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The stability of bi-derivations and bihomomorphisms in Banach algebras

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Abstract

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of bi-derivations and bihomomorphisms in Banach algebras, associated with the bi-additive functional inequality

$$\begin{aligned}&\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\|\\&\leqslant \|s\left(2f\left(x+y,z-w\right)+2f\left(x-y,z+w\right)-4f(x,z)+4f(y,w)\right)\|,\end{aligned} \tag{1}$$

where s is a fixed nonzero complex number with $|s|<1$.

Keywords: Hyers-Ulam stability, biderivation on Banach algebra, bihomomorphism in Banach algebra, fixed point method, bi-additive functional inequality.

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1. Introduction

Ulam [55] initially proposed the concept of a stability problem for a functional equation concerning the stability of group homomorphisms in 1940 at the University of Wisconsin's Mathematics Club, in relation to the stability of group homomorphisms. Hyers [22] presented a partial response to Ulam's question for additive groups in the next year, assuming that groups are Banach spaces. The direct method, which was introduced by Hyers in [22], has been used to investigate the stability of numerous functional equations. Hyers' Theorem was generalized by Aoki [5] for additive mappings and by Rassias [48] for linear mappings by considering an unbounded Cauchy difference.

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Theorem 1.1 ([48]). *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$*, where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $p < 1$*. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$*. If* $p < 0$*, then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$*. Also, if for each* $x \in E$ *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$*, then* L *is* $\mathbb{R}$*-linear.*

A generalization of the Rassias theorem was obtained by Găvruta [20] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Rassias [49] during the 27$^{\text{th}}$ International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. Gajda [19] following the same approach as in Rassias [48], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [19], as well as by Rassias and Šemrl [51] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [19], as well as of Rassias and Šemrl [51] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [20], Jung [27], who among others studied the Hyers-Ulam stability of functional equations.

Park [41, 42, 44] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations, functional inequalities and differential equations have been extensively investigated by a number of authors (see [1–4, 7, 8, 12–14, 16–18, 21, 23, 24, 26, 28–35, 39, 46, 50, 52–54, 57]).

In 1996, Isac and Rassias [25] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations and differential equations have been extensively investigated by a number of authors (see [9–11, 15, 43, 47]).

Maksa [37, 38] introduced and investigated bi-derivations and symmetric bi-derivations on rings. Öztürk and Sapanci [40], Vukman [56], and Yazarli [58] investigated some properties of symmetric bi-derivations on rings.

Definition 1.2 ([37, 38]). Let A be a ring. A bi-additive mapping $D : A \times A \to A$ is called a *symmetric bi-derivation* on A if D satisfies

$$D(xy, z) = D(x, z)y + xD(y, z), \quad D(x, y) = D(y, x),$$

for all $x, y, z \in A$.

In this paper, we introduce bi-derivations and bihomomorphisms in Banach algebras.

Let A be a complex Banach algebra. Suppose that a $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is a derivation in each variable, i.e.,

$$D(xy, z) = D(x, z)y + xD(y, z), \quad D(x, zw) = D(x, z)w + zD(x, w),$$

for all $x, y, z, w \in A$. It is easy to show that

$$D(xy, zw) = D(x, z)wy + zD(x, w)y + xD(y, z)w + xzD(y, w)$$

for all $x, y, z, w \in A$.

Definition 1.3. Let A be a complex Banach algebra. A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a *biderivation* on A if D satisfies

$$D(xy, zw) = D(x,z)wy + zD(x,w)y + xD(y,z)w + xzD(y,w)$$

for all $x, y, z, w \in A$.

Definition 1.4. Let A and B be complex Banach algebras. A $\mathbb{C}$-bi-linear mapping $H : A \times A \to B$ is called a *bihomomorphism* if H satisfies

$$H(xy, zw) = H(x,z)H(y,w)$$

for all $x, y, z, w \in A$.

This paper is organized as follows. In Sections 2 and 3, we prove the Hyers-Ulam stability of bi-derivations and bi-homomorphisms in Banach algebras associated with the bi-additive s-functional inequality (1) by using the direct method. In Sections 4 and 5, we prove the Hyers-Ulam stability of bi-derivations and bi-homomorphisms in Banach algebras associated with the bi-additive s-functional inequality (1) by using the fixed point method.

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Let A and B be complex Banach algebras. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Hyers-Ulam stability of bi-derivations on Banach algebras: direct method

We investigate the bi-additive s-functional inequality (1) in complex normed spaces.

Lemma 2.1 ([36, Lemma 2.1])**.** *If a mapping $f : X^2 \to Y$ satisfies $f(0,z) = f(x,0) = 0$ and*

$$\begin{aligned}&\|f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\|\\ &\quad \leqslant \|s\left(2f\left(x+y,z-w\right) + 2f\left(x-y,z+w\right) - 4f(x,z) + 4f(y,w)\right)\|\end{aligned}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

In [45], Park proved the Hyers-Ulam stability of the bi-additive s-functional inequality (1) in complex Banach spaces.

Theorem 2.2 ([45, Theorem 2.2])**.** *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.1}$$

for all $x, y \in X$ and let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\begin{aligned}&\|f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\|\\ &\quad \leqslant \|s\left(2f\left(x+y,z-w\right) + 2f\left(x-y,z+w\right) - 4f(x,z) + 4f(y,w)\right)\| + \varphi(x,y)\varphi(z,w)\end{aligned} \tag{2.2}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z) - P(x,z)\| \leqslant \frac{1}{4(1-|s|)}\Psi(x,x)\varphi(z,0) \tag{2.3}$$

for all $x, z \in X$, where

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x, y \in X$.

Theorem 2.3 ([45, Theorem 2.3]). *Let* $\varphi : X^2 \to [0,\infty)$ *be a function satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \tag{2.4}$$

for all $x, y \in X$ *and let* $f : X^2 \to Y$ *be a mapping satisfying* (2.2) *and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in X$*. Then there exists a unique bi-additive mapping* $P : X^2 \to Y$ *such that*

$$\|f(x,z) - P(x,z)\| \leqslant \frac{1}{2(1-|s|)}\Psi(x,x)\varphi(z,0) \tag{2.5}$$

for all $x, z \in X$.

Now, we investigate bi-derivations on complex Banach algebras associated with the bi-additive s-functional inequalitiy (1).

Lemma 2.4 ([6, Lemma 2.1]). *Let* $f : X^2 \to Y$ *be a bi-additive mapping such that* $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ *for all* $x, z \in X$ *and* $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} \, : \, |\nu| = 1\}$*. Then* f *is* $\mathbb{C}$*-bi-linear.*

Theorem 2.5. *Let* $\varphi : A^2 \to [0,\infty)$ *be a function satisfying* (2.1) *with* $X = A$ *and* $f : A^2 \to A$ *be a mapping satisfying* $f(x,0) = f(0,z) = 0$ *and*

$$\begin{aligned} &\|f(\lambda(x+y),\mu(z+w)) + f(\lambda(x+y),\mu(z-w)) + f(\lambda(x-y),\mu(z+w)) \\ &\quad + f(\lambda(x-y),\mu(z-w)) - 4\lambda\mu f(x,z)\| \\ &\leqslant \|s\left(2f\left(x+y,z-w\right) + 2f\left(x-y,z+w\right) - 4f(x,z) + 4f(y,w)\right)\| + \varphi(x,y)\varphi(z,w) \end{aligned} \tag{2.6}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (2.3) *with* $X = A$*, where* P *is replaced by* D *in* (2.3)*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* $f(2x,z) = 2f(x,z)$ *and*

$$\|f(xy,zw) - f(x,z)wy - zf(x,w)y - xf(y,z)w - xzf(y,w)\| \leqslant \varphi(x,y)\varphi(z,w) \tag{2.7}$$

for all $x, y, z, w \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-derivation.*

Proof. Let $\lambda = \mu = 1$ in (2.6). By Theorem 2.2, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (2.3) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$. Letting $y = w = 0$ in (2.6), we get $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x, z \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. By Lemma 2.4, the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-bi-linear. If $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then we can easily show that $D(x,z) = f(x,z)$ for all $x, z \in A$. It follows from (2.7) that

$$\begin{aligned} &\|D(xy,zw) - D(x,z)wy - zD(x,w)y - xD(y,z)w - xzD(y,w)\| \\ &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}, zw\right) - f\left(\frac{x}{2^n}, z\right)\frac{wy}{2^n} - zf\left(\frac{x}{2^n}, w\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}, z\right) w - \frac{xz}{2^n} f\left(\frac{y}{2^n}, w\right) \right\| \\ &\leqslant \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\varphi(z,w) = 0 \end{aligned}$$

for all $x, y, z, w \in A$. Thus

$$D(xy,zw) = D(x,z)wy + zD(x,w)y + xD(y,z)w + xzD(y,w)$$

for all $x, y, z, w \in A$. Hence the mapping $f : A^2 \to A$ is a bi-derivation. □

Corollary 2.6. *Let* $r > 2$ *and* θ *be nonnegative real numbers, and* $f : A^2 \to A$ *be a mapping satisfying* $f(x, 0) = f(0, z) = 0$ *and*

$$\begin{aligned} &\|f(\lambda(x+y), \mu(z+w)) + f(\lambda(x+y), \mu(z-w)) + f(\lambda(x-y), \mu(z+w)) \\ &\quad + f(\lambda(x-y), \mu(z-w)) - 4\lambda\mu f(x, z)\| \\ &\leqslant \|s\left(2f\left(x+y, z-w\right) + 2f\left(x-y, z+w\right) - 4f(x, z) + 4f(y, w)\right)\| + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{2.8}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *such that*

$$\|f(x, z) - D(x, z)\| \leqslant \frac{\theta}{(1-|s|)(2^r-2)}\|x\|^r\|z\|^r \tag{2.9}$$

for all $x, z \in A$*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* $f(2x, z) = 2f(x, z)$ *and*

$$\|f(xy, zw) - f(x, z)wy - zf(x, w)y - xf(y, z)w - xzf(y, w)\| \leqslant \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \tag{2.10}$$

for all $x, y, z, w \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-derivation.*

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

Theorem 2.7. *Let* $\varphi : A^2 \to [0, \infty)$ *be a function satisfying* (2.4) *with* $X = A$ *and* $f : A^2 \to A$ *be a mapping satisfying* (2.6) *and* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (2.5) *with* $X = A$*.*

If, in addition, the mapping $f : A^2 \to A$ *satisfies* (2.7) *and* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-derivation.*

Proof. The proof is similar to the proof of Theorem 2.5. □

Corollary 2.8. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and* $f : A^2 \to A$ *be a mapping satisfying* (2.8) *and* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *such that*

$$\|f(x, z) - D(x, z)\| \leqslant \frac{\theta}{(1-|s|)(2-2^r)}\|x\|^r\|z\|^r \tag{2.11}$$

for all $x, z \in A$*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* (2.10) *and* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-derivation.*

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

3. Hyers-Ulam stability of bi-homomorphisms in Banach algebras: direct method

Now, we investigate bi-homomorphisms in complex Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 3.1. *Let* $\varphi : A^2 \to [0, \infty)$ *be a function satisfying* (2.1) *with* $X = A$ *and* $f : A^2 \to B$ *be a mapping satisfying* (2.6) *and* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $H : A^2 \to B$ *satisfying* (2.3) *with* $X = A$ *and* $Y = B$*, where* P *is replaced by* H *in* (2.3)*. If, in addition, the mapping* $f : A^2 \to B$ *satisfies* $f(2x, z) = 2f(x, z)$ *and*

$$\|f(xy, zw) - f(x, z)f(y, w)\| \leqslant \varphi(x, y)\varphi(z, w) \tag{3.1}$$

for all $x, y, z, w \in A$*, then the mapping* $f : A^2 \to B$ *is a bi-homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.5, there is a unique $\mathbb{C}$-bi-linear mapping $H : A^2 \to B$, which is defined by

$$H(x, z) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$. If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $H(x, z) = f(x, z)$ for all $x, z \in A$. It follows from (3.1) that

$$\begin{aligned}\|H(xy, zw) - H(x, z)H(y, w)\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}, zw\right) - f\left(\frac{x}{2^n}, z\right) f\left(\frac{y}{2^n}, w\right) \right\| \\ &\leqslant \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \varphi(z, w) = 0\end{aligned}$$

for all $x, y, z, w \in A$. Thus

$$H(xy, zw) = H(x, z)H(y, w)$$

for all $x, y, z, w \in A$. Hence the mapping $f : A^2 \to B$ is a bi-homomorphism. □

Corollary 3.2. *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (2.8) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (2.9) with $X = A$ and $Y = B$, where P is replaced by H in (2.9). If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ and*

$$\|f(xy, zw) - f(x, z)f(y, w)\| \leqslant \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \tag{3.2}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.

Theorem 3.3. *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (2.4) with $X = A$ and $f : A^2 \to B$ be a mapping satisfying (2.6) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (2.5) with $X = A$ and $Y = B$, where P is replaced by H in (2.5). If, in addition, the mapping $f : A^2 \to B$ satisfies (3.1) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bi-homomorphism.*

Proof. The proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (2.8) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bi-linear mapping $H : A^2 \to B$ satisfying (2.11) with $X = A$ and $Y = B$, where D is replaced by H in (2.11). If, in addition, the mapping $f : A^2 \to B$ satisfies (3.2) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bi-homomorphism.*

4. Hyers-Ulam stability of bi-derivations on Banach algebras: fixed point method

Using the fixed point method, Park [45] proved the Hyers-Ulam stability of the bi-additive s-functional inequality (1) in complex Banach spaces.

Theorem 4.1 ([45, Theorem 4.1])**.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leqslant \frac{L}{4}\varphi(x, y) \leqslant \frac{L}{2}\varphi(x, y) \tag{4.1}$$

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying (2.2) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x, z) - P(x, z)\| \leqslant \frac{L}{4(1 - |s|)(1 - L)}\varphi(x, x)\varphi(z, 0) \tag{4.2}$$

for all $x, z \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of bi-derivations on complex Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 4.2. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (4.1) with $A = X$ and $f : A^2 \to A$ be a mapping satisfying (2.6) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bi-linear mapping $D : A^2 \to A$ satisfying (4.2) with $X = A$. If, in addition, the mapping $f : A^2 \to A$ satisfies (2.7) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a bi-derivation.*

Proof. Let $\lambda = \mu = 1$ in (2.4). By Theorem 4.1, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (4.2) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$. Letting $y = w = 0$ in (2.4), we get $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x, z \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. By Lemma 2.4, the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-bi-linear. The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 4.3. *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying (2.7) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bi-linear mapping $D : A^2 \to A$ satisfying (2.9). If, in addition, the mapping $f : A^2 \to A$ satisfies (2.8), (2.9), and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a bi-derivation.*

Proof. The proof follows from Theorem 4.2 by taking $L = 2^{1-r}$ and $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

Theorem 4.4 ([45, Theorem 4.4])**.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leqslant 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{4.3}$$

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying (2.3) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z) - P(x,z)\| \leqslant \frac{1}{4(1-|s|)(1-L)} \varphi(x,x)\, \varphi(z,0) \tag{4.4}$$

for all $x, z \in X$.

Theorem 4.5. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (4.3) with $X = A$ and $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and (2.4). Then there exists a unique $\mathbb{C}$-bi-linear mapping $D : A^2 \to A$ satisfying (4.4).*

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x,z) = 2f(x,z)$ and (2.6), then the mapping $f : A^2 \to A$ is a bi-derivation.

Proof. The proof is similar to the proof of Theorem 4.2. □

Corollary 4.6. *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying (2.7) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bi-linear mapping $D : A^2 \to A$ satisfying (2.11). If, in addition, the mapping $f : A^2 \to A$ satisfies (2.8), (2.9), and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a bi-derivation.*

Proof. The proof follows from Theorem 4.5 by taking $L = 2^{r-1}$ and $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

5. Hyers-Ulam stability of bi-homomorphisms in Banach algebras: fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of bi-homomorphisms in complex Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 5.1. *Let* $\varphi : A^2 \to [0, \infty)$ *be a function satisfying* (4.1) *with* $X = A$ *and* $f : A^2 \to A$ *be a mapping satisfying* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$ *and* (2.4)*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (4.2) *with* $X = A$*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* (3.1) *and* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-homomorphism.*

Proof. By Theorem 4.2, there is a unique $\mathbb{C}$-bi-linear mapping $D : A^2 \to A$ satisfying (4.2) defined by

$$D(x, z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$. The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 5.2. *Let* $r > 2$ *and* θ *be nonnegative real numbers, and* $f : A^2 \to A$ *be a mapping satisfying* (2.7) *and* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (2.9)*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* (3.2) *and* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-homomorphism.*

Theorem 5.3. *Let* $\varphi : A^2 \to [0, \infty)$ *be a function satisfying* (4.3) *with* $X = A$ *and* $f : A^2 \to A$ *be a mapping satisfying* $f(x, 0) = f(0, z) = 0$ *and* (2.4)*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (4.4) *with* $X = A$*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$ *and* (3.1)*, then the mapping* $f : A^2 \to A$ *is a bi-homomorphism.*

Proof. The proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and* $f : A^2 \to A$ *be a mapping satisfying* (2.7) *and* $f(x, 0) = f(0, z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bi-linear mapping* $D : A^2 \to A$ *satisfying* (2.11)*. If, in addition, the mapping* $f : A^2 \to A$ *satisfies* (3.2) *and* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a bi-homomorphism.*

6. Conclusion

Using the fixed point method and the direct method, we proved the Hyers-Ulam stability of bi-derivations and bi-homomorphisms in Banach algebras, associated with the bi-additive functional inequality (1).

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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