s) ds + \int_{t-h(t)}^{t} e^{2\alpha(s-t)} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix} ds \\
&\quad + \int_{t-h_M}^{t} e^{2\alpha(s-t)} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} \xi(s) \\ f(\xi(s)) \end{bmatrix} ds, \\
V_5(\xi(t), t) &= h_M \int_{-h_M}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \xi^T(\theta) P_6 \xi(\theta) d\theta ds + \int_{-h_M}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} y^T(\theta) P_7 y(\theta) d\theta ds, \\
V_6(\xi(t), t) &= \rho_M \int_{-\rho_M}^{0} \int_{t+\rho_M}^{t} e^{2\alpha(\theta-t)} f(\xi(\theta))^T P_8 f(\xi(\theta)) d\theta ds, \\
V_7(\xi(t), t) &= h_M \int_{-h_M}^{0} \int_{t+s}^{t} e^{2\alpha(\theta-t)} \begin{bmatrix} \xi(\theta) \\ y(\theta) \end{bmatrix}^T \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \begin{bmatrix} \xi(\theta) \\ y(\theta) \end{bmatrix} d\theta ds, \\
V_8(\xi(t), t) &= \frac{h_M^2}{2} \int_{-h_M}^{0} \int_{\lambda}^{0} \int_{t+s}^{t} e^{2\alpha(\theta+s-t)} \xi^T(\theta) P_9 \xi(\theta) d\theta ds d\lambda \\
&\quad + \frac{h_M^2}{2} \int_{-h_M}^{0} \int_{\lambda}^{0} \int_{t+s}^{t} e^{2\alpha(\theta+s-t)} y^T(\theta) P_{10} y(\theta) d\theta ds d\lambda,
\end{aligned}$$

$$V_9(\xi(t),t) = \frac{h_M^3}{6}\int_{-h_M}^{0}\int_{s}^{0}\int_{u}^{0}\int_{t+s+\lambda}^{t} e^{2\alpha(\theta+\lambda+s-t)}\xi^T(\theta)P_{11}\xi(\theta)d\theta d\lambda du ds$$
$$+ \frac{h_M^3}{6}\int_{t-h_M}^{0}\int_{s}^{0}\int_{u}^{0}\int_{t+s+\lambda}^{t} e^{2\alpha(\theta+\lambda+s-t)}y^T(\theta)P_{12}y(\theta)d\theta d\lambda du ds,$$

with

$$E = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} Q_1 & 0 & 0 & 0 & 0 \\ Q_2 & Q_5 & Q_8 & Q_{11} & Q_{14} \\ Q_3 & Q_6 & Q_9 & Q_{12} & Q_{15} \\ Q_4 & Q_7 & Q_{10} & Q_{13} & Q_{16} \end{bmatrix}, \quad \varpi(t) = \begin{bmatrix} \xi(t) \\ \xi(t-h(t)) \\ \int_{t-h(t)}^{t}\xi(s)ds \\ \xi(t-h_M) \\ y(t) \end{bmatrix}.$$

Calculating the time derivative of $V(t)$ while following the delayed NNs' trajectory (3.1), we have

$$\dot{V}(t) = \sum_{i=1}^{9}\dot{V}_i(\xi(t),t).$$

Notably, $\varpi^T(t)EP_1\varpi(t)$ is equal to $\xi^T(t)Q_1\xi(t)$. Finding the differential along the system's trajectory (3.1) for $V_1(\xi(t),t)$, we obtain

$$\begin{aligned}
\dot{V}_1(\xi(t),t) &= 2\xi^T(t)Q_1\dot{\xi}(t) + 2\dot{\xi}^T(t)Q_1\big[-\dot{\xi}(t)+y(t)\big] + 2\alpha\xi^T(t)Q_1\xi(t) - 2\alpha V_1(t) \\
&= 2\xi^T(t)P_1^T\begin{bmatrix}\dot{\xi}(t)\\0\\0\\0\end{bmatrix} + 2\dot{\xi}^T(t)Q_1\big[-\dot{\xi}(t)+y(t)\big] + 2\alpha\xi^T(t)Q_1\xi(t) - 2\alpha V_1(t) \\
&= 2\begin{bmatrix}\xi(t)\\ \xi(t-h(t)) \\ \int_{t-h(t)}^{t}\xi(s)ds \\ \xi(t-h_M) \\ y(t)\end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2^T & Q_3^T & Q_4^T \\ 0 & Q_5^T & Q_6^T & Q_7^T \\ 0 & Q_8^T & Q_9^T & Q_{10}^T \\ 0 & Q_{11}^T & Q_{12}^T & Q_{13}^T \\ 0 & Q_{14}^T & Q_{15}^T & Q_{16}^T \end{bmatrix}\begin{bmatrix}\dot{\xi}(t)\\0\\0\\0\end{bmatrix} \\
&\quad + 2\dot{\xi}^T(t)Q_1\big[-\dot{\xi}(t)+y(t)\big] + 2\alpha\xi^T(t)Q_1\xi(t) - 2\alpha V_1(t) \\
&= 2\xi^T(t)Q_1\dot{\xi}(t) + 2\Big[\xi^T(t)Q_2^T + \xi^T(t-h(t))Q_5^T \\
&\quad + \Big(\int_{t-h(t)}^{t}\xi(s)ds\Big)^TQ_8^T + \xi^T(t-h_M)Q_{11}^T + y^T(t)Q_{14}^T\Big] \\
&\quad \times \Big[-y(t) - A\xi(t) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t}h(\xi(s))ds + u(t)\Big] \\
&\quad + 2\Big[\xi^T(t)Q_3^T + \xi^T(t-h(t))Q_6^T + \Big(\int_{t-h(t)}^{t}\xi(s)ds\Big)^TQ_9^T + \xi^T(t-h_M)Q_{12}^T + y^T(t)Q_{15}^T\Big] \\
&\quad \times \Big[\xi(t) - \xi(t-h(t)) - \int_{t-h(t)}^{t}y(s)ds\Big] \\
&\quad + 2\Big[\xi^T(t)Q_4^T + \xi^T(t-h(t))Q_7^T + \Big(\int_{t-h(t)}^{t}\xi(s)ds\Big)^TQ_{10}^T + \xi^T(t-h_M)Q_{13}^T + y^T(t)Q_{16}^T\Big] \\
&\quad \times \Big[\xi(t-h(t)) - \xi(t-h_M) - \int_{t-h_M}^{t-h(t)}y(s)ds\Big] \\
&\quad + 2\dot{\xi}^T(t)Q_1\big[-\dot{\xi}(t)+y(t)\big] + 2\alpha\xi^T(t)Q_1\xi(t) - 2\alpha V_1(t).
\end{aligned} \tag{3.4}$$

The representation of $V_2(\xi(t),t)$'s time derivative is as follows:

$$\begin{aligned}\dot{V}_2(\xi(t),t) \leqslant\ & 2e^{2\alpha h_M}\Big(\xi(t)-A\int_{t-\delta}^{t}\dot{\xi}(s)ds\Big)^T P_2\Big(-A\xi(t)+Bf(\xi(t))+Ck(\xi(t-h(t)))\\ & +D\int_{t-\rho(t)}^{t}h(\xi(s))ds+u(t)\Big)-2\alpha V_2(t).\end{aligned} \tag{3.5}$$

By calculating $V_3(\xi(t),t)$ and taking its time derivative, we obtain

$$\begin{aligned}\dot{V}_3(\xi(t),t) =\ & \xi^T(t)P_3\xi^T(t)+\delta^2\xi^T(t)P_4\big)\xi(t)+e^{-2\alpha\delta}\xi^T(t-\delta)(-P_3)\xi(t-\delta)\\ & -\int_{t-\delta}^{t}\delta e^{2\alpha\delta}\xi^T(\delta)P_4\xi(\delta)d\theta-2\alpha V_3(t)\\ =\ & \xi^T(t)(P_3+\delta^2P_4)\xi^T(t)+e^{2\alpha\delta}\xi^T(t-\delta)(-P_3)\xi(t-\delta)-\delta^2e^{2\alpha\delta}\xi^T(\delta)P_4\xi(\delta)-2\alpha V_3(t).\end{aligned} \tag{3.6}$$

After $V_4(\xi(t),t)$ is differentiated, we obtain

$$\begin{aligned}\dot{V}_4(\xi(t),t) \leqslant\ & \xi^T(t)P_5\xi(t)-e^{-2\alpha h_M}\xi^T(t-h_M)P_5\xi(t-h_M)+\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}\\ & -e^{-2\alpha h_M}(1-h_d)\begin{bmatrix}\xi(t-h(t))\\ f(\xi(t-h(t)))\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t-h(t))\\ f(\xi(t-h(t)))\end{bmatrix}\\ & +\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}\\ & -e^{-2\alpha h_M}\begin{bmatrix}\xi(t-h_M)\\ f(\xi(t-h_M))\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t-h_M)\\ f(\xi(t-h_M))\end{bmatrix}-2\alpha V_4(t).\end{aligned} \tag{3.7}$$

Applying Lemmas 2.6 and 2.8, the expression for $V_5(\xi(t),t)$ is determined as

$$\begin{aligned}\dot{V}_5(\xi(t),t) =\ & h_M\int_{-h_M}^{0}\Big(\xi^T(t)P_6\xi(t)-e^{2\alpha s}\xi^T(t+s)P_6\xi(t+s)\Big)ds\\ & +\int_{-h_M}^{0}\Big(y^T(t)P_7y(t)-e^{2\alpha s}y^T(t+s)P_7y(t+s)\Big)ds-2\alpha V_5(t)\\ =\ & h_M\int_{-h_M}^{0}\xi^T(t)P_6\xi(t)ds-h_M\int_{-h_M}^{0}e^{2\alpha s}\xi^T(t+s)P_6\xi(t+s)ds\\ & +\int_{-h_M}^{0}y^T(t)P_7y(t)ds-\int_{-h_M}^{0}e^{2\alpha s}y^T(t+s)P_7y(t+s)ds-2\alpha V_5(t)\\ \leqslant\ & h_M^2\xi^T(t)P_6\xi(t)-h_Me^{-2\alpha h_M}\int_{t-h_M}^{t}\xi^T(s)P_6\xi(s)ds\\ & +h_My^T(t)P_7y(t)-e^{-2\alpha h_M}\int_{t-h_M}^{t}\dot{\xi}^T(s)P_7\dot{\xi}(s)ds-2\alpha V_5(t)\\ \leqslant\ & h_M^2\xi^T(t)P_6\xi(t)+h_My^T(t)P_7y(t)-h_Me^{-2\alpha h_M}\int_{t-h(t)}^{t}\xi^T(s)dsP_6\int_{t-h(t)}^{t}\xi(s)ds\\ & -h_Me^{-2\alpha h_M}\int_{t-h_M}^{t-h(t)}\xi^T(s)dsP_6\int_{t-h_M}^{t-h(t)}\xi(s)ds+e^{-2\alpha h_M}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}^T\\ & \times\begin{bmatrix}M_1+M_1^T & -M_1^T+M_2 & 0\\ -M_1+M_2^T & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ 0 & -M_1+M_2^T & -M_2-M_2^T\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}\end{aligned} \tag{3.8}$$

$$+h_M e^{-2\alpha h_M}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}^T\begin{bmatrix}M_3 & M_4 & 0\\ M_4^T & M_3+M_5 & M_4\\ 0 & M_4^T & M_5\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\end{bmatrix}-2\alpha V_5(t).$$

Calculating $\dot{V}_6(\xi(t),t)$, we have

$$\begin{aligned}V_6(\xi(t),t) &= \rho_M^2 f(\xi^T(t))P_8 f(\xi(t)) - \rho_M\int_{-\rho_M}^{0} e^{-2\alpha\rho_M} f(\xi^T(t+\rho_M))P_8 f(\xi(t+\rho_M))ds - 2\alpha V_6(t)\\ &= \rho_M^2 f(\xi^T(t))P_8 f(\xi(t)) - \rho_M^2 e^{-2\alpha\rho_M} f(\xi^T(t+\rho_M))P_8 f(\xi(t+\rho_M)) - 2\alpha V_6(t).\end{aligned} \tag{3.9}$$

It is based on Lemma 2.7 that we can derive the following:

$$\begin{aligned}\dot{V}_7(\xi(t),t) &= h_M\int_{-h_M}^{0}\Big(\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix} - e^{2\alpha s}\begin{bmatrix}\xi(t+s)\\ y(t+s)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(t+s)\\ y(t+s)\end{bmatrix}\Big)ds - 2\alpha V_7(t)\\
&= h_M^2\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix} - h_M\int_{t-h_M}^{t} e^{2\alpha s}\begin{bmatrix}\xi(s)\\ y(s)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(s)\\ y(s)\end{bmatrix}ds - 2\alpha V_7(t)\\
&\leqslant h_M^2\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix} - e^{-2\alpha h_M}h_M\int_{t-h_M}^{t}\begin{bmatrix}\xi(s)\\ y(s)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(s)\\ y(s)\end{bmatrix}ds - 2\alpha V_7(t)\\
&\leqslant h_M^2\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}^T\begin{bmatrix}R_7 & R_8\\ R_8^T & R_9\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix} + e^{-2\alpha h_M}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\\ \int_{t-h(t)}^{t}\xi(s)ds\\ \int_{t-h_M}^{t-h(t)}\xi(s)ds\end{bmatrix}^T\\
&\quad\times\begin{bmatrix}-R_9 & R_9 & 0 & -R_8^T & 0\\ R_9^T & -R_9-R_9^T & R_9 & R_8^T & -R_8^T\\ 0 & R_9^T & -R_9 & 0 & R_8^T\\ -R_8 & R_8 & 0 & -R_7 & 0\\ 0 & -R_8 & R_8 & 0 & -R_7\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-h(t))\\ \xi(t-h_M)\\ \int_{t-h(t)}^{t}\xi(s)ds\\ \int_{t-h_M}^{t-h(t)}\xi(s)ds\end{bmatrix} - 2\alpha V_7(t).\end{aligned} \tag{3.10}$$

It is based on Lemma 2.4 that we can derive the following:

$$\begin{aligned}V_8(\xi(t),t) &\leqslant e^{2\alpha h_M}\frac{h_M^2}{2}\Big(\frac{h_M^2}{2}\xi^T(t)P_9\xi(t) - e^{4\alpha h_M}\frac{h_M^2}{2}\int_{-h_M}^{0}\int_{t+\lambda}^{t}\xi^T(s)P_9\xi(s)dsd\lambda\Big)\\
&\quad + e^{2\alpha h_M}\frac{h_M^2}{2}\Big(\frac{h_M^2}{2}y^T(t)P_{10}y(t) - e^{4\alpha h_M}\frac{h_M^2}{2}\int_{-h_M}^{0}\int_{t+\lambda}^{t}\dot{\xi}^T(s)P_{10}\dot{\xi}(s)dsd\lambda\Big) - 2\alpha V_8(t)\\
&\leqslant e^{2\alpha h_M}\frac{h_M^4}{4}\xi^T(t)P_9\xi(t) - e^{-4\alpha h_M}\int_{-h_M}^{0}\int_{t+\lambda}^{t}\xi^T(s)dsd\lambda P_9\int_{-h_M}^{t}\int_{t+\lambda}^{t}\xi(s)dsd\lambda\\
&\quad + e^{2\alpha h_M}\frac{h_M^4}{4}y^T(t)P_{10}y(t)\\
&\quad - e^{-4\alpha h_M}\int_{-h_M}^{0}\int_{t+\lambda}^{t}\dot{\xi}^T(s)dsd\lambda P_{10}\int_{-h_M}^{t}\int_{t+\lambda}^{t}\dot{\xi}(s)dsd\lambda - 2\alpha V_8(t)\\
&= e^{2\alpha h_M}\frac{h_M^4}{4}\xi^T(t)P_9\xi(t) + e^{2\alpha h_M}\frac{h_M^4}{4}y^T(t)P_{10}y(t)\\
&\quad - e^{-4\alpha h_M}\int_{-h_M}^{0}\int_{t+\lambda}^{t}\xi^T(s)dsd\lambda P_9\int_{-h_M}^{t}\int_{t+\lambda}^{t}\xi(s)dsd\lambda\\
&\quad - e^{-4\alpha h_M}[h_M\xi^T(t) - \int_{t-h_M}^{t}\xi^T(\lambda)d\lambda]P_{10}[h_M\xi(t) - \int_{t-h_M}^{t}\xi(\lambda)d\lambda] - 2\alpha V_8(t).\end{aligned} \tag{3.11}$$

Applying Lemma 2.5 and calculating $V_9(\xi(t),t)$, we have

$$\begin{aligned}
V_9(\xi(t),t) \leqslant{}& e^{2\alpha h_M}\frac{h_M^3}{6}\Big(\frac{h_M^3}{6}\xi^T(t)P_{11}\xi(t) - e^{4\alpha h_M}\frac{h_M^3}{6}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi^T(\lambda)P_{11}\xi(\lambda)d\lambda duds\Big)\\
&+ e^{2\alpha h_M}\frac{h_M^3}{6}\Big(\frac{h_M^3}{6}y^T(t)P_{12}y(t)\\
&- e^{4\alpha h_M}\frac{h_M^3}{6}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}y^T(\lambda)P_{12}y(\lambda)d\lambda duds\Big) - 2\alpha V_9(t)\\
\leqslant{}& e^{2\alpha h_M}\frac{h_M^6}{36}\xi^T(t)P_{11}\xi(t) + e^{2\alpha h_M}\frac{h_M^6}{36}y^T(t)P_{12}y(t)\\
&- e^{4\alpha h_M}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi^T(\lambda)d\lambda dudsP_{11}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi(\lambda)d\lambda duds \qquad (3.12)\\
&- e^{4\alpha h_M}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}y^T(\lambda)d\lambda dudsP_{12}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}y(\lambda)d\lambda duds - 2\alpha V_9(t)\\
={}& e^{2\alpha h_M}\frac{h_M^6}{36}\xi^T(t)P_{11}\xi(t) + e^{2\alpha h_M}\frac{h_M^6}{36}y^T(t)P_{12}y(t)\\
&- e^{4\alpha h_M}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi^T(\lambda)d\lambda dudsP_{11}\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi(\lambda)d\lambda duds\\
&- e^{-4\alpha h_M}\Big[-\int_{t+s-h_M}^{t+s}s\xi^T(t+s)ds - \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\xi^T(t+s+u)duds\Big]P_{12}\\
&\times\Big[-\int_{t+s-h_M}^{t+s}s\xi(t+s)duds - \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\xi(t+s+u)duds\Big] - 2\alpha V_9(t).
\end{aligned}$$

Based on assumption (A1), it follows that $\left[f_i(\xi_i(t)) - F_i^-\xi_i(t)\right]\left[f_i(\xi_i(t)) - F_i^+\xi_i(t)\right] \leqslant 0$ for each instance of $i = 1,2,\dots,n$, which signifies

$$\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}F_i^-F_i^+e_ie_i^T & -\frac{F_i^-+F_i^+}{2}e_ie_i^T\\ -\frac{F_i^-+F_i^+}{2}e_ie_i^T & e_ie_i^T\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix} \leqslant 0,$$

in each case when $i = 1,2,\dots,n$. $Y_1 = \text{diag}\{y_1,y_2,\dots,y_n\} > 0$ is defined. We have

$$\sum_{i=1}^{n} y_i\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}F_i^-F_i^+e_ie_i^T & -\frac{F_i^-+F_i^+}{2}e_ie_i^T\\ -\frac{F_i^-+F_i^+}{2}e_ie_i^T & e_ie_i^T\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix} \leqslant 0,$$

which is equivalent to

$$\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T\begin{bmatrix}-F_1Y_1 & F_2Y_1\\ F_2Y_1 & -Y_1\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix} \geqslant 0. \qquad (3.13)$$

Similarly, based on Assumptions (A2) and (A3), we establish $Y_2 = \text{diag}(\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_n) > 0$ and $Y_3 = \text{diag}(\hat{y}_1,\hat{y}_2,\dots,\hat{y}_n) > 0$. This results in:

$$\begin{bmatrix}\xi(t-h(t))\\ k(\xi(t-h(t)))\end{bmatrix}^T\begin{bmatrix}-K_1Y_2 & K_2Y_2\\ K_2Y_2 & -Y_2\end{bmatrix}\begin{bmatrix}\xi(t-h(t))\\ k(\xi(t-h(t)))\end{bmatrix} \geqslant 0, \qquad (3.14)$$

$$\begin{bmatrix}\xi(t)\\ h(\xi(t))\end{bmatrix}^T\begin{bmatrix}-H_1Y_3 & H_2Y_3\\ H_2Y_3 & -Y_3\end{bmatrix}\begin{bmatrix}\xi(t)\\ h(\xi(t))\end{bmatrix} \geqslant 0. \qquad (3.15)$$

Applying the Leibniz-Newton formula, the equations hold true for any real constant matrices E_i, $i = 1, 2, \dots, 9$ with appropriate dimensions as follows:

$$\begin{aligned}
&2h_M\left[\xi^T(t)E_1 + \xi^T(t-h(t))E_2 + \int_{t-h(t)}^{t} y^T(s)dsE_3\right]\left[\xi(t) - \xi(t-h(t)) - \int_{t-h(t)}^{t} y(s)ds\right] = 0,\\
&2\rho_M\left[\xi^T(t-h(t))E_4 + \xi^T(t-h_M)E_5 + \int_{t-h_M}^{t-h(t)} y^T(s)dsE_6\right]\\
&\qquad\times\left[\xi(t-h(t)) - \xi(t-h_M) - \int_{t-h_M}^{t-h(t)} y(s)ds\right] = 0,\\
&2\delta\left[\xi^T(t)E_7 + \xi^T(t-\delta)E_8 + \int_{t-\delta}^{t} \dot{\xi}^T(s)dsE_9\right]\left[\xi(t) - \xi(t-\delta) - \int_{t-\delta}^{t} \dot{\xi}(s)ds\right] = 0.
\end{aligned} \tag{3.16}$$

From (3.1), we have

$$\begin{aligned}
&2\int_{t-\rho(t)}^{t} h^T(\xi(s))dsE_{10} \times \left[\dot{\xi}(t) + A\xi(t-\delta) - Bf(\xi(t)) - Ck(\xi(t-h(t)))\right.\\
&\quad\left. - D\int_{t-\rho(t)}^{t} h(\xi(s))ds - u(t)\right] = 0.
\end{aligned} \tag{3.17}$$

By referring to equations (3.4)-(3.17) along with (3.1), it becomes evident that

$$\dot{V}(t) + 2\alpha V(t) - 2z^T(t)u(t) \leqslant \eta^T(t)\sum\eta(t),$$

where

$$\begin{aligned}
\eta(t) = [&\xi(t), \dot{\xi}(t), \xi(t-h(t)), \int_{t-h(t)}^{t}\xi(s)ds, \xi(t-h_M), y(t), f(\xi(t)), k(\xi(t-h(t))), \int_{t-\rho(t)}^{t}\xi(s)ds,\\
&\int_{t-h(t)}^{t} y(s)ds, \int_{t-h_M}^{t-h(t)} y(s)ds, u(t), \int_{t-\delta}^{t}\dot{\xi}(s)ds, \xi(s), \xi(t-\delta), f(\xi(t-h_M)),\\
&\xi(t-h(t)), f(\xi(t-h(t))), \int_{t-h_M}^{t-h(t)}\xi_s ds, f(\xi(t+\rho_M)), \int_{t-h_M}^{t}\xi(\lambda)d\lambda, \int_{h_M}^{0}\int_{t+\lambda}^{t}\xi(s)dsd\lambda,\\
&\int_{t+s-h_M)}^{t+s} s\xi(t+s)ds, \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\xi(t+s+u)duds,\\
&\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s}\xi(\lambda)d\lambda duds, h(\xi(t))].
\end{aligned}$$

Given that $\sum$ is a negative definite matrix and satisfying conditions (3.2), we obtain

$$\dot{V}(t) + 2\alpha V(t) \leqslant 2z^T(t)u(t), \ \forall t \in \mathbb{R}^+.$$

Therefore, in accordance with Definition (2.1), the delayed NNs (3.1) is exponentially passive. This completes the proof. □

Remark 3.2. We have independently established Lemmas 2.6-2.8 to estimate the LKFs. To determine the solution values of the various system variables, we utilize the LMI editor and the MATLAB program. Providing upper bound values in multiple forms for comparison with historical data. Additionally, novel LKFs with single, double, triple, and quadruple integral terms are shown along with the application of integral inequality. These techniques are in opposition to the methods used in the studies that have been listed [9–12, 14, 15, 18, 25–27, 39, 45, 49, 50]. Our methodology involves the use of the Leibniz-Newton formula and the zero equation notion. Notably, compared to earlier findings, the methodology used in this investigation yields less conservative results [9–12, 14, 15, 18, 25–27, 39, 45, 49, 50].

3.2. Analyzing H_∞ performance and exponential stability in neural networks

In this section, we will first define the required criteria that guarantee the neural networks' H_∞ performance analysis and exponential stability. This entails taking into account the subsequent model:

$$\begin{aligned}
\dot{\xi}(t) &= -A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds + w(t),\\
z(t) &= C_1(\xi(t)) + C_2(\xi(t-h(t))) + C_3 w(t),\\
\xi(t) &= \phi(t), \quad t \in [-\tau_{max}, 0], \quad \tau_{max} = \max\{h_M, \rho_M\},
\end{aligned} \tag{3.18}$$

where $w(t) \in \mathbb{R}^n$ is the input vector which belongs to $L_2[0,\infty]$.

The following words and notations are introduced in this work and will be used later:

$$\bar{\sum} = \left[\bar{\sum}_{(i,j)}\right]_{25\times 25},$$

where $\bar{\sum}_{(i,j)} = \bar{\sum}^T_{(i,j)}$, $i, j = 1, 2, \ldots, 25$,

$$\begin{aligned}
\bar{\sum}_{(1,1)} &= -Q_2^T A - A^T Q_2 + Q_3^T + Q3 + \alpha Q_1 + Q_1^T \alpha - e^{2\alpha h_M} P_2 A - e^{2\alpha h_M} A^T P_2^T + P_3 + \delta^2 P_4\\
&\quad + P_5 + R_1 + R_4 + h_m^2 P_6 + (M_1 + M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M} M_3 + h_M^2 R_7 - e^{2\alpha h_M} R_9\\
&\quad + e^{2\alpha h_M} \frac{{h_M}^4}{4} P_9 - e^{-4\alpha h_M} h_M^2 P_{10} + e^{2\alpha h_M} \frac{h_M^6}{36} P_{11} - F_1 Y_1 - H_1 Y_3 + 2h_M E_1 + 2\delta E_7,\\
\bar{\sum}_{(1,2)} &= Q_1,\\
\bar{\sum}_{(1,3)} &= -A^T Q_5 - Q_3^T + Q_6 + Q_4^T + (M_2 - M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M} M_4^T\\
&\quad + e^{2\alpha h_M} R_9 - h_M E_1 + h_M E_2^T,\\
\bar{\sum}_{(1,4)} &= -A^T Q_8 + Q_9 - e^{2\alpha h_M} R_8^T,\\
\bar{\sum}_{(1,5)} &= -A^T Q_{11} + Q_{12} - Q_4^T,\\
\bar{\sum}_{(1,6)} &= -Q_2^T - A^T Q_{14} + Q_{15} + h_M^2 R_5,\\
\bar{\sum}_{(1,7)} &= Q_2^T B + e^{2\alpha h_M} B + R_2 + R_5 + F_2 Y_1,\\
\bar{\sum}_{(1,8)} &= Q_2^T C + e^{2\alpha h_M} C,\\
\bar{\sum}_{(1,9)} &= Q_2^T D + e^{2\alpha h_M} D,\\
\bar{\sum}_{(1,10)} &= -Q_3^T - h_M E_1 + h_M E_3^T,\\
\bar{\sum}_{(1,11)} &= -Q_4^T,\\
\bar{\sum}_{(1,12)} &= e^{2\alpha h_M} A^T A^T - \delta E_7 + \delta E_9^T,\\
\bar{\sum}_{(1,14)} &= \delta E_7 + \delta E_8^T,\\
\bar{\sum}_{(1,20)} &= e^{-4\alpha h_M} h_M,\\
\bar{\sum}_{(1,25)} &= H_2 Y_3,\\
\bar{\sum}_{(2,2)} &= -Q_1^T - Q_1,
\end{aligned}$$

$$
\begin{aligned}
\bar{\sum}_{(2,6)} &= Q_1,\\
\bar{\sum}_{(2,9)} &= E_{10}^T,\\
\bar{\sum}_{(3,3)} &= -Q_6^T - Q_6 + Q_7^T + Q_7 + (M_1 + M_1^T - M_2 - M_2^T)e^{-2\alpha h_M}\\
&\quad + h_M e^{-2\alpha h_M}(M_3 + M_5) - K_1 Y_2 - 2h_M E_2 + 2\rho_M E_4 + (-R_9 - R_9^T)e^{-2\alpha h_M},\\
\bar{\sum}_{(3,4)} &= -Q_9 + Q_{10} + e^{2\alpha h_M} R_8^T,\\
\bar{\sum}_{(3,5)} &= -Q_{12} - Q_7^T + Q_{13} + (-M_1^T + M_2)e^{-2\alpha h_M}\\
&\quad + h_M e^{-2\alpha h_M} M_4 + e^{2\alpha h_M} R_9 - \rho_M E_4 + \rho_M E_5^T,\\
\bar{\sum}_{(3,6)} &= -Q_5^T - Q_{15} + Q_{16},\\
\bar{\sum}_{(3,7)} &= Q_5^T B,\\
\bar{\sum}_{(3,8)} &= Q_5^T C + K_2 Y_2,\\
\bar{\sum}_{(3,9)} &= Q_5^T D,\\
\bar{\sum}_{(3,10)} &= -Q_6^T - h_M E_2 - h_M E_3^T,\\
\bar{\sum}_{(3,11)} &= -Q_7^T - \rho_M E_4 + \rho_M E_6^T,\\
\bar{\sum}_{(3,18)} &= -e^{2\alpha h_M} R_8^T,\\
\bar{\sum}_{(4,4)} &= -h_M e^{-2\alpha h_M} P_6 - e^{2\alpha h_M} R_7,\\
\bar{\sum}_{(4,5)} &= -Q_{10}^T,\\
\bar{\sum}_{(4,6)} &= -Q_8^T,\\
\bar{\sum}_{(4,7)} &= Q_8^T B,\\
\bar{\sum}_{(4,8)} &= Q_8^T C,\\
\bar{\sum}_{(4,9)} &= Q_8^T D,\\
\bar{\sum}_{(4,10)} &= -Q_9^T,\\
\bar{\sum}_{(4,11)} &= -Q_{10}^T,\\
\bar{\sum}_{(5,5)} &= -Q_{13} - Q_{13}^T - e^{-2\alpha h_M} P_5\\
&= -e^{-2\alpha h_M} R_4 + (-M_2 - M_2^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M} M_5 - e^{2\alpha h_M} R_9 - 2\rho_M E_5,\\
\bar{\sum}_{(5,6)} &= -Q_{11}^T - Q_{16},\\
\bar{\sum}_{(5,7)} &= Q_{11}^T B,\\
\bar{\sum}_{(5,8)} &= Q_{11}^T C,\\
\bar{\sum}_{(5,9)} &= Q_{11}^T D,
\end{aligned}
$$

$$\bar{\sum}_{(5,10)} = -Q_{12}^T,$$
$$\bar{\sum}_{(5,11)} = -Q_{13}^T - \rho_M E_5 - \rho_M E_6^T,$$
$$\bar{\sum}_{(5,15)} = -e^{-2\alpha h_M} R_5,$$
$$\Omega_{(5,18)} = e^{2\alpha h_M} R_8^T,$$
$$\bar{\sum}_{(6,6)} = -Q_{14}^T - Q_{14} + h_M P_7 + h_M^2 R_9 + e^{2\alpha h_M} \frac{h_M^4}{4} P_{10} + e^{2\alpha h_M} \frac{h_M^6}{36} P_{12},$$
$$\bar{\sum}_{(6,7)} = Q_{14}^T B,$$
$$\bar{\sum}_{(6,8)} = Q_{14}^T C,$$
$$\bar{\sum}_{(6,9)} = Q_{14}^T D,$$
$$\bar{\sum}_{(6,10)} = -Q_{15}^T,$$
$$\bar{\sum}_{(6,11)} = -Q_{16}^T,$$
$$\bar{\sum}_{(7,7)} = R_3 + R_6 + \rho_M^2 P_8 - Y_1,$$
$$\bar{\sum}_{(7,9)} = -E_{10}^T B^T,$$
$$\bar{\sum}_{(7,12)} = e^{2\alpha h_M} B^T A^T,$$
$$\bar{\sum}_{(8,8)} = -Y_2,$$
$$\bar{\sum}_{(8,9)} = -E_{10}^T C^T,$$
$$\bar{\sum}_{(8,12)} = e^{2\alpha h_M} C^T A^T,$$
$$\bar{\sum}_{(9,9)} = -E_{10} D - D^T E_{10}^T,$$
$$\bar{\sum}_{(9,12)} = e^{2\alpha h_M} D^T A^T,$$
$$\bar{\sum}_{(9,14)} = A E_{10},$$
$$\bar{\sum}_{(10,10)} = -2h_M E_3,$$
$$\bar{\sum}_{(11,11)} = -2\rho_M E_6,$$
$$\bar{\sum}_{(12,12)} = -2\delta E_9,$$
$$\bar{\sum}_{(12,14)} = -\delta E_8^T + \delta E_9,$$
$$\bar{\sum}_{(13,13)} = -\delta^2 e^{2\alpha\delta} P_4,$$
$$\bar{\sum}_{(14,14)} = -e^{2\alpha\delta} P_3 - 2\delta E_8,$$
$$\bar{\sum}_{(15,15)} = -e^{-2\alpha h_M} R_6,$$
$$\bar{\sum}_{(16,16)} = -e^{-2\alpha h_M}(1 - h_d) R_1,$$

$$\bar{\sum}_{(16,17)} = -e^{-2\alpha h_M}(1-h_d)R_2,$$
$$\bar{\sum}_{(17,17)} = -e^{-2\alpha h_M}(1-h_d)R_3,$$
$$\bar{\sum}_{(18,18)} = -h_M e^{-2\alpha h_M} - e^{2\alpha h_M}R_7,$$
$$\bar{\sum}_{(19,19)} = -e^{2\alpha \rho_M}\rho_M^2 P_8,$$
$$\bar{\sum}_{(20,20)} = -e^{-4\alpha h_M}P_{10},$$
$$\bar{\sum}_{(21,21)} = -e^{-4\alpha h_M}P_9,$$
$$\bar{\sum}_{(22,22)} = -e^{-4\alpha h_M}P_{12},$$
$$\bar{\sum}_{(22,23)} = -e^{-4\alpha h_M}P_{12},$$
$$\bar{\sum}_{(23,23)} = -e^{4\alpha h_M}P_{12},$$
$$\bar{\sum}_{(24,24)} = -e^{4\alpha h_M}P_{11},$$
$$\bar{\sum}_{(25,25)} = -Y_3,$$

and the others are 0.

$$\widehat{\sum} = \left[\hat{\sum}_{(i,j)}\right]_{26\times 26},$$

where $\hat{\sum}_{(i,j)} = \hat{\sum}_{(i,j)}^T = \bar{\sum}_{(i,j)}$, $i,j = 1,2,\dots,25$, except

$$\begin{aligned}\hat{\sum}_{(1,1)} = {} & -Q_2^T A - A^T Q_2 + Q_3^T + Q3 + \alpha Q_1 + Q_1^T\alpha - e^{2\alpha h_M}P_2 A - e^{2\alpha h_M}A^T P_2^T + P_3 \\ & + \delta^2 P_4 + P_5 + R_1 + R_4 + h_m^2 P_6 + (M_1 + M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M}M_3 + h_M^2 R_7 \\ & - e^{2\alpha h_M}R_9 + e^{2\alpha h_M}\frac{h_M{}^4}{4}P_9 - e^{-4\alpha h_M}h_M^2 P_{10} + e^{2\alpha h_M}\frac{h_M^6}{36}P_{11} - F_1 Y_1 - H_1 Y_3 \\ & + 2h_M E_1 + 2\delta E_7 + C_1^T C_1,\end{aligned}$$
$$\begin{aligned}\hat{\sum}_{(1,3)} = {} & -A^T Q_5 - Q_3^T + Q_6 + Q_4^T + (M_2 - M_1^T)e^{-2\alpha h_M} + h_M e^{-2\alpha h_M}M_4^T \\ & + e^{2\alpha h_M}R_9 - h_M E_1 + h_M E_2^T + C_1^T C_2,\end{aligned}$$
$$\hat{\sum}_{(1,26)} = C_1^T C_3,$$
$$\begin{aligned}\hat{\sum}_{(3,3)} = {} & -Q_6^T - Q_6 + Q_7^T + Q_7 + (M_1 + M_1^T - M_2 - M_2^T)e^{-2\alpha h_M} \\ & + h_M e^{-2\alpha h_M}(M_3 + M_5) - K_1 Y_2 - 2h_M E_2 + 2\rho_M E_4 + (-R_9 - R_9^T)e^{-2\alpha h_M} + C_2^T C_2,\end{aligned}$$
$$\hat{\sum}_{(3,26)} = C_2^T C_3,$$
$$\hat{\sum}_{(26,26)} = C_3^T C_3 - \gamma^2 I,$$

and the others are 0.

Theorem 3.3. *The system* (3.18) *is exponentially stable for a decay rate* $\alpha > 0$ *with the* H_∞ *performance* γ*, provided positive scalars* h_M*,* h_d*, and* ρ_M *with a prescribed scalar* $\delta > 0$*. If any suitable dimensional matrices* Q_m *and*

$m = 1, 2, \dots, 16$ *exist, such that the following LMIs hold, and there exist positive definite matrices* Q_1, R_7, R_9, Y_1, Y_2, Y_3 P_i, $i \in \{1, 2, \dots, 12\}$

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_7 & R_8 \\ * & R_9 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_7 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \quad \widehat{\sum} < 0. \tag{3.19}$$

Proof. We first show the exponential stability of system (3.18) under the constraints of the theorem. The system (3.18) with $w(t) = 0$ and $z(t) \equiv 0$ is obtained using the model transformation approach, yielding the following form:

$$\dot{\xi}(t) = y(t), \quad 0 = -y(t) - A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t - h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds.$$

Using the same format as in Theorem 3.1, construct a LKF candidate for the system (3.18). According to (3.4)-(3.17), it becomes evident that

$$\dot{V}(t) + 2\alpha V(t) \leqslant \bar{\xi}^T(t)\bar{\Sigma}\bar{\xi}(t),$$

where

$$\begin{aligned}
\bar{\xi}(t) = [&\xi(t), \dot{\xi}(t), \xi(t-h(t)), \int_{t-h(t)}^{t} \xi(s)ds, \xi(t-h_M), y(t), f(\xi(t)), k(\xi(t-h(t))), \int_{t-\rho(t)}^{t} \xi(s)ds, \\
&\int_{t-h(t)}^{t} y(s)ds, \int_{t-h_M}^{t-h(t)} y(s)ds, \int_{t-\delta}^{t} \dot{\xi}(s)ds, \xi(s), \xi(t-\delta), f(\xi(t-h_M)), \\
&\xi(t-h(t)), f(\xi(t-h(t))), \int_{t-h_M}^{t-h(t)} \xi_s ds, f(\xi(t+\rho_M)), \int_{t-h_M}^{t} \xi(\lambda)d\lambda, \int_{h_M}^{0}\int_{t+\lambda}^{t} \xi(s)dsd\lambda, \\
&\int_{t+s-h_M)}^{t+s} s\xi(t+s)ds, \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s} \xi(t+s+u)duds, \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s} \xi(\lambda)d\lambda duds, h(\xi(t))].
\end{aligned}$$

When both conditions (3.19) and $\bar{\Sigma} < 0$ are satisfied, we have

$$\dot{V}(t) + 2\alpha V(t) \leqslant 0, \quad \forall t \in \mathbb{R}^+.$$

So, we lead to

$$\|\varphi(t, \phi)\| \leqslant M\|\phi\|e^{-\kappa t}, \quad t \in \mathbb{R}^+,$$

where $M, \alpha \in \mathbb{R}^+$. This implies that the system described by equation (3.18) with $w(t) = 0$ and $z(t) \equiv 0$ is exponentially stable. Subsequently, we will analyze the H_∞ performance of the system (3.18) when subjected to zero initial conditions. We express the performance index J as follows:

$$J(t) = \int_0^t [z^T(s)z(s) - \chi^2 w^T(s)w(s)]ds, \quad t > 0. \tag{3.20}$$

Under the zero initial condition, (3.20) becomes

$$\begin{aligned}
J(t) &= \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]ds \\
&= \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]ds + \int_0^t \dot{V}(s)ds - V(t) + V(0) \\
&= \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s) + \dot{V}(s)]ds - V(t)
\end{aligned}$$

$$\leqslant \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s) + \dot{V}(s)]ds,$$

where $V(t)$ is define in (3.3). After some algebraic manipulations, we obtain

$$z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t) \leqslant \hat{\xi}^T(t)\widehat{\sum}\hat{\xi}(t), \tag{3.21}$$

where

$$\begin{aligned}\hat{\xi}(t) = [&\xi(t), \dot{\xi}(t), \xi(t-h(t)), \int_{t-h(t)}^t \xi(s)ds, \xi(t-h_M), y(t), f(\xi(t)), k(\xi(t-h(t))), \int_{t-\rho(t)}^t \xi(s)ds,\\ &\int_{t-h(t)}^t y(s)ds, \int_{t-h_M}^{t-h(t)} y(s)ds, \int_{t-\delta}^t \dot{\xi}(s)ds, \xi(s), \xi(t-\delta), f(\xi(t-h_M)),\\ &\xi(t-h(t)), f(\xi(t-h(t))), \int_{t-h_M}^{t-h(t)} \xi_s ds, f(\xi(t+\rho_M)), \int_{t-h_M}^t \xi(\lambda)d\lambda, \int_{h_M}^0 \int_{t+\lambda}^t \xi(s)dsd\lambda,\\ &\int_{t+s-h_M)}^{t+s} s\xi(t+s)ds, \int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s} \xi(t+s+u)duds,\\ &\int_{t+s-h_M}^{t+s}\int_{t+2s}^{t+s}\int_{t+s+u}^{t+s} \xi(\lambda)d\lambda duds, h(\xi(t)), w(t)].\end{aligned}$$

We can confirm that condition (3.21) ensures $z^T(s)z(s) - \chi^2 w^T(s)w(s) + \dot{V}(t) < 0$. Consequently, $J(t) < 0$, which implies $\|z(t)\|2 \leqslant \gamma\|w(t)\|2$ for any nonzero $w(t) \in L_2[0, \infty)$. Therefore, the NNs (3.18) is both exponentially stable and possesses an H_∞ performance index of γ. □

Remark 3.4. For NNs with leakage and mixed time-varying delays, we suggest new sufficient conditions to achieve exponential passivity, exponential stability, and H_∞ performance, covering a wide range of performances. Furthermore, none of these elements have previously been combined in any prior work.

4. Numerical examples

Example 4.1. Examine the subsequent neural networks, which exhibit discrete and distributed time-varying delays, taking into account the impact of leaking delay, as indicated by (3.1). We use Theorem 3.1 to study the exponential passivity of system (3.1). The following is a definition of system (3.1) parameters:

$$A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}0.2 & -0.1\\-0.5 & 0.1\end{bmatrix}, \quad C = \begin{bmatrix}-0.5 & 0\\-0.3 & -0.2\end{bmatrix},$$
$$D = \begin{bmatrix}0.15 & 0.1\\0 & -0.3\end{bmatrix}, \quad C_1 = \begin{bmatrix}0.5 & 0\\0 & 0.3\end{bmatrix}, \quad I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$
$$C_2 = C_3 = 0.1I, \quad F_1 = K_1 = H_1 = -0.4I, \quad F_2 = K_2 = H_2 = 0.4I,$$
$$f_i(x_i) = \tanh(x_i), \quad h_i(x_i) = k_i(x_i) = 0.3(|\, x_i + 1\,| - |\, x_i + 1\,|), \quad h(t) = |\cos(t)|,$$
$$\rho(t) = \cos^2(0.5t), \quad \phi(t) = \begin{bmatrix}0.5\\1\end{bmatrix}, \quad t \in [-1, 0].$$

It is easy to see that $\delta = 0.2$, $\alpha = 0.6$, $h_M = 0.3$, $h_d = 0.5$, $\rho_M = 0.7$. Utilizing MATLAB LMI Toolbox, we apply (3.2) from Theorem 3.1. This example demonstrates that the solutions to the LMIs are provided as follows:

$$P_1 = 10^7 \times \begin{bmatrix}9.6180 & 0\\0 & 9.6180\end{bmatrix}, \quad P_2 = 10^8 \times \begin{bmatrix}4.54281 & 0.0001\\0.0001 & 4.5424\end{bmatrix}, \quad P_3 = 10^7 \times \begin{bmatrix}7.1721 & -0.0004\\-0.0004 & 7.1693\end{bmatrix},$$
$$P_4 = 10^8 \times \begin{bmatrix}1.5752 & -0.0000\\-0.0000 & 1.5752\end{bmatrix}, \quad P_5 = 10^7 \times \begin{bmatrix}8.0735 & -0.0002\\-0.0002 & 8.0742\end{bmatrix}, \quad P_6 = 10^7 \times \begin{bmatrix}9.0261 & 0.0000\\0.0000 & 9.0261\end{bmatrix},$$

$$P_7 = \begin{bmatrix} 0.1111 & 0.0000 \\ 0.0000 & 0.1111 \end{bmatrix}, \quad P_8 = 10^7 \times \begin{bmatrix} 7.1568 & -0.0019 \\ -0.0019 & 7.1555 \end{bmatrix}, \quad P_9 = 10^5 \times \begin{bmatrix} 4.5587 & 0.0001 \\ 0.0001 & 4.5591 \end{bmatrix},$$

$$P_{10} = 10^{-4} \times \begin{bmatrix} 0.4727 & 0.0000 \\ 0.0000 & 0.4727 \end{bmatrix}, \quad P_{11} = 10^3 \times \begin{bmatrix} 4.5689 & 0.0001 \\ 0.0001 & 4.5693 \end{bmatrix}, \quad P_{12} = 10^{-6} \times \begin{bmatrix} 0.7053 & -0.0000 \\ -0.000 & 0.7053 \end{bmatrix},$$

$$R_1 = 10^8 \times \begin{bmatrix} 1.1438 & -0.0003 \\ -0.0003 & 1.1435 \end{bmatrix}, \quad R_2 = 10^7 \times \begin{bmatrix} -3.1382 & 0.0073 \\ 0.0073 & -3.1338 \end{bmatrix}, \quad R_3 = 10^7 \times \begin{bmatrix} 7.2209 & -0.0116 \\ -0.0116 & 7.2139 \end{bmatrix},$$

$$R_4 = 10^7 \times \begin{bmatrix} 8.9565 & 0.0006 \\ 0.0006 & 8.9573 \end{bmatrix}, \quad R_5 = 10^7 \times \begin{bmatrix} -2.6063 & -0.0008 \\ -0.0008 & -2.6100 \end{bmatrix}, \quad R_6 = 10^7 \times \begin{bmatrix} 6.5397 & -0.0011 \\ -0.0011 & 6.5402 \end{bmatrix},$$

$$R_7 = 10^7 \times \begin{bmatrix} 9.3803 & -0.0001 \\ -0.0001 & 9.3803 \end{bmatrix}, \quad R_8 = \begin{bmatrix} -2.6369 & -2.5794 \\ -2.5794 & -2.9858 \end{bmatrix}, \quad R_9 = \begin{bmatrix} 0.3702 & 0.0000 \\ 0.0000 & 0.3702 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0666 & 0.0000 \\ 0.0000 & 0.0666 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -1.4643 & -1.4019 \\ -1.4019 & -1.6810 \end{bmatrix}, \quad Q_3 = 10^5 \times \begin{bmatrix} -1.7335 & 0.0265 \\ 0.0265 & -1.8441 \end{bmatrix},$$

$$Q_4 = 10^6 \times \begin{bmatrix} 2.5350 & 0.0143 \\ 0.0143 & 2.5435 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 0.1240 & 0.0107 \\ 0.0107 & 0.1015 \end{bmatrix}, \quad Q_6 = 10^7 \times \begin{bmatrix} 6.3746 & 0.0004 \\ 0.0004 & 6.3745 \end{bmatrix},$$

$$Q_7 = 10^6 \times \begin{bmatrix} -4.3617 & -0.0062 \\ -0.0062 & -4.3545 \end{bmatrix}, \quad Q_8 = \begin{bmatrix} 0.0011 & -0.0001 \\ -0.0001 & 0.0000 \end{bmatrix}, \quad Q_9 = 10^3 \times \begin{bmatrix} 4.4295 & 1.4398 \\ 1.4398 & 1.0142 \end{bmatrix},$$

$$Q_{10} = 10^3 \times \begin{bmatrix} 5.4291 & 2.4485 \\ 2.4485 & 4.4017 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} -0.0162 & 0.0046 \\ 0.0046 & -0.0090 \end{bmatrix}, \quad Q_{12} = 10^6 \times \begin{bmatrix} -2.8329 & 0.0017 \\ 0.0017 & -2.8216 \end{bmatrix},$$

$$Q_{13} = 10^6 \times \begin{bmatrix} -1.1355 & -0.0091 \\ -0.0091 & -1.1322 \end{bmatrix}, \quad Q_{14} = \begin{bmatrix} 0.2000 & -0.0000 \\ -0.0000 & 0.2000 \end{bmatrix}, \quad Q_{15} = \begin{bmatrix} -0.3779 & -0.3327 \\ -0.3327 & -0.4159 \end{bmatrix},$$

$$Q_{16} = \begin{bmatrix} 0.0000 & -0.0354 \\ -0.0354 & -0.0125 \end{bmatrix}, \quad Y_1 = 10^8 \times \begin{bmatrix} 2.2036 & -0.0008 \\ -0.0008 & 2.2030 \end{bmatrix}, \quad Y_2 = 10^7 \times \begin{bmatrix} 8.1003 & 0.0005 \\ 0.0005 & 8.0906 \end{bmatrix},$$

$$Y_3 = 10^7 \times \begin{bmatrix} 8.7738 & 0.0069 \\ 0.0069 & 8.7670 \end{bmatrix}, \quad E_1 = 10^4 \times \begin{bmatrix} -5.2005 & 0.0796 \\ 0.0796 & -5.5323 \end{bmatrix}, \quad E_2 = 10^8 \times \begin{bmatrix} -2.0288 & 0.0002 \\ 0.0002 & -2.0292 \end{bmatrix},$$

$$E_3 = 10^7 \times \begin{bmatrix} 8.1815 & 0.0015 \\ 0.0015 & 8.1806 \end{bmatrix}, \quad E_4 = 10^6 \times \begin{bmatrix} -3.0532 & -0.0044 \\ -0.0044 & -3.0482 \end{bmatrix}, \quad E_5 = 10^5 \times \begin{bmatrix} -7.9485 & -0.0638 \\ -0.0638 & -7.9257 \end{bmatrix},$$

$$E_6 = 10^7 \times \begin{bmatrix} 4.1405 & 0.0005 \\ 0.0005 & 4.1419 \end{bmatrix}, \quad E_7 = 10^7 \times \begin{bmatrix} 2.0505 & 0.0020 \\ 0.0020 & 2.0633 \end{bmatrix}, \quad E_8 = 10^7 \times \begin{bmatrix} 5.0183 & 0.0007 \\ 0.0007 & 5.0244 \end{bmatrix},$$

$$E_9 = 10^8 \times \begin{bmatrix} 1.1552 & 0.0001 \\ 0.001 & 1.1554 \end{bmatrix}, \quad E_{10} = \begin{bmatrix} 0.0344 & 0.0110 \\ 0.0110 & -0.0677 \end{bmatrix}, \quad K = 10^7 \times \begin{bmatrix} 9.6049 & -0.0026 \\ -0.0026 & 9.6054 \end{bmatrix}.$$

According to the aforementioned findings, every need listed in Theorem 3.1 has been met. As a result, the system (3.1) with the previously specified parameters is exponentially passive.

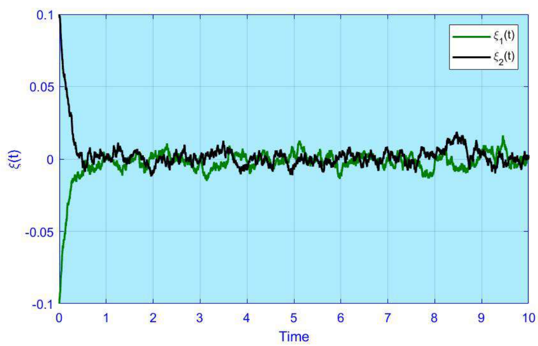

Figure 1: The state responses of $\xi_1(t)$ and $\xi_2(t)$ in Example 4.1.

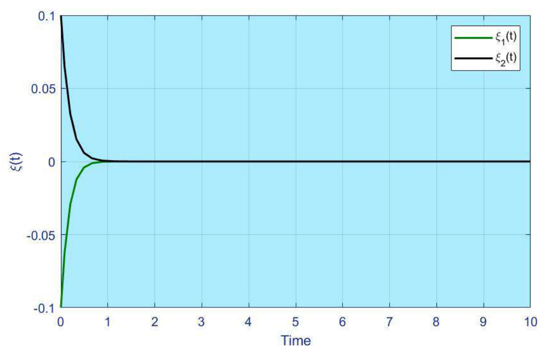

Figure 2: The state responses of $\xi_1(t)$ and $\xi_2(t)$ with $u(t) = 0$ in Example 4.1.

Example 4.2. Take into consideration the following NNs under the influence of leaking delay (3.18), which have discrete and distributed time-varying delays. Using Theorem 3.3, we examine the H_∞ performance and exponential stability of the system (3.18). The following is a definition of system (3.18) parameters:

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.21 & -0.32 \\ 0.13 & 0.22 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_1 = K_1 = H_1 = -0.4I,$$

$$F_2 = K_2 = H_2 = 0.4I, \quad f_i(x_i) = \tanh(x_i), \quad h_i(x_i) = k_i(x_i) = 0.2(|\, x_i + 1 \,| - |\, x_i - 1 \,|)$$

$$h(t) = |\sin(t)|, \quad \rho(t) = \sin^2(0.3t), \quad \phi(t) = \begin{bmatrix} 0.3 \\ 1 \end{bmatrix}, \ t \in [-1, 0].$$

It is easy to see that $\gamma = 0.9$, $\alpha = 0.8$, $\delta = 0.5$, $h_M = 0.2$, $h_d = 0.5$, $\rho_M = 0.3$. Utilizing MATLAB LMI Toolbox, we apply (3.19) from Theorem 3.3. This example demonstrates that the solutions to the LMIs are provided as follows:

$$P_1 = 10^4 \times \begin{bmatrix} 5.0666 & 0 \\ 0 & 5.0666 \end{bmatrix}, \quad P_2 = 10^5 \times \begin{bmatrix} 4.1951 & 0.0976 \\ 0.0976 & 5.7630 \end{bmatrix}, \quad P_3 = 10^4 \times \begin{bmatrix} 2.7855 & -0.0054 \\ -0.0054 & 2.5702 \end{bmatrix},$$

$$P_4 = 10^4 \times \begin{bmatrix} 6.2955 & -0.0006 \\ -0.0006 & 6.0508 \end{bmatrix}, \quad P_5 = 10^4 \times \begin{bmatrix} 4.1727 & -0.0926 \\ -0.0926 & 3.7140 \end{bmatrix}, \quad P_6 = 10^4 \times \begin{bmatrix} 4.8769 & -0.0002 \\ -0.0002 & 4.8300 \end{bmatrix},$$

$$P_7 = 10^4 \times \begin{bmatrix} 1.3778 & -1.1203 \\ -1.1203 & 2.9914 \end{bmatrix}, \quad P_8 = 10^4 \times \begin{bmatrix} 1.6828 & -0.1001 \\ -0.1001 & 1.4171 \end{bmatrix}, \quad P_9 = \begin{bmatrix} 138.0156 & -0.2003 \\ -0.2003 & 62.1563 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 1.8934 & -2.6099 \\ -2.6099 & 5.6519 \end{bmatrix}, \quad P_{11} = \times \begin{bmatrix} 0.6161 & -0.0009 \\ -0.0009 & 0.2778 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 0.0123 & -0.0172 \\ -0.0172 & 0.0372 \end{bmatrix},$$

$$R_1 = 10^5 \times \begin{bmatrix} 1.4861 & 0.1040 \\ 0.1040 & 1.2844 \end{bmatrix}, \quad R_2 = 10^4 \times \begin{bmatrix} -1.8766 & -0.1141 \\ -0.1141 & -1.5861 \end{bmatrix}, \quad R_3 = 10^3 \times \begin{bmatrix} 4.1958 & -0.0037 \\ -0.0037 & 3.4484 \end{bmatrix},$$

$$R_4 = 10^4 \times \begin{bmatrix} 8.5778 & 0.5553 \\ 0.5553 & 7.8499 \end{bmatrix}, \quad R_5 = 10^4 \times \begin{bmatrix} -1.2086 & -0.0473 \\ -0.0473 & -1.0741 \end{bmatrix}, \quad R_6 = 10^3 \times \begin{bmatrix} 3.3310 & -0.0787 \\ -0.0787 & 2.8012 \end{bmatrix},$$

$$R_7 = 10^4 \times \begin{bmatrix} 4.1388 & 0.0001 \\ 0.0001 & 4.1318 \end{bmatrix}, \quad R_8 = \begin{bmatrix} 587.0411 & -469.3463 \\ -469.3463 & -860.3650 \end{bmatrix}, \quad R_9 = 10^4 \times \begin{bmatrix} 3.6083 & -0.6550 \\ -0.6550 & 4.0146 \end{bmatrix},$$

$$Q_1 = 10^4 \times \begin{bmatrix} 0.6673 & -0.5148 \\ -0.5148 & 1.0844 \end{bmatrix}, \quad Q_2 = 10^4 \times \begin{bmatrix} -2.6939 & 0.9243 \\ 0.9243 & -4.5298 \end{bmatrix}, \quad Q_3 = 10^4 \times \begin{bmatrix} 0.1935 & -1.1885 \\ -1.1885 & 0.8690 \end{bmatrix},$$

$$Q_4 = 10^4 \times \begin{bmatrix} 1.3561 & 0.8556 \\ 0.8556 & 1.5624 \end{bmatrix}, \quad Q_5 = 10^4 \times \begin{bmatrix} 0.3650 & -2.7472 \\ -2.7472 & -8.5309 \end{bmatrix}, \quad Q_6 = 10^4 \times \begin{bmatrix} 3.4542 & 0.3307 \\ 0.3307 & 4.1608 \end{bmatrix},$$

$$Q_7 = 10^5 \times \begin{bmatrix} -0.8072 & -0.2332 \\ -0.2332 & -1.9184 \end{bmatrix}, \quad Q_8 = \begin{bmatrix} -169.8442 & 56.3088 \\ 56.3088 & 383.6442 \end{bmatrix}, \quad Q_9 = \begin{bmatrix} 455.4791 & -361.0859 \\ -361.0859 & -768.2308 \end{bmatrix},$$

$$Q_{10} = \begin{bmatrix} -164.0528 & 54.2174 \\ 54.2174 & -46.4057 \end{bmatrix}, \quad Q_{11} = 10^3 \times \begin{bmatrix} -3.3911 & 2.4082 \\ 2.4082 & -6.4005 \end{bmatrix}, \quad Q_{12} = 10^4 \times \begin{bmatrix} -0.1021 & -0.3515 \\ -0.3515 & -1.6332 \end{bmatrix},$$

$$Q_{13} = 10^4 \times \begin{bmatrix} -2.2828 & -0.5856 \\ 0.5856 & -1.0771 \end{bmatrix}, \quad Q_{14} = 10^4 \times \begin{bmatrix} 2.0712 & -1.9203 \\ -1.9203 & 7.2283 \end{bmatrix}, \quad Q_{15} = 10^3 \times \begin{bmatrix} 0.5837 & 7.5168 \\ 7.5168 & 9.3752 \end{bmatrix},$$

$$Q_{16} = 10^4 \times \begin{bmatrix} 0.1826 & -0.8527 \\ -0.8527 & -1.8049 \end{bmatrix}, \quad Y_1 = 10^4 \times \begin{bmatrix} 1.3695 & 0.2237 \\ 0.2237 & 1.5152 \end{bmatrix}, \quad Y_2 = 10^4 \times \begin{bmatrix} 0.7577 & 0.2949 \\ 0.2949 & 2.2772 \end{bmatrix},$$

$$Y_3 = 10^3 \times \begin{bmatrix} 4.5529 & -0.0023 \\ -0.0023 & 3.6899 \end{bmatrix}, \quad E_1 = 10^3 \times \begin{bmatrix} 0.3869 & -2.3770 \\ -2.3770 & 1.7379 \end{bmatrix}, \quad E_2 = 10^5 \times \begin{bmatrix} -2.4136 & -0.8051 \\ 0.8051 & -7.7578 \end{bmatrix},$$

$$E_3 = 10^5 \times \begin{bmatrix} 1.0134 & 0.0285 \\ 0.0285 & 1.9918 \end{bmatrix}, \quad E_4 = 10^6 \times \begin{bmatrix} 0.3482 & 0.1148 \\ 0.1148 & 1.1186 \end{bmatrix}, \quad E_5 = 10^3 \times \begin{bmatrix} -6.8484 & 1.7569 \\ 1.7569 & -3.2313 \end{bmatrix},$$

$$E_6 = 10^5 \times \begin{bmatrix} 0.8929 & 0.0762 \\ 0.0762 & 1.6572 \end{bmatrix}, \quad E_7 = 10^4 \times \begin{bmatrix} 2.1286 & -0.0305 \\ -0.0305 & 1.4598 \end{bmatrix}, \quad E_8 = 10^4 \times \begin{bmatrix} 1.2178 & 0.0066 \\ 0.0066 & 1.1964 \end{bmatrix},$$

$$E_9 = 10^4 \times \begin{bmatrix} 3.7466 & -0.0017 \\ -0.0017 & 3.4059 \end{bmatrix}, \quad E_{10} = 10^3 \times \begin{bmatrix} 1.8131 & -2.2532 \\ -2.2532 & 0.2958 \end{bmatrix}, \quad K = 10^4 \times \begin{bmatrix} 6.2281 & 1.8862 \\ 1.8862 & 8.9686 \end{bmatrix}.$$

The results above demonstrate that all of the requirements listed in Theorem 3.3 have been met, resulting in an exponentially stable system with H_∞ performance index γ for system (3.18) given the specified parameters.

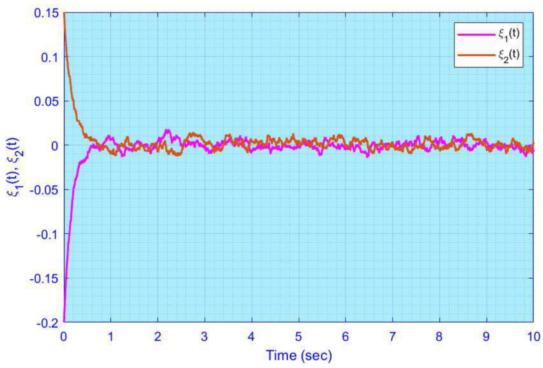

Figure 3: The state responses of $\xi_1(t)$ and $\xi_2(t)$ in Example 4.2.

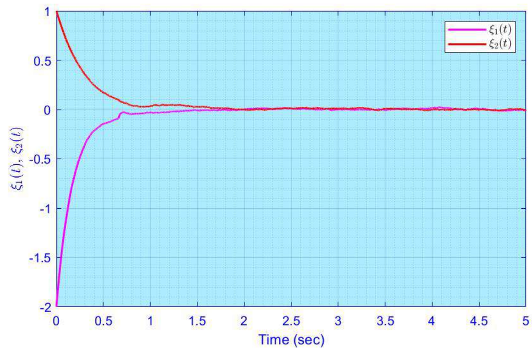

Figure 4: The state responses of $\xi_1(t)$ and $\xi_2(t)$ with $w(t) = 0$ in Example 4.2.

Example 4.3. Our attention is directed toward system (3.1), which refers to NNs subjected to leakage delay and having discrete and distributed time-varying delays,

$$\begin{cases} \dot{\xi}(t) = -A\xi(t-\delta) + Bf(\xi(t)) + Ck(\xi(t-h(t))) + D\int_{t-\rho(t)}^{t} h(\xi(s))ds + u(t), \\ z(t) = C_1(\xi(t)) + C_2(\xi(t-h(t))) + C_3u(t), \\ \xi(t) = \phi(t), \quad t \in [-\tau_{max}, 0], \quad \tau_{max} = \max\{h_M, \rho_M\}, \end{cases}$$

with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad f_i(x_i) = \tanh(x_i), \quad h_i(x_i) = k_i(x_i) = 0.2(|x_i+1| - |x_i-1|),$$

for $h(t) = 0.2 + \frac{\sin^2(t)}{2}$, $\rho(t) = 0.4 + \frac{|\cos(t)|}{4}$. In this instance, we are concentrating on the system's exponential passivity (3.1). Table 1 provides the permitted upper bound h_M that has been determined.

Table 1: For fixed $\delta = 0.5$, $\rho_M = 0.7$, and varying h_d and α, the delay upper bound h_M was calculated (see Example 4.3).

h_d	$\alpha = 0.20$	$\alpha = 0.40$	$\alpha = 0.60$	$\alpha = 0.80$	$\alpha = 1.00$	$\alpha = 1.10$	$\alpha = 1.13$
0.10	1.3117	1.4370	1.9636	1.4642	1.1579	1.0236	1.0121
0.30	1.3066	1.4390	1.9650	1.4641	1.1621	1.0432	1.0201
0.50	1.5030	1.4332	1.9629	1.4619	1.1600	1.0610	1.0068
0.70	1.6010	1.4329	1.9622	1.4619	1.1641	1.0501	1.0019
0.90	1.5010	1.4361	1.9629	1.4690	1.1601	1.0199	1.0129
0.95	1.4304	1.4333	1.0999	1.4629	1.1710	1.0570	1.0222
0.99	1.5001	1.4410	1.9634	1.4626	1.1633	1.0403	1.10173

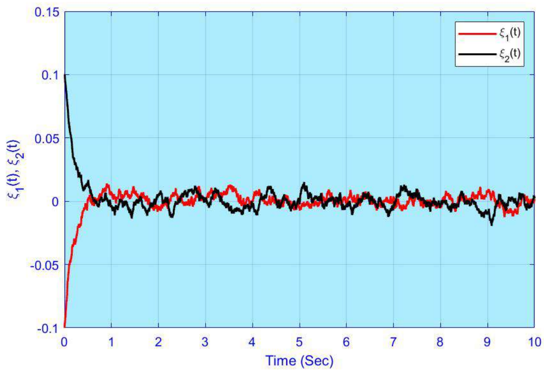

Figure 5: The state responses of $\xi_1(t)$ and $\xi_2(t)$ in Example 4.3.

Example 4.4. Examine the delayed NNs (3.1), which are NNs with time-varying delay, where $u(t) = z(t) = 0$ and $\delta = D = 0$,

$$\begin{cases} \dot{\xi}(t) = -A\xi(t) + Bf(\xi(t)) + Ck(\xi(t-h(t))), \\ \xi(t) = \phi(t), \quad t \in [-h_M, 0], \end{cases} \tag{4.1}$$

with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad f_i(x_i) = \tanh(x_i), \quad h_i(x_i) = 0.4(|\, x_i + 1 \,| - |\, x_i - 1 \,|).$$

A comparison of the exponential convergence rates of system (4.1) utilizing different approaches is shown in Table 2. It is evident that our results surpass those from [11, 14, 15, 27, 39, 45]. Table 3 shows a comparison of the allowable upper bound of h_M for various α and $h_d = 0$ in Example 4.4. It can be seen that our results outperform those found in [9, 18, 25–27].

Table 2: Permitted exponential convergence rate α in Example 4.4 for different h_d and $h_M = 1$.

Method	$h_d = 0.8$	$h_d = 0.9$	Unknown h_d
[45]	0.8643	0.8344	0.8169
[14]	0.8696	0.8354	0.8169
[15]	0.8784	0.8484	0.8217
[11]	0.8841	0.8570	0.8260
[39]	1.0214	1.2010	1.1011
[27]	4.1450	4.1450	4.1420
Theorem 3.3	7.1040	6.9601	5.2029

Table 3: Acceptable upper bound of h_M in Example 4.4 for different α and $h_d = 0$.

Method	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$
[26]	2.5900	0.9700	0.3500
[9]	2.8200	1.1800	0.5400
[25]	2.9000	1.3200	0.7200
[18]	2.9400	1.3500	0.7200
[27]	6.6021	3.6051	2.5198
Theorem 3.3	19.6010	8.9010	6.0020

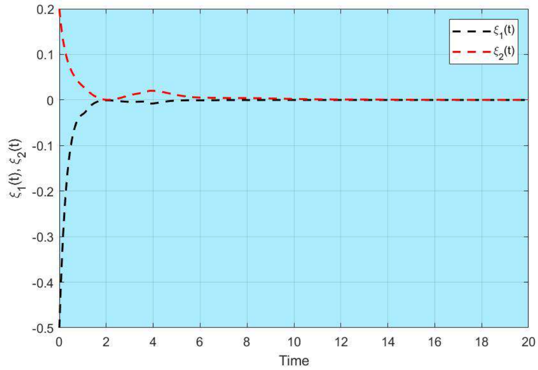

Figure 6: The state responses of $\xi_1(t)$ and $\xi_2(t)$ in Example 4.4.

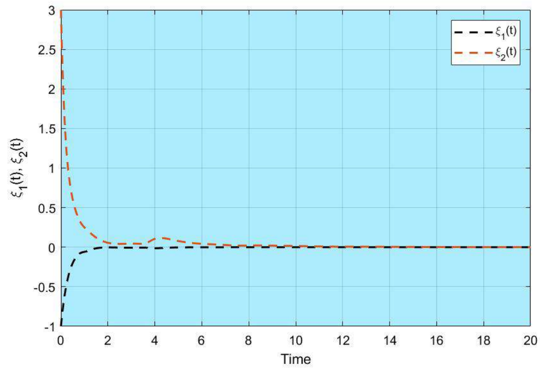

Figure 7: The state responses of $\xi_1(t)$ and $\xi_2(t)$ in Example 4.5.

Table 4: Acceptable maximum h_M for different h_d and $\alpha = 0$ in Example 4.5.

Method	$h_d = 0.8$	$h_d = 0.9$
[12]	2.3534	1.6050
[49]	3.2160	2.1995
[10]	2.8980	1.9562
[50]	3.1409	1.6375
[11]	3.7756	2.2201
Theorem 3.3	4.5546	5.6544

Example 4.5. Consider exponential stability for the delayed NNs (4.1) with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad f_i(x_i) = \tanh(x_i), \quad h_i(x_i) = 0.2(| x_i + 2 | - | x_i - 2 |).$$

A comparison of the permitted upper bound h_M for different h_d and $\alpha = 0$ in Example 4.5 is shown in Table 4. Our results clearly outperform the ones reported in [10–12, 49, 50].

5. Conclusions

The issue of exponential passivity and H_∞ performances of NNs with mixed time-varying delays under the influence of leakage delay is presented in this study. Novel augmented LKFs with single, double, triple, and quadruple integral terms are created based on the Lyapunov stability theory. In addition, the Leibniz-Newton formula, mixed model transformation, zero equation application, and different inequalities are used to obtain less conservative findings for certain examples of NNs with mixed delays. Furthermore, this work offers novel delay-dependent requirements for H_∞ performances, exponential passivity, and exponential stability for NNs with leakage and mixed time-varying delays, covering a wide range of performance characteristics. In conclusion, numerical examples are provided to demonstrate the efficacy of the findings and surpass some previous findings in the literature. This paper can be applied to fuzzy control systems, T-S fuzzy NNs [37], uncertain NNs [48], neutral-type NNs [24], and other dynamical systems in future work. Furthermore, it is evident that the technique offered can be expanded to include Takagi-Sugeno fuzzy non-homogeneous Markovian jump systems [35] in neural network systems, coupled reaction-diffusion synchronization [44], and coupled memristive synchronization [13].

Acknowledgment

This research project is supported by Science Research and Innovation Fund, agreement No. FF67/P1-004.

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