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On the numerical solution of second order delay differential equations via a novel approach

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Abstract

Delay differential equations belong to an important class of differential equations in which the evolution of the state depends on the previous time. This work proposes a novel approach for the numerical solution of delay differential equations of second order. The suggested numerical scheme is based on Laplace transform (LT) technique. In the suggested technique, first, the given equation is transformed using the LT method to an algebraic expression. The expression is then solved for the unknown transformed function and finally the well-known Weeks method is utilized to convert the solution back to time domain. Functional analysis was used to examine the existence and uniqueness of the considered equations and to generate sufficient requirements for Ulam-Hyers (UH) type stability. Furthermore, we consider different numerical example from literature to validate our method.

Keywords: Delay differential equation, Laplace transform, uniqueness and existence, Ulam-Hyers stability, Weeks method.

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1. Introduction

The delay differential equations (DDEs) of second order are of the form

$$y''(\upsilon) = g(\upsilon) + c_1 y(\upsilon) + c_2 y(\upsilon - \alpha),\ \upsilon \in [0,1], \tag{1.1}$$

$$y(\upsilon) = \psi(\upsilon), -\alpha \leqslant \upsilon \leqslant 0,\ y(0) = \beta_1,\ y'(0) = \beta_2, \tag{1.2}$$

where $g : [0,1] \to \mathbb{R}$ is linear continuous function, $\psi(\upsilon)$ is given real valued and sufficiently smooth function, β_1, β_2 are real numbers, and α is a positive constant large delay that will be considered in this work. Ordinary differential equations (ODEs) and DDEs arise in many science fields. DDEs are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the function at previous times. A time-invariant system is one that runs independent of time. The ODEs with constant coefficients can defined such systems [34]. Recently, there are numerous research undertaken in engineering and other science subjects with a focus on the development of mathematical models

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using DDEs. DDEs have applications in a large number of real life problems, including immune system [8], dynamics of the population [19], cell proliferation [7], the bistable device [31], control theory[10], etc.

Many techniques have been developed to study the solution of DDEs such as in [16], the authors explored the uniqueness and existence of a solution to DDEs. Jafari et al. [15], studied the numerical solution of pantograph type DDEs via the transferred Legendre pseudospectral, they also discussed the convergence of the method. In [24], the authors proposed a hybrid scheme based on Picard and Krasnoselskii iterative schemes to study the numerical solution of nonlinear DDEs. Ali et al. [3] proposed an accurate and efficient numerical method for the approximate solution of pantograph-type DDEs with vanishing proportional delays, they also proved the exponential convergence of the suggested method. Mechee et al. [22] developed a Runge-Kutta-Nyström to study the solution of a 2nd order DDEs. Zi-ada [35] obtained the solution of a nonlinear DDE of fractional order with multi terms via the Adomian decomposition method. The author also proved the existence and stability of the unique solution of the considered problem. Alshehri et al. [4] obtained the numerical solution of ODEs and DDEs via the Reproducing Kernel Hilbert Spaces method. The authors of [6], utilized the explicit RKM for solving the DDEs, they also discussed the determination of stability regions. Sedaghat et al. [27], developed a numerical method based on Chebyshev polynomials to study the approximate solution of DDEs. Ogunlaran and Olagunju [23] solved the DDEs by using the modified power series method. Senu et al. [28], studied the numerical solution of the DDEs using the two-derivative RKM with Newton interpolation. The authors of [18], developed a novel method coupling the Laplace transform (LT) with the Fourier series for the solution DDEs. Sherman et al. [30] compared the performance of MAPLE and MATLAB for computing the method of steps and LT solutions for neutral and retarded linear DDEs. Akhmet et al. [2] considered a general linear impulsive system of DEs with distributed delay. They showed that the trivial solution of the considered system is asymptotically stable under Perron condition. Alzabut et al. [5] studied the asymptotic stability of Pantograph equation. They derived their main results using the Krasnoselskii's and generalized Banach fixed point theorems. Saini et al. [25] studied the numerical solution of singularly perturbed Robin type parabolic reaction diffusion multiple scale problems with large delay in time. The authors of [20] solved the time-delayed parabolic partial differential equations with a small diffusion parameter.

Recently, the authors of [9] have develop an efficient LT method to establish the exact solution of a class of 2nd order DDEs. In their suggested scheme, they apply the LT to the considered DDE and with the help of some excellent theoretical results they obtain a transformed problem in LT domain, and then invert it analytically to obtain the exact solution of the problem. However, for many complex problems using their method the analytical inversion becomes hard. Thus, a numerical method is needed to obtain the solution of the problem [26]. The aim of this article is to extend the work of [9], and obtain the approximate solution of 2nd order DDEs using numerical inversion of the LT. The numerical inversion of LT is generally an ill-posed problem and has led to numerous numerical methods. Every numerical method has its own benefits and is suitable for a specific problem. In the present study, the numerical inversion of the LT is performed via the Weeks method [17, 33].

2. Preliminaries

Let $\mathcal{J} = [0, 1]$ and $\Omega = C(\mathcal{J}, \mathbb{R})$, then for any function $y \in \Omega$ the supremum norm $\|.\|$ on Ω is defined as

$$\|y\| = \sup_{\upsilon \in \mathcal{J}} |y(\upsilon)|.$$

Definition 2.1. Let $y(\upsilon)$ is a real valued piecewise continuous function defined for $\upsilon > 0$ and is of exponential order. Then the LT of $y(\upsilon)$ exists, which is denoted and defined as

$$\widehat{y}(p) = \mathscr{L}\{y(\upsilon)\} = \int_0^\infty e^{-p\upsilon} y(\upsilon) d\upsilon.$$

Theorem 2.2. *Suppose that* $y(\upsilon), y'(\upsilon), \ldots, y^{(n-1)}(\upsilon)$ *are real functions, and are continuous on* $(0, \infty)$*, and of exponential order* μ*, while* $y^{n}(\upsilon)$ *is piecewise continuous on* $[0, \infty)$*. Then*

$$\mathscr{L}\{y^{n}(\upsilon)\} = p^{n}\mathscr{L}\{y(\upsilon)\} - p^{n-1}y(0) - \cdots - y^{(n-1)}(0).$$

Theorem 2.3 ([9], Gronwall's inequality]). *Let* $y(\upsilon)$, $\sigma(\upsilon) \geqslant 0$, $\rho(\upsilon) \geqslant 0$ *are real functions and continuous on* $(0, \infty)$*. If*

$$\nu(\upsilon) \leqslant y(\upsilon) + \sigma(\upsilon)\int_0^{\upsilon} \rho(\tau)\nu(\tau)d\tau,$$

then

$$\nu(\upsilon) \leqslant y(\upsilon) + \sigma(\upsilon)\int_0^{\upsilon} y(\tau)\rho(\tau)e^{\int_{\tau}^{\upsilon}\rho(\zeta)\sigma(\zeta)d\zeta}d\tau.$$

Theorem 2.4 ([9]). *Let* $\psi(\upsilon), \psi'(\upsilon)$ *are continuous on* $[-\alpha, 0]$*, then the LT of* $y(\upsilon - \alpha)$ *is given as*

$$\mathscr{L}\{y(\upsilon - \alpha)\} = \bar{\psi}(p) + \exp(-p\alpha)\mathscr{L}\{y(\upsilon)\},$$

and the LT of $y'(\upsilon - \alpha)$ *is given as*

$$\mathscr{L}\{y'(\upsilon - \alpha)\} = \bar{\bar{\psi}}(p) + \exp(-p\alpha)\mathscr{L}\{y'(\upsilon)\},$$

where

$$\bar{\psi}(p) = \int_{-\alpha}^{0} e^{-p(\upsilon+\alpha)}\psi(\upsilon)d\upsilon, \qquad \bar{\bar{\psi}}(p) = \int_{-\alpha}^{0} e^{-p(\upsilon+\alpha)}\psi'(\upsilon)d\upsilon.$$

Lemma 2.5 ([32]). *The solution of*

$$y''(\upsilon) = g(\upsilon), \quad 0 < \upsilon \leqslant 1, \quad y(0) = \beta_1, \quad y'(0) = \beta_2,$$

can be expressed as

$$y(\upsilon) = \beta_1 + \upsilon\beta_2 + \int_0^{\upsilon} (\upsilon - s)g(s)ds.$$

3. Existence of solution

In this section, we study the existence and uniqueness of solution to problem (1.1)-(1.2). A detailed discussion on the uniqueness existence of solution to differential equations, Ulam-Hyers stability, and error analysis can be found in [11–14].

Lemma 3.1. *Let* $g \in C(\mathcal{J}, R)$*, then the solution of the problem*

$$\begin{aligned} y''(\upsilon) &= g(\upsilon) + C_1 y(\upsilon) + C_2 y(\upsilon - \alpha)), \quad \upsilon \in \mathcal{J}, \\ y(\upsilon) &= \psi(\upsilon), \quad -\alpha \leqslant \upsilon \leqslant 0, \quad y(0) = \beta_1, \quad y'(0) = \beta_2, \quad \upsilon \in \mathcal{J}, \end{aligned} \tag{3.1}$$

can be expressed as

$$y(\upsilon) = \beta_1 + \upsilon\beta_2 + \int_0^{\upsilon} (\upsilon - s)[g(s) + C_1 y(s) + C_2 y(s - \alpha)]ds, \ \upsilon \in \mathcal{J}.$$

Proof. *Using* (2.5) *we have*

$$y(\upsilon) = \beta_1 + \upsilon\beta_2 + \int_0^{\upsilon} (\upsilon - s)[g(s) + C_1 y(s) + C_2 y(s - \alpha)]ds, \ \upsilon \in \mathcal{J}.$$

To prove the existence of solution to problem (1.1)-(1.2). Let us define an operator $\mathcal{S} : \Omega \to \Omega$ by:

$$\mathcal{S}y(\upsilon) = \beta_1 + \upsilon\beta_2 + \int_0^{\upsilon} (\upsilon - s)[g(s) + C_1 y(s) + C_2 y(s - \alpha)]ds, \ \upsilon \in \mathcal{J}.$$

The following assumptions are needed for further analysis.

(H1) For continuous function $\mathcal{K}(y)$ and $\mathcal{M} > 0$, $|\mathcal{K}(y_1) - \mathcal{K}(y_2)| \leqslant \mathcal{M}|y_1 - y_2|$.
(H2) For $\mu_1 > 0, \mu_2 > 0$, $|f(s, y_1, y_2) - f(s, \widehat{y}_1, \widehat{y}_2)| \leqslant \mu_1|y_1 - \widehat{y}_1| + \mu_2|y_2 - \widehat{y}_2|$.

Theorem 3.2 ([1, The Brouwer's fixed point theorem])**.** *Suppose that* M *is a nonempty, convex, and compact subset of a Banach space* Ω *and* $\mathcal{S} : M \to M$ *is a continuous mapping. Then* $\mathcal{S}$ *has a fixed point in* M.

From the considered problem (1.1)-(1.2), the equivalent integral form is obtained as

$$y(\upsilon) = \begin{cases} \beta_1 + \beta_2\nu + \int_0^\upsilon (\upsilon - s)g(s)ds + \int_0^\upsilon (\upsilon - s)[C_1 y(s) + C_2 y(s - \alpha)]ds, & \upsilon \in \mathcal{J}, \\ \psi(\upsilon), & -\alpha \leqslant \upsilon \leqslant 0. \end{cases}$$

Since g is linear bounded function so we have $|g(\nu)| \leqslant K_g,\ K_g > 0$.

Theorem 3.3. *Under the hypothesis* $(\mathbf{H}_1)$*, the considered problem has a solution.*

Proof. Let us define the Banach space Ω under the norm described by $\|y\| = \sup_{\upsilon \in \mathcal{J}} |y(\upsilon)|$. Consider a nonempty, convex, and compact subset M of Ω defined by $M = \{y \in \Omega : \ \|y\| \leqslant r\}$, where

$$r \geqslant \frac{2(|\beta_1| + |\beta_2|) + K_g}{2 - (|C_1| + |C_2|)}.$$

Define the operator $\mathcal{S} : M \to M$ by

$$\mathcal{S}[y(\upsilon)] = \beta_1 + \beta_2\nu + \int_0^\upsilon (\upsilon - s)g(s)ds + \int_0^\upsilon (\upsilon - s)[C_1 y(s) + C_2 y(s - \alpha)]ds,\ \upsilon \in \mathcal{J}. \tag{3.2}$$

Let $y \in M$, then to show that M is bounded, we have by using (3.2),

$$\begin{aligned} |y(\upsilon)| &= \left|\beta_1 + \beta_2\nu + \int_0^\upsilon (\upsilon - s)g(s)ds + \int_0^\upsilon (\upsilon - s)[C_1 y(s) + C_2 y(s - \alpha)]ds\right| \\ &\leqslant |\beta_1| + |\beta_2||\upsilon| + \int_0^1 (1 - s)|g(s)|ds + \int_0^1 (1 - s)[|C_1||y(s)| + |C_2||y(s - \alpha)|]ds \\ &\leqslant |\beta_1| + |\beta_2| + \frac{K_g}{2} + \frac{|C_1| + |C_2|}{2}\|y\| \leqslant r \end{aligned}$$

which implies that $\|y\| \leqslant r$. Thus M is bounded. Obviously $\mathcal{S}$ is also bounded by following the same assertion that $\|\mathcal{S}y\| \leqslant r$. Thus $\mathcal{S}$ maps bounded set to bounded. To show that $\mathcal{S}$ is continuous, let y_n in M, since M is compact and contains all of its limit points. Therefore, $y_n \to y,$ as $n \to \infty$. Therefore, we take

$$\begin{aligned} \|\mathcal{S}y_n - \mathcal{S}y\| &= \sup_{\upsilon \in \mathcal{J}} \left| \int_0^\upsilon (\nu - s)[C_1(y_n(s) - y(s)) + C_2(y_n(s - \alpha) - y(s - \alpha))]ds \right| \\ &\leqslant \sup_{\upsilon \in \mathcal{J}} C_1 \int_0^\upsilon (\upsilon - s)|y_n(s) - y(s)|ds + \sup_{\upsilon \in \mathcal{J}} C_2 \int_0^\upsilon (\upsilon - s)|y_n(s - \alpha) - y(s - \alpha)|ds \\ &\leqslant C_1 \int_0^1 (1 - s)\|y_n(s) - y(s)\|ds + C_2 \int_0^1 (1 - s)\|y_n(s - \alpha) - y(s - \alpha)\|ds. \end{aligned} \tag{3.3}$$

According to Lebesgue dominated convergence theorem, we see that at $n \to \infty$, the integrals on right side go to zero. Therefore, we have

$$\|\mathcal{S}y_n - \mathcal{S}y\| \to 0 \text{ as } n \to \infty.$$

Hence, $\mathcal{S}$ is continuous. Thus inview of Theorem 3.2, $\mathcal{S}$ has a fixed point in M. Consequently, the problem under consideration has a solution. □

4. Stability

In this section, we establish the stability results of the considered problem. The idea of UH stability is important for practical issues in economics, biological and physical problems. Consider the problem

$$\begin{aligned} y''(\upsilon) &= g(\upsilon) + C_1 y(\upsilon) + C_2 y(\upsilon-\alpha)) + h(\upsilon), \quad \upsilon \in \mathcal{J}, \\ y(\upsilon) &= \psi(\upsilon), \quad -\alpha \leqslant \upsilon \leqslant 0, \quad y(0)=\beta_1, \quad y'(0)=\beta_2. \end{aligned} \tag{4.1}$$

where, $h \in \Omega$ such that $|h(\upsilon)| \leqslant \epsilon$, for $\epsilon > 0$, then (4.1) has a solution

$$y(\upsilon) = \beta_1 + \upsilon\beta_2 + \int_0^\upsilon (\upsilon - s)[g(s) + C_1 y(s) + C_2 y(s-\alpha) + h(s)]ds. \tag{4.2}$$

Using Theorem (3.3), (4.2) can be written as

$$y(\upsilon) = \mathcal{S}y(\upsilon) + \int_0^\upsilon (\upsilon - s)h(s)ds, \ \upsilon \in \mathcal{J}.$$

From equation (4.2), using (4.1), one has

$$|\mathcal{S}y(\upsilon) - y(\upsilon)| \leqslant \epsilon\frac{\upsilon^2}{2}.$$

Theorem 4.1. *Problem* (3.1) *is UH and generalized UH stable if* $\frac{(|C_1|+|C_2|)}{2} < 1$.

Proof. Let $y, \bar{y} \in \Omega$ and unique solution of (3.1), then

$$\begin{aligned} \|y - \bar{y}\| &= \sup_{\upsilon\in\mathcal{J}} |y(\upsilon) - \mathcal{S}\bar{y}(\upsilon)| \\ &\leqslant \sup_{\upsilon\in\mathcal{J}} |y(\upsilon) - \mathcal{S}y(\upsilon)| + \sup_{\upsilon\in\mathcal{J}} |\mathcal{S}y(\upsilon) - \mathcal{S}\bar{y}(\upsilon)| \leqslant \frac{\upsilon^2}{2}\epsilon + \frac{C_1 + C_2}{2}\|y - \bar{y}\| \leqslant \frac{\frac{\upsilon^2}{2}\epsilon}{1 - \frac{|C_1|+|C_2|}{2}}. \end{aligned}$$

□

5. Methodology

In the proposed scheme, first we employ the LT to (1.1)-(1.2), we have

$$\begin{gathered} \mathscr{L}\{y''(\upsilon)\} = \mathscr{L}\{g(\upsilon) + C_1 y(\upsilon) + C_2 y(\upsilon - \alpha)\} \\ \Rightarrow \\ p^2\mathscr{L}\{y(\upsilon)\} - py(0) - y'(0) = \mathscr{L}\{g(\upsilon)\} + C_1\mathscr{L}\{y(\upsilon)\} + C_2\mathscr{L}\{y(\upsilon-\alpha)\}, \end{gathered}$$

then using theorem (2.4), we have

$$\Rightarrow p^2\widehat{y}(\upsilon) - p\beta_1 - \beta_2 = \widehat{g}(p) + C_1\widehat{y}(p) + C_2\bar{\psi}(p) + C_2\exp(-p\alpha)\widehat{y}(\upsilon),$$

or

$$\Rightarrow \Big[p^2 - C_1 - C_2\exp(-p\alpha)\Big]\widehat{y}(\upsilon) = p\beta_1 + \beta_2 + \widehat{g}(p) + C_2\bar{\psi}(p). \tag{5.1}$$

Eq. (5.1) can be simplified as

$$\widehat{y}(\upsilon) = \frac{p\beta_1 + \beta_2 + \widehat{g}(p) + C_2\bar{\psi}(p)}{p^2 - C_1 - C_2\exp(-p\alpha)}. \tag{5.2}$$

Taking inverse Laplace transform of (5.2), we have

$$y(\upsilon) = \mathscr{L}^{-1}\left\{\frac{p\beta_1 + \beta_2 + \widehat{g}(p) + C_2\bar{\psi}(p)}{p^2 - C_1 - C_2\exp(-p\alpha)}\right\}. \tag{5.3}$$

Our next target is to evaluate the inverse LT in (5.3). Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; thus, a numerical inversion method must be used. If a direct formulation is available, a numerical inversion could still be useful. The aim of this article is to use Weeks method for the numerical inversion of LT.

5.1. Weeks method

Weeks technique is one of the most straightforward easy and accurate techniques for numerically inverting the LT, provided that the parameters involved in Laguerre expansion are chosen with the best values. In comparison with trapezoidal rule and Talbot's technique, the Weeks procedure has a distinct benefit: it offers a function expansion, particularly the Laguerre series expansion. This suggests that the unknowns in the series may be identified for any given $\widehat{y}(p)$. In Weeks approach we select $p = \rho + i\vartheta$, $\vartheta \in \mathbb{R}$ to get

$$y_n(\upsilon) = \frac{e^{\rho\upsilon}}{2\pi}\int_{-\infty}^{\infty} e^{i\upsilon\vartheta}\widehat{y}(\rho + i\vartheta)d\vartheta. \tag{5.4}$$

The transform function $\widehat{y}(\rho + i\vartheta)$ is expanded as

$$\widehat{y}(\rho + i\vartheta) = \sum_{\kappa=-\infty}^{\infty} a_\kappa \frac{(-\gamma + i\vartheta)^\kappa}{(\gamma + i\vartheta)^{\kappa+1}},\ \gamma > 0,\ \vartheta \in \mathbb{R}. \tag{5.5}$$

Using (5.5) in (5.4), we obtain

$$y_n(\upsilon) = \frac{e^{\rho\upsilon}}{2\pi}\sum_{\kappa=-\infty}^{\infty} a_\kappa \delta_\kappa(\upsilon;\gamma),$$

where

$$\delta_\kappa(\upsilon;\gamma) = \int_{-\infty}^{\infty} e^{i\upsilon\vartheta}\frac{(-\gamma + i\vartheta)^\kappa}{(\gamma + i\vartheta)^{\kappa+1}}d\vartheta.$$

Residues can be utilized to compute the Fourier integral, and for $\upsilon > 0$ one gets

$$\delta_\kappa(\upsilon;\gamma) = \begin{cases} 2\pi e^{-\gamma\upsilon}\mathcal{L}_\kappa(2\gamma\upsilon), & \kappa \geqslant 0, \\ 0, & \kappa < 0. \end{cases}$$

Here $\mathcal{L}_\kappa(\upsilon)$ denotes the κ^{th} degree Laguerre polynomial, $\rho > \rho_0$, ρ_0 is the convergence abscissa, and $\rho, \gamma \in \mathbb{R}^+$. The $\mathcal{L}_\kappa(\upsilon)$ are expressed as

$$\mathcal{L}_\kappa(\upsilon) = \frac{e^\upsilon}{\kappa!}\frac{d^\kappa}{d\upsilon^\kappa}(e^{-\upsilon}\upsilon^\kappa),$$

a_κ are the Taylor series coefficients

$$\mathcal{C}(\phi) = \frac{2\gamma}{1-\phi}\widehat{y}\left(\rho + \frac{2\gamma}{1-\phi} - \gamma\right) = \sum_{\kappa=0}^{\infty} a_\kappa \phi^\kappa,\ |\phi| < R, \tag{5.6}$$

where R is the radius of convergence of Maclaurin series (5.6). The coefficients a_κ are obtained as

$$a_\kappa = \frac{1}{2\pi i}\int_{|\phi|=1}\frac{\mathcal{C}(\phi)}{\phi^{\kappa+1}}d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{C}(e^{i\beta})e^{-i\kappa\beta}d\beta. \tag{5.7}$$

In (5.7), the integral is the widely recognized Cauchy's formula, which can be computed as

$$\tilde{a}_\kappa = \frac{e^{-i\kappa\lambda/2}}{2n}\sum_{j=-n}^{n-1}\mathcal{C}(e^{i\beta_{j+1/2}})e^{-i\kappa\beta_j},\ \kappa = 0, 1, 2, \ldots, n-1,$$

where $\beta_j = j\lambda$, $\lambda = \frac{\pi}{n}$.

Theorem 5.1. *Let $\mathcal{N} : \mathcal{T} \to \mathcal{T}$ be a contraction mapping with constant $0 < L < 1$, and Banach space $\mathcal{T}$, then the solution using inverse LT method can be expressed in the series form as*

$$y_n = \mathcal{N}(y_{n-1}),\ \ y_{n-1} = \sum_{i=1}^{n-1} y_i,\ n = 1, 2, \ldots$$

and

$$y_n \in S_r(y) = \{\overline{y} \in \mathcal{T} : \|y - \overline{y}\| < r\}, \quad \lim_{n\to\infty} y_n = y, \quad \textit{where} \quad y_0 = y(0).$$

Proof. To prove the required result, we use mathematical induction, for $n = 1$, we have

$$\|y_1 - y\| = \|\mathcal{N}y_0 - \mathcal{N}y\| \leqslant L\|y_0 - y\|.$$

Let for $n-1$ the result is true, then

$$\|y_{n-1} - y\| \leqslant L^{n-1}\|y_0 - y\|. \tag{5.8}$$

Now, we have

$$\|y_n - y\| = \|\mathcal{N}(y_{n-1}) - \mathcal{N}(y)\| \leqslant L\|y_{n-1} - y\|. \tag{5.9}$$

Using Eqs. (5.8) and (5.9) implies that

$$\|y_n - y\| \leqslant LL^{n-1}\|y_0 - y\| \leqslant L^n.r < r,$$

which yields that $y_n \in S_r(y)$. Also as $n \to \infty$, $L^n \to 0$. Therefore

$$\lim_{n\to\infty} y_n = y,$$

which completes the proof. □

5.2. *Error analysis*

The authors of [33] studied the error of the Weeks method. They made the following observations:

$$y(\upsilon) = \exp(\rho\upsilon) \sum_{\kappa=0}^{\infty} a_\kappa \exp(-\gamma\upsilon)\mathcal{L}_\kappa(2\gamma\upsilon). \tag{5.10}$$

Three main factors of error were identified as following.

- 1st truncation of the series at n terms;
- 2nd numerically evaluating the unknown coefficients;
- 3rd the numerical inversion of LT.

The real expansion for modelling the aforementioned three errors is

$$\tilde{y}(\upsilon) = \exp(\rho\upsilon) \sum_{\kappa=0}^{n-1} \tilde{a}_\kappa(1+\delta_\kappa) \exp(-\gamma\upsilon)\mathcal{L}_\kappa(2\gamma\upsilon), \tag{5.11}$$

here δ_κ denotes the relative error in the floating-point representation of the evaluated coefficients, i.e., $fl(\tilde{a}_\kappa) = \tilde{a}_\kappa(1+\delta_\kappa)$. Subtracting (5.11) from (5.10) and assuming $\sum_{\kappa=0}^{\infty} |a_\kappa| < \infty$ yields

$$|y(\upsilon) - \tilde{y}(\upsilon)| \leqslant e^{(\rho\upsilon)}\Big(E_T + E_D + E_C\Big),$$

where the truncation, discretization, and conditioning error bounds are, respectively, defined by

$$er^T = \sum_{\kappa=n}^{\infty} |a_\kappa|, \quad er^D = \sum_{\kappa=0}^{n-1} |a_\kappa - \tilde{a}_\kappa|, \quad er^C = \chi \sum_{\kappa=0}^{n-1} |\tilde{a}_\kappa|.$$

Here δ is the machine roundoff unit, which satisfies $\max_{0\leqslant\kappa\leqslant n-1}|\delta_\kappa| \leqslant \delta$. It is important to note that we have used the fact $|\exp(-\gamma\upsilon)\mathcal{L}_\kappa(2\gamma\upsilon)| \leqslant 1$. For the purposes of minimizing the error bound we shall disregard the discretization error er^D in comparison with er^T and er^C [33]. So, we refer to er^T and er^C. For er^T and er^C the upper bound derived by the author of [33] are as

$$er^T \leqslant \frac{\kappa(\upsilon)}{\upsilon^n(\upsilon-1)}, \quad er^C \leqslant \delta\frac{\upsilon\kappa(\upsilon)}{\upsilon-1},$$

which is true for $\upsilon \in (1, R)$. Hence, we have the error bound given as

$$\text{error}_{\text{est}} \leqslant \frac{\kappa(\upsilon)}{\upsilon^n(\upsilon-1)} + \delta\frac{\upsilon\kappa(\upsilon)}{\upsilon-1}.$$

6. Application

In this section we present the numerical results of the proposed numerical method on five problems. The selected problems have been widely discussed in literature and their numerical solutions are available for comparison. Two kinds of error measures: maximum absolute error E^∞ and root mean squared error E^{rms} are considered as

$$E^\infty = \max_{1\leqslant j\leqslant n} |y(\upsilon_j) - y_n(\upsilon_j))| \quad \text{and} \quad E^{rms} = \sqrt{\frac{\sum_{j=1}^{n}(y(\upsilon_j)-y_n(\upsilon_j))^2}{n}},$$

where $y(\upsilon)$ and $y_n(\upsilon)$ represent exact and numerical solutions of the given problem.

Problem 6.1. Consider the second order DDE of the form

$$y''(\upsilon) + 2y(\upsilon) - y(\upsilon-1) = 3e^\upsilon - e^{(\upsilon-1)}, \ \upsilon \in \mathcal{J},$$

with delay condition

$$y(\upsilon) = e^\upsilon, \ -1 \leqslant \upsilon \leqslant 0,$$

and initial conditions

$$y(0) = y'(0) = 1.$$

The analytic solution of the problem is $y(\upsilon) = e^\upsilon$.

Table 1: Optimal values of (σ, β), E^∞, E^{rms}, and E^{est} using the suggested method for Problem 6.1.

n	(σ,β)	E^∞	E^{rms}	E^{est}	CPU time(sec)
20	(3.6384, 1.9574)	4.4409×10^{-16}	9.9301×10^{-17}	1.7853×10^{-14}	0.360449
22	(3.6384, 1.8197)	4.4409×10^{-16}	9.4680×10^{-17}	1.7158×10^{-14}	0.358295
24	(3.6384, 1.7260)	0	0	1.5354×10^{-14}	0.400068
26	(3.6598, 1.8480)	4.4409×10^{-16}	8.7093×10^{-17}	1.7606×10^{-14}	0.391232
28	(3.8007, 1.6554)	8.8818×10^{-16}	1.6785×10^{-16}	1.7950×10^{-14}	0.371862
30	(3.5855, 1.6474)	8.8818×10^{-16}	1.6216×10^{-16}	1.6164×10^{-14}	0.374024
32	(3.5652, 1.5466)	4.4409×10^{-16}	7.8505×10^{-17}	1.7641×10^{-14}	0.375128
34	(3.8168, 1.4590)	1.3323×10^{-15}	2.2848×10^{-16}	2.1274×10^{-14}	0.454199
36	(3.5517, 1.2361)	0	0	1.5923×10^{-14}	0.379240

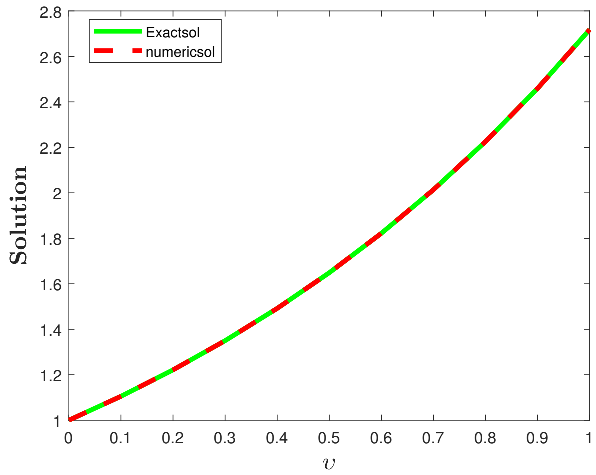

Figure 1: The numerical and exact solutions of Problem 1.1.

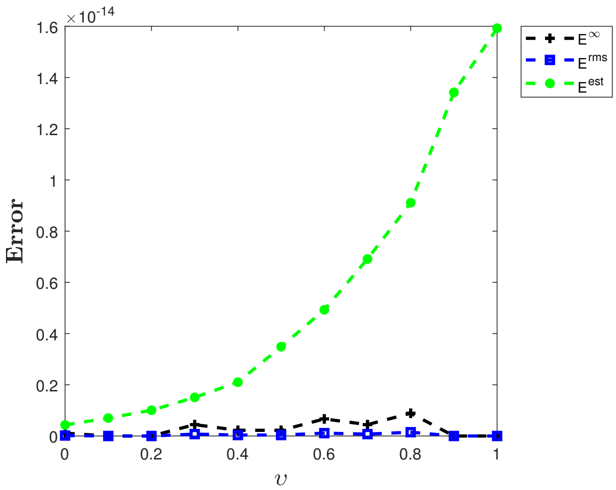

Figure 2: Comparison of E^{∞}, E^{rms}, and E^{est} with $\upsilon \in [0,1]$ for Problem 6.1.

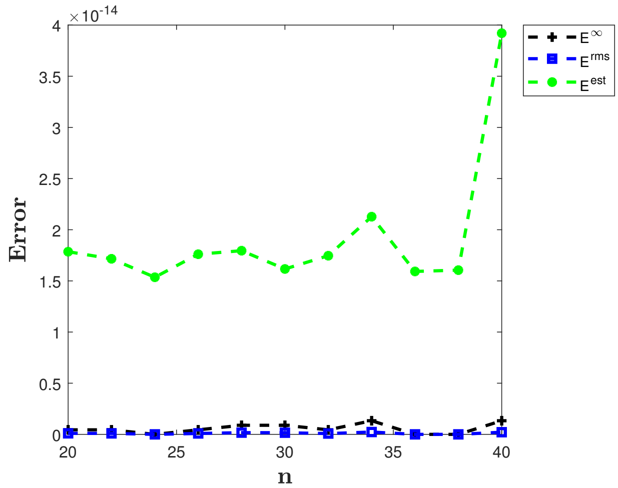

Figure 3: Comparison of E^{∞}, E^{rms}, and E^{est} versus n corresponding to Problem 6.1.

Problem 6.2. Here we consider the following DDE

$$y''(\upsilon)+2y(\upsilon)+\frac{1}{2}y(\upsilon-1)=2+2\upsilon^2+\frac{1}{2}(\upsilon-1)^2,\ \ \upsilon\in\mathcal{J},$$

with delay condition

$$y(\upsilon)=\upsilon^2,\ \ -1\leqslant\upsilon\leqslant 0,$$

and initial condition

$$y(0)=0,\ y'(0)=0.$$

The analytic solution is $y(\upsilon)=\upsilon^2$.

Table 2: Optimal values of (σ, β), E^{∞}, E^{rms}, and E^{est} using the suggested method for Problem 6.2.

n	(σ, β)	E^{∞}	E^{rms}	E^{est}	CPU time(sec)
20	(3.5855, 1.9737)	3.1252×10^{-10}	6.9881×10^{-11}	4.4867×10^{-8}	0.323433
22	(3.5855, 1.9737)	5.8835×10^{-10}	1.2544×10^{-10}	4.6052×10^{-9}	0.348041
24	(3.5855, 1.9737)	7.9861×10^{-11}	1.6301×10^{-11}	4.6418×10^{-10}	0.360817
26	(3.5855, 1.9737)	5.1661×10^{-12}	1.0132×10^{-12}	4.6085×10^{-11}	0.355712
28	(3.5855, 1.9737)	2.5979×10^{-14}	4.9096×10^{-15}	4.5188×10^{-12}	0.418108
30	(3.5855, 1.9737)	4.9960×10^{-14}	9.1214×10^{-15}	4.3953×10^{-13}	0.352980
32	(3.5855, 1.9737)	7.1054×10^{-15}	1.2561×10^{-15}	4.4175×10^{-14}	0.449753
34	(3.5855, 1.9737)	6.6613×10^{-16}	1.1424×10^{-16}	6.2609×10^{-15}	0.408586
36	(3.5855, 1.9737)	2.2204×10^{-16}	3.7007×10^{-17}	2.2602×10^{-15}	0.352346

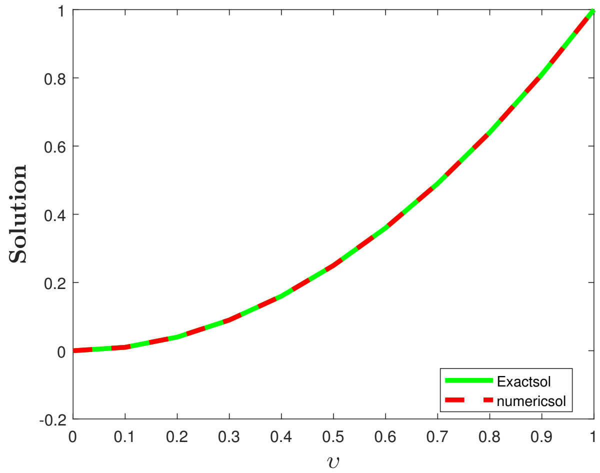

Figure 4: The numerical and exact solutions of Problem 6.2.

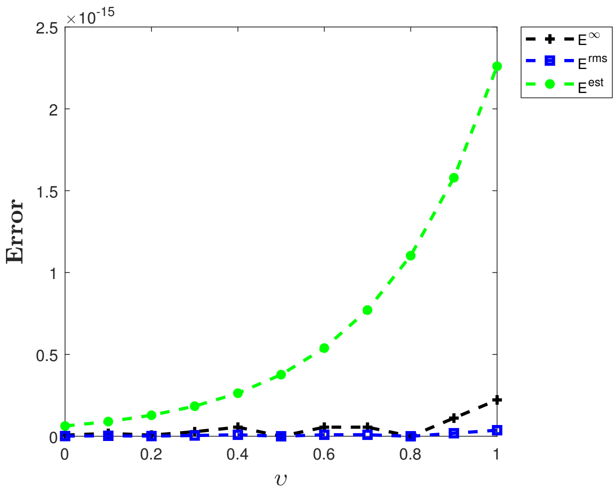

Figure 5: Comparison of E^{∞}, E^{rms}, and E^{est} with $\upsilon \in [0, 1]$ for Problem 6.2.

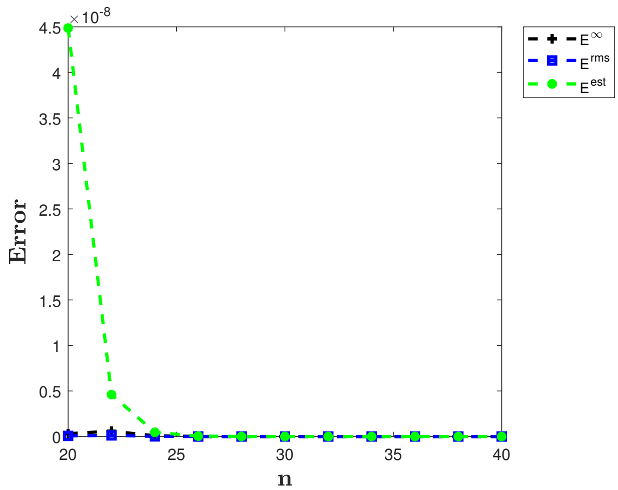

Figure 6: Comparison of E^{∞}, E^{rms}, and E^{est} versus n corresponding to Problem 6.2.

Problem 6.3. Here we consider the following DDE

$$y''(\upsilon) = y(\upsilon - \pi),\ \upsilon \in [0, \pi],$$

with delay condition

$$y(\upsilon) = \sin(\upsilon),\ \ -\pi \leqslant \upsilon \leqslant 0,$$

and initial condition

$$y'(0) = 1,\ y(0) = 0.$$

The analytic solution of the problem is $y(\upsilon) = \sin(\upsilon)$.

Table 3: Optimal values of (σ, β), E^{∞}, E^{rms}, and E^{est} using the suggested method for Problem 6.3.

n	(σ, β)	E^{∞}	E^{rms}	E^{est}	CPU time(sec)
20	$(1.9918, 1.9307)$	8.5154×10^{-14}	1.9041×10^{-14}	2.3488×10^{-12}	0.767043
22	$(2.0777, 1.9044)$	1.0658×10^{-14}	2.2723×10^{-15}	9.5793×10^{-14}	0.397119
24	$(1.8657, 1.8752)$	2.1094×10^{-15}	4.3058×10^{-16}	1.6647×10^{-14}	0.362369
26	$(2.1721, 1.9297)$	0	0	2.0330×10^{-15}	0.458420
28	$(2.0249, 1.8407)$	1.1102×10^{-16}	2.0981×10^{-17}	1.4493×10^{-15}	0.334794
30	$(1.9917, 1.6701)$	1.1102×10^{-16}	2.0270×10^{-17}	1.4432×10^{-15}	0.374849
32	$(1.7619, 1.7315)$	0	0	1.3290×10^{-15}	0.360192
34	$(1.6124, 1.4735)$	1.1102×10^{-16}	1.9040×10^{-17}	1.3437×10^{-15}	0.335430
36	$(1.5311, 1.3013)$	1.1102×10^{-16}	1.8504×10^{-17}	1.2449×10^{-15}	0.527198
[34]		7.31435×10^{-8}			
[29]		2.072245×10^{-11}			

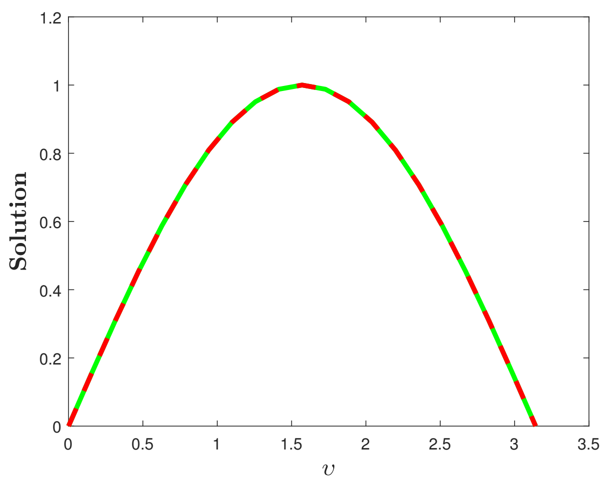

Figure 7: The numerical and exact solutions of Problem 6.3.

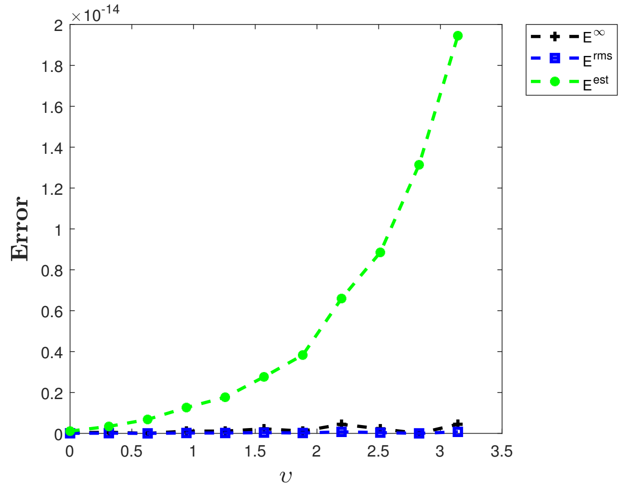

Figure 8: Comparison of E^{∞}, E^{rms}, and E^{est} with $\upsilon \in [0, 1]$ for Problem 6.3.

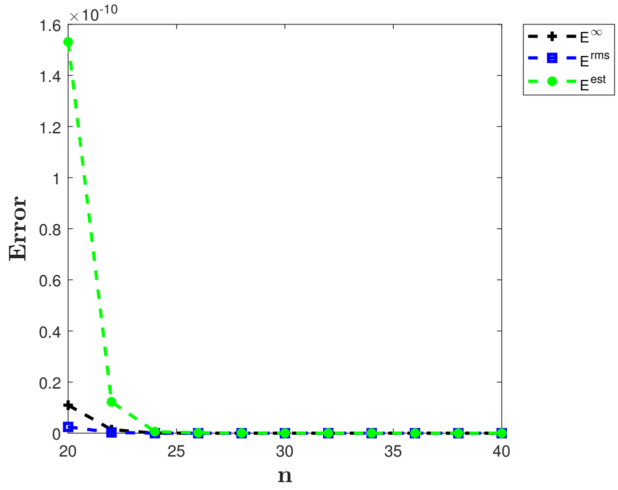

Figure 9: Comparison of E^{∞}, E^{rms}, and E^{est} versus n corresponding to Problem 6.3.

Problem 6.4. We consider the following DDE

$$y''(\upsilon) = -\frac{1}{2}y(\upsilon) + \frac{1}{2}y(\upsilon - \pi), \upsilon \in \ [0, 8\pi],$$

with delay condition

$$y(\upsilon) = \sin(\upsilon), \ -\pi \ \leqslant \ \upsilon \ \leqslant \ 0,$$

and initial condition

$$y(0) = 0, \ y'(0) = 1.$$

The analytic solution of the problem is $y(\upsilon) = \sin(\upsilon)$.

Table 4: Optimal values of (σ, β), E^{∞}, E^{rms}, and E^{est} using the suggested method for Problem 6.4.

n	(σ, β)	E^{∞}	E^{rms}	E^{est}	CPU time(sec)
20	$(1.9917, 1.9314)$	7.8604×10^{-14}	1.7576×10^{-14}	2.2698×10^{-12}	0.359800
22	$(2.0876, 1.9178)$	1.2434×10^{-14}	2.6510×10^{-15}	1.1183×10^{-13}	0.312349
24	$(1.8657, 1.8752)$	2.1094×10^{-15}	4.3058×10^{-16}	1.6584×10^{-14}	0.341719
26	$(1.9965, 1.9125)$	2.2204×10^{-16}	4.3547×10^{-17}	2.1598×10^{-15}	0.325819
28	$(1.9917, 1.8034)$	0	0	1.2786×10^{-15}	0.351125
30	$(1.8604, 1.7324)$	1.1102×10^{-16}	2.0270×10^{-17}	1.3577×10^{-15}	0.332441
32	$(1.6886, 1.5869)$	0	0	1.2057×10^{-15}	0.345769
34	$(1.5794, 1.5412)$	0	0	1.1606×10^{-15}	0.361486
36	$(1.4651, 1.4709)$	0	0	1.0788×10^{-15}	0.417640
[34]		7.68531×10^{-8}			

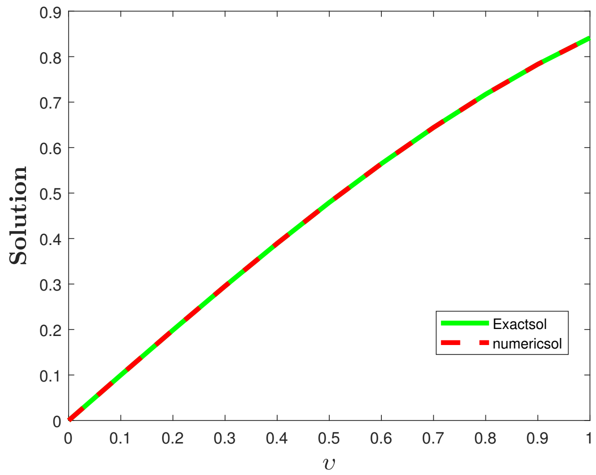

Figure 10: The numerical and exact solutions of Problem 6.4.

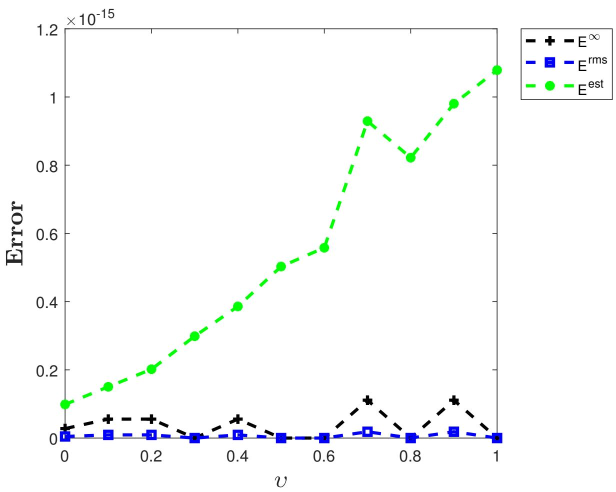

Figure 11: Comparison of E^{∞}, E^{rms}, and E^{est} with $\upsilon \in [0,1]$ for Problem 6.4.

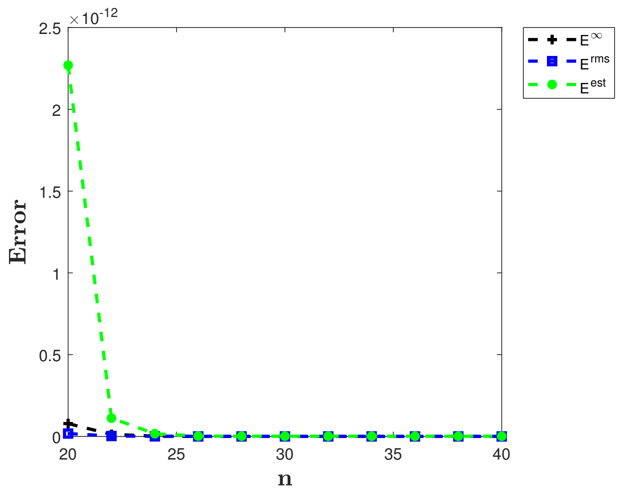

Figure 12: Comparison of E^{∞}, E^{rms}, and E^{est} versus n corresponding to Problem 6.4.

Problem 6.5. We consider the following DDE

$$y''(\upsilon) + y(\upsilon) = y(\upsilon - 1), \ \upsilon \in [0,1],$$

with initial conditions

$$y(0) = 2, \ y'(0) = 0,$$

and delay condition

$$y(\upsilon) = \upsilon^2 + 3\upsilon + 2, \ -1 \leqslant \upsilon \leqslant 0.$$

The analytic solution of the problem is $y(\upsilon) = \upsilon^2 + \upsilon - 2 + 4\cos(t) - \sin(t)$.

Table 5: Optimal values of (σ, β), E^{∞}, E^{rms}, and E^{est} using the suggested method for Problem 6.5.

n	(σ, β)	E^{∞}	E^{rms}	E^{est}	CPU time(sec)
50	$(72.607, 23.779)$	3.7238×10^{-6}	5.2662×10^{-7}	4.5177×10^{-5}	0.542118
100	$(12.899, 24.778)$	9.3516×10^{-7}	9.3516×10^{-8}	1.4430×10^{-5}	0.399772
150	$(14.738, 24.571)$	4.9185×10^{-7}	4.0159×10^{-8}	8.1668×10^{-6}	0.415064
200	$(14.885, 24.801)$	3.1447×10^{-7}	2.2237×10^{-8}	5.4459×10^{-6}	0.485590
250	$(14.210, 24.417)$	2.3376×10^{-7}	1.4784×10^{-8}	4.1396×10^{-6}	0.440651
300	$(14.838, 24.671)$	1.7543×10^{-7}	1.0129×10^{-8}	3.2425×10^{-6}	0.549823
350	$(14.738, 24.671)$	1.3980×10^{-7}	7.4724×10^{-9}	2.6639×10^{-6}	0.455142
400	$(14.738, 24.671)$	1.1449×10^{-7}	5.7247×10^{-9}	2.2383×10^{-6}	0.456882
450	$(14.588, 24.801)$	9.4704×10^{-8}	4.4644×10^{-9}	1.8957×10^{-6}	0.493372
500	$(14.833, 24.671)$	8.1274×10^{-8}	3.6347×10^{-9}	1.6643×10^{-6}	0.474972
[34]		1.27698×10^{-7}			
[29]		2.6012×10^{-6}			
[21]		2.3419×10^{-6}			

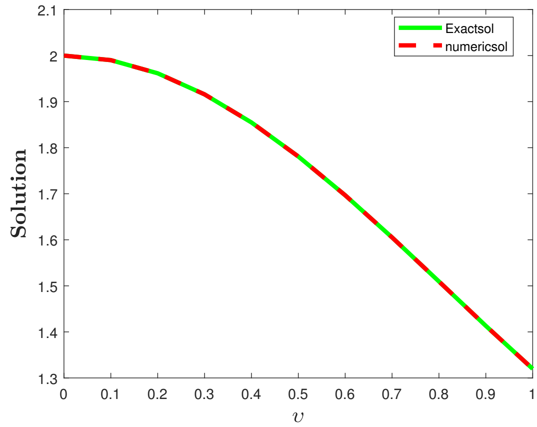

Figure 13: The numerical and exact solutions of Problem 6.5.

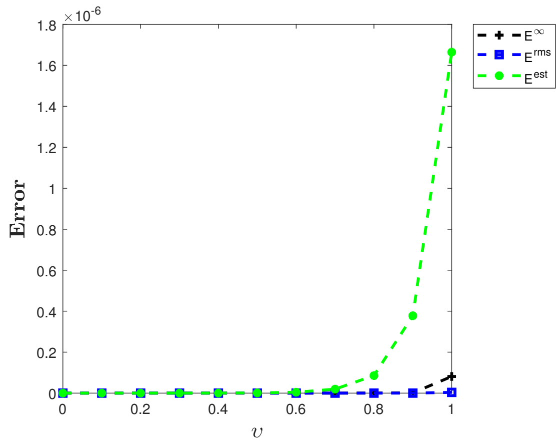

Figure 14: Comparison of E^{∞}, E^{rms}, and E^{est} with $\upsilon \in [0, 1]$ for Problem 6.5.

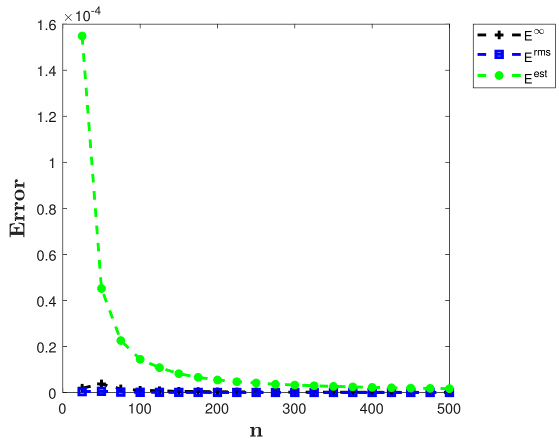

Figure 15: Comparison of E^{∞}, E^{rms}, and E^{est} versus n corresponding to Problem 6.5.

7. Discussion

The E^{∞}, E^{rms}, and E^{est} obtained for various values of n corresponding to Problems 6.1-6.5 using the suggested scheme are presented in Tables 1-5. It is observed that the accuracy of the method improves by increasing the number of nodes n. The comparison of numerical and exact solutions of Problems 6.1-6.5 are compared in Figs. 1, 4, 7, 10, and 13, respectively. The comparison of E^{∞}, E^{rms}, and E^{est} for various values of υ corresponding to Problems 6.1-6.5 are depicted in Fig. 2, 5, 8, 11, and 14, respectively. Similarly the comparison of E^{∞}, E^{rms}, and E^{est} for various values of n corresponding to Problems 6.1-6.5 are depicted in Figs. 3, 6, 9, 12, and 15, respectively. It is evident form the computational results that the suggested scheme has excellent accuracy. The obtained results are compared with other methods available in literature. It can be seen that our method has produced accurate results.

8. Conclusion

In the current work, the LT method was applied to linear 2nd order DDEs. In the suggested method the DDEs were transformed to equivalent problems in LT domain, which were then solved for the unknown function. Then Weeks method was utilized to invert the obtain solution back into real domain. Numerical examples were considered for the illustration of the suggested method. The results show that the suggested method can efficiently solve 2nd order linear DDEs. In our future work, we want to extend the suggested scheme to high order linear DDEs and system of linear DDEs.

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