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Some generalizations of dynamic Opial-type inequalities on time scales

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Abstract

In this paper, we prove some dynamic Opial type inequalities on time scales. The functions involved in these Opial type inequalities are positive and monotone. In addition to these generalizations, some integral and discrete inequalities will be obtained as special cases of our results.

Keywords: Opial type inequalities, dynamic inequalities, time scale.

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1. Introduction

In 1960, Opial [23] proved the following inequality:

$$\int_a^b |x(t)||x'(t)|dt \leqslant \frac{b-a}{4}\int_a^b |x'(t)|^2 dt, \tag{1.1}$$

where x is absolutely continuous on $[a, b]$ and $x(a) = x(b) = 0$, and the constant $\frac{b-a}{4}$ is the best possible. Equality holds in (1.1) if and only if

$$x(t) = c(t-a) \quad \text{for} \quad a \leqslant x \leqslant \frac{b-a}{2},$$

and

$$x(t) = c(b-t) \quad \text{for} \quad \frac{b-a}{2} \leqslant x \leqslant b,$$

where c is a constant. Opial's inequality together with its numerous generalizations, extensions, discretizations and other types has been playing a fundamental role in the study of the existence and uniqueness properties of solutions of initial and boundary value problems for differential equations as well as

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difference equations [1, 4, 6, 8, 12–14, 19]. In further simplifying the proof of Opial's inequality, which has already been simplified by Olech [22], Beescak [5], Levison [18], Mallows [20], and Pederson [24], it is proved that if x is real absolutely continuous on $(0, b)$ and with $x(0) = 0$, then

$$\int_0^b |x(t)||x'(t)|dt \leqslant \frac{b}{2}\int_0^b |x'(t)|^2 dt. \tag{1.2}$$

For a generalization of (1.1), Beesack [5] proved that if x is an absolutely continuous function on $[a, \tau]$ with $x(a) = 0$, then

$$\int_a^\tau |x(t)||x'(t)|dt \leqslant \frac{1}{2}\int_a^\tau \frac{1}{r(t)}dt \int_a^\tau r(t)|x'(t)|^2 dt, \tag{1.3}$$

where $r(t)$ is a positive and continuous function with $\int_a^\tau \frac{dt}{r(t)} \leqslant \infty$, and if $x(b) = 0$, then

$$\int_\tau^b |x(t)||x'(t)|dt \leqslant \frac{1}{2}\int_\tau^b \frac{1}{r(t)}dt \int_\tau^b r(t)|x'(t)|^2 dt.$$

Yang [25] simplified Beesack's proof and extended inequality (1.3) and proved that: if x is an absolutely continuous function on (a, b) with $x(a) = 0$, then

$$\int_a^b q(t)|x(t)||x'(t)|dt \leqslant \frac{1}{2}\int_a^b \frac{1}{r(t)}dt \int_a^b r(t)q(t)|x'(t)|^2 dt,$$

where $r(t)$ is a positive and continuous function with $\int_a^\tau \frac{dt}{r(t)} \leqslant \infty$, and $q(t)$ is a positive bounded and nonincreasing function on $[a, b]$. Hua [15] extended inequality (1.2) and proved that: if x is an absolutely continuous function with $x(a) = 0$, then

$$\int_a^b |x(t)^p||x'(t)|dt \leqslant \frac{(b-a)^p}{p+1}\int_a^b |x'(t)|^{p+1} dt,$$

where p is a positive integer. We mentioned here that the result in [15] failed to apply for general values of p. Maroni [21] generalized (1.3) and proved that: if x is an absolutely continuous function on $[a, b]$ with $x(a) = 0 = x(b)$, then

$$\int_a^b |x(t)||x'(t)|dt \leqslant \frac{1}{2}\left(\int_a^b \left(\frac{1}{r(t)}\right)^{\alpha-1} dt\right)^{\frac{2}{\alpha}} \left(\int_a^b r(t)|x'(t)|^\nu dt\right)^{\frac{2}{\nu}},$$

where $\int_a^b \left(\frac{1}{r(t)}\right)^{\alpha-1} \leqslant \infty$, $\alpha \geqslant 1$, and $\frac{1}{\alpha} + \frac{1}{\nu} = 1$. In fact, the discrete analogy of (1.1), which has been proved by Lasota [17], is given by

$$\sum_{i=1}^{h-1} |x_i \Delta x_i| \leqslant \frac{1}{2}\left[\frac{h+1}{2}\right]\sum_{i=0}^{h-1} |\Delta x_i|^2,$$

where $[x_i]_{0\leqslant i\leqslant h}$ is a sequence of real numbers with $x_0 = x_h = 0$. The discrete analogy of (1.2) is proved in [3, Theorem 5.2.2] and given by

$$\sum_{i=1}^{h-1} |x_i \Delta x_i| \leqslant \frac{h-1}{2}\sum_{i=0}^{h-1} |\Delta x_i|^2, \tag{1.4}$$

where $[x_i]_{0\leqslant i\leqslant h}$ is a sequence of real numbers with $x_0 = 0$.

In this paper, we are concerned with a certain class of Opial-type dynamic inequalities on time scales and their extensions. If the time scale equals the real (or the integers), the results represent the classical results for differential (or difference) inequalities. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [16]), that is, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$, where $q \geqslant 1$. For more details on time scale analysis, we refer the reader to the two books by Bohner and Peterson [9, 10] which summarize and organize much of the time scale calculus. In [7], Bohner and Kaymakcalan introduced the dynamic Opial inequality on time scales, which unifies the continuous version (1.2) and the discrete version (1.4), and proved that if $x : [0, b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $x(0) = 0$, then

$$\int_0^h \left|x(t) + x^{\sigma}(t)\right|\left|x^{\Delta}(t)\right|\Delta t \leqslant h \int_0^h \left|x^{\Delta}(t)\right|^2 \Delta t.$$

In the following, we recall some of the related results that have been established for differential inequalities and dynamic inequalities on time scales that serve and motivate the contents of this paper.

Agarwal and Pang [3] proved the following inequality, which is generalization of Opial's inequalities and some extensions of Beesack's. They proved that, if x_1, x_2 are absolutely continuous functions on $[a, \tau]$ with $x_1(a) = x_2(a) = 0$, then

$$\int_a^{\tau} q(t)\left[|x_1(t)x_2'(t)| + |x_1'(t)x_2(t)|\right] dt \leqslant \frac{1}{2}\int_a^{\tau} \frac{1}{p(t)} dt \int_a^{\tau} p(t)q(t)\left[|x_1'(t)|^2 + |x_2'(t)|^2\right] dt, \tag{1.5}$$

where p is a positive and continuous function with $\int_a^{\tau} \frac{dt}{p(t)} \leqslant \infty$, and q is bounded, positive, and non-increasing on $[a, \tau]$. Also, Agarwal and Pang [3] proved that, if $x(t) \in C^{(n-1)}[0, a]$ and $x^{(i)}(0) = 0$, $0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. Further, if $x^{(n-1)}(t)$ is absolutely continuous and $\int_a^b |x^{(n)}(t)|^2 dt \leqslant \infty$, then the following inequality holds:

$$\int_0^a \left[|x(t)x^{(n)}(t)|\right] dt \leqslant c_n a^n \int_0^a \left[|x^{(n)}(t)|^2\right] dt, \tag{1.6}$$

where

$$c_n = \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}}.$$

And also in [3] the authors proved that for $j = 1, 2$, let $x_j(t) \in C^{(n-1)}[0, a]$, such that $x_j^{(i)}(0) = 0$, $0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$, further, $x_j^{(n-1)}(t)$ be absolutely continuous, and $\int_a^b |x_j^{(n)}(t)|^2 dt \leqslant \infty$, then the following inequality holds:

$$\int_0^a \left[|x_1(t)x_2^{(n)}(t)| + |x_1^{(n)}(t)x_2(t)|\right] dt \leqslant c_n a^n \int_0^a \left[|x_1^{(n)}(t)|^2 + |x_2^{(n)}(t)|^2\right] dt,$$

where c_n defined by (1.6). Further, equality holds if and only if $n = 1$ and $x_1^{(n)}(t) = x_2^{(n)}(t) = c$. Zhao and An [26] generalized (1.1) and proved that: if $F \in C^1([a, b], \mathbb{R})$, $F(a) = F(b) = 0$, and $\lambda_1, \lambda_2 \geqslant 1$, then

$$\int_a^b \left|(F(t))^{\lambda_1}(F'(t))^{\lambda_2}\right| dt \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \int_a^b \left|F'(t)\right|^{\lambda_1+\lambda_2} dt.$$

Also in the same paper, the authors generalized (1.5) and proved that if $F, G : [a, b] \to \mathbb{R}$ are real-valued absolutely continuous functions on $[a, b]$, $\lambda_1, \lambda_2 \geqslant 1$,

(1) if $F(a) = G(a) = 0$, then we have

$$\begin{aligned}&\int_a^b \left(\left|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(F'(t))^{\lambda_2}\right|\right) dt\\ &\qquad \leqslant \frac{(b-a)^{\lambda_1}}{4}\int_a^b \left(|F'(t)|^{2\lambda_1} + |F'(t)|^{2\lambda_2} + |G'(t)|^{2\lambda_1} + |G'(t)|^{2\lambda_2}\right) dt;\end{aligned}$$

(2) if $F(a) = F(b) = G(a) = G(b) = 0$, then we have

$$\int_a^b \left(\left|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(F'(t))^{\lambda_2}\right|\right) dt$$
$$\leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1+2}} \int_a^b \left(|F'(t)|^{2\lambda_1} + |F'(t)|^{2\lambda_2} + |G'(t)|^{2\lambda_1} + |G'(t)|^{2\lambda_2}\right) dt.$$

In this article, we will state and prove some dynamic Opial type inequalities on time scales. Our results generalize some results of [3, 26] to time scales. After each result, we will study as special cases when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ to obtain some continuous and discrete results. This paper is organized as follows. In Section 2, we briefly present the basic definitions and concepts related to the calculus on time scales. In Section 3, we present some new dynamic Opial type inequalities via time scales integrals concerning first order derivatives. In Section 4, we present some new dynamic Opial type inequalities via time scales integrals concerning higher order derivatives. Finally, in the concluding Section 5, we summaries our results.

2. Preliminaries and lemmas

In this section, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We suppose throughout the article that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. In [11], Bohner and Peterson defined the forward jump operator σ and the graininess function μ by $\sigma(x) := \inf\{t \in \mathbb{T} : t > x\}$ and $\mu(x) := \sigma(x) - x \geqslant 0$, respectively.

In the following, we use the notations $f^\sigma(x) = f(\sigma(x))$, for any function $f : \mathbb{T} \to \mathbb{R}$ and $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$ for any interval on $\mathbb{T}$.

Definition 2.1 ([11]). $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$, the collection of rd-continuous functions is symbolized as $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([11]). Assuming $f : \mathbb{T} \to \mathbb{R}$ and $x \in \mathbb{T}$, we define the delta derivative $f^\Delta(x)$ to be the number if it exists, as follows: for any $\epsilon > 0$ there is a neighborhood $U = (x - t, x + t) \cap \mathbb{T}$ for some $t > 0$ of x, such that

$$|f(\sigma(x)) - f(t) - f^\Delta(x)(\sigma(x) - t)| \leqslant \epsilon|\sigma(x) - t|, \quad \forall t \in U, \ t \neq \sigma(x).$$

Theorem 2.3 ([11]). *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be differentiable at* $x \in \mathbb{T}$. *Then*

1. $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* x *and the "product rule"holds as*

$$(fg)^\Delta(x) = f^\Delta(x)g(x) + f(\sigma(x))g^\Delta(x) = f(x)g^\Delta(x) + f^\Delta(x)g(\sigma(x));$$

2. *if* $g(x)g(\sigma(x)) \neq 0$, *then* $f/g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* x *and the "quotient rule"holds as*

$$\left(\frac{f}{g}\right)^\Delta(x) = \frac{f^\Delta(x)g(x) - f(x)g^\Delta(x)}{g(x)g(\sigma(x))}.$$

Definition 2.4 ([11]). $f : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $g : \mathbb{T} \to \mathbb{R}$ if

$$f^\Delta(x) = g(x) \quad \text{holds} \quad \forall x \in \mathbb{T}^k,$$

in this case, the delta integral of g is

$$\int_a^b g(x)\Delta x = f(b) - f(a), \quad \forall a, b \in \mathbb{T}.$$

Theorem 2.5 ([11]). *If* $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, *then*

1. $\int_a^b [\alpha f(t) + \beta g(t)]\,\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
2. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;
3. $\left|\int_a^b f(t)\Delta t\right| \leqslant \int_a^b |f(t)|\,\Delta t$;
4. *If* $f(t) \geqslant 0$, $\forall t \in [a, b]_{\mathbb{T}}$, *then* $\int_a^b f(t)\Delta t \geqslant 0$.

Theorem 2.6 ([11]). *Every rd-continuous function* $g : \mathbb{T} \to \mathbb{R}$ *has an anti-derivative and if* $x_0 \in \mathbb{T}$, *then*

$$\left(\int_{x_0}^{x} g(t)\Delta t\right)^{\Delta} = g(x), \qquad \forall x \in \mathbb{T}.$$

The following key relations between $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ are used as special cases of our results.

1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$.
2. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $f^{\Delta}(t) = f(t + 1) - f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$.
3. If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $\sigma(t) = qt$, $f^{\Delta}(t) = \frac{f(qt)-f(t)}{(q-1)t}$, $\int_a^b f(t)\Delta t = (q - 1)\sum_{t=\log_q a}^{\log_q b-1} q^t f(q^t)$.

Lemma 2.7 (Chain Rule, [11, Theorem 1.90]). *Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable, then*

$$(f \circ g)^{\Delta}(x) = f'(g(c))\, g^{\Delta}(x), \quad c \in [x, \sigma(x)].$$

Lemma 2.8 ([2, Integration by parts]). *If* $a, b \in \mathbb{T}$ *and* $u, v \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t.$$

Lemma 2.9 ([2, Hölder's inequality]). *If* $a, b \in \mathbb{T}$ *and* $f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^p \Delta t\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q \Delta t\right]^{\frac{1}{q}}, \tag{2.1}$$

where $p > 1$ *and* $1/p + 1/q = 1$. (2.1) *is reversed if* $0 < p < 1$ *or* $p < 0$.

The special case $p = q = 2$ yields the Cauchy-Schwarz inequality.

Lemma 2.10 ([2, Cauchy-Schwarz inequality]). *If* $a, b \in \mathbb{T}$ *and* $f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^2 \Delta t\right]^{\frac{1}{2}} \left[\int_a^b |g(t)|^2 \Delta t\right]^{\frac{1}{2}}. \tag{2.2}$$

3. Opial type inequalities concerning first order derivatives

In what follows, all considered parameters λ_1, λ_2 will mean positive integer and $a, b, \frac{a+b}{2} \in \mathbb{T}$.

Lemma 3.1. *Let* $\mathbb{T}$ *be a time scale,* $F \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ *with* $F(a) = 0$, *and* $\lambda_1, \lambda_2 \geqslant 1$. *Then*

$$\int_a^b \left|(F(t))^{\lambda_1} \left(F^{\Delta}(t)\right)^{\lambda_2}\right| \Delta t \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}} (b - a)^{\lambda_1}}{\lambda_1 + \lambda_2} \int_a^b |F^{\Delta}(t)|^{\lambda_1+\lambda_2} \Delta t. \tag{3.1}$$

Proof. By hypothesis, we have

$$|F(t)| \leqslant \int_a^t |F^\Delta(s)|\Delta s, \quad t \in [a,b]_{\mathbb{T}}.$$

Thanks to Hölder's inequality (2.1), it follows that

$$\int_a^t |F^\Delta(s)|\Delta s \leqslant \left(\int_a^t \Delta s\right)^{\frac{\lambda_1+\lambda_2-1}{\lambda_1+\lambda_2}} \left(\int_a^t |F^\Delta(s)|^{\lambda_1+\lambda_2}\Delta s\right)^{\frac{1}{\lambda_1+\lambda_2}}.$$

Let $G(t) = \int_a^t |F^\Delta(s)|^{\lambda_1+\lambda_2}\Delta s$. Then $G^\Delta(t) = |F^\Delta(t)|^{\lambda_1+\lambda_2}$. Consequently, applying Hölder's inequality again with indices $p = \frac{\lambda_1+\lambda_2}{\lambda_1}$, $q = \frac{\lambda_1+\lambda_2}{\lambda_2}$, we have

$$\begin{aligned}
\int_a^b \left|(F(t))^{\lambda_1}(F^\Delta(t))^{\lambda_2}\right| \Delta t &= \int_a^b |F(t)|^{\lambda_1} \left|F^\Delta(t)\right|^{\lambda_2} \Delta t \\
&\leqslant \int_a^b \left\{\int_a^t \left|F^\Delta(s)\right| \Delta s\right\}^{\lambda_1} \left(G^\Delta(t)\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \Delta t \\
&\leqslant \int_a^b \left\{\int_a^t \Delta s\right\}^{\frac{\lambda_1(\lambda_1+\lambda_2-1)}{\lambda_1+\lambda_2}} \left\{\int_a^t |F^\Delta(s)|^{\lambda_1+\lambda_2}\Delta s\right\}^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(G^\Delta(t)\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \Delta t \\
&\leqslant \int_a^b \left\{ \left\{\int_a^t \Delta s\right\}^{\frac{\lambda_1(\lambda_1+\lambda_2-1)}{\lambda_1+\lambda_2}} \{G(t)\}^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(G^\Delta(t)\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \right\} \Delta t \\
&\leqslant \left(\int_a^b \left(\int_a^t \Delta s\right)^{\lambda_1+\lambda_2-1} \Delta t\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\int_a^b G^{\frac{\lambda_1}{\lambda_2}}(t) G^\Delta(t)\Delta t\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} \\
&\leqslant \left(\int_a^b (t-a)^{\lambda_1+\lambda_2-1}\Delta t\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} G(b) \\
&\leqslant \left(\frac{(b-a)^{\lambda_1+\lambda_2}}{\lambda_1+\lambda_2}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{\frac{\lambda_2}{\lambda_1+\lambda_2}} G(b) \\
&\leqslant \frac{(\lambda_2)^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{\lambda_1+\lambda_2} G(b) = \frac{(\lambda_2)^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{\lambda_1+\lambda_2} \int_a^b \left|F^\Delta(t)\right|^{\lambda_1+\lambda_2} \Delta t.
\end{aligned}$$

The proof is complete. □

Corollary 3.2. *If* $\mathbb{T} = \mathbb{R}$ *in Lemma* 3.1, *we have*

$$\int_a^b \left|(F(t))^{\lambda_1} \left(F'(t)\right)^{\lambda_2}\right| dt \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{\lambda_1+\lambda_2} \int_a^b |F'(t)|^{\lambda_1+\lambda_2} dt,$$

which is inequality (3.1) *in* [26].

Corollary 3.3. *If* $\mathbb{T} = \mathbb{Z}$ *in Lemma* 3.1, *we get*

$$\sum_{t=a}^{b-1} \left|(F(t))^{\lambda_1} (\Delta F(t))^{\lambda_2}\right| \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{t=a}^{b-1} |\Delta F(t)|^{\lambda_1+\lambda_2}.$$

Corollary 3.4. *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Lemma* 3.1, *then*

$$\sum_{t=(\log_q a)}^{\log_q b-1} q^t \left|\left(F(q^t)\right)^{\lambda_1} \left(F^\Delta(q^t)\right)^{\lambda_2}\right| \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{t=(\log_q a)}^{\log_q b-1} q^t |F^\Delta(q^t)|^{\lambda_1+\lambda_2}.$$

Theorem 3.5. *Let $\mathbb{T}$ be a time scale, $F \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, $F(a) = F(b) = 0$, and $\lambda_1, \lambda_2 \geqslant 1$. Then*

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{\Delta}(t))^{\lambda_2}\right| \Delta t \leqslant \frac{(\lambda_2)^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \int_a^b \left|F^{\Delta}(t)\right|^{\lambda_1+\lambda_2} \Delta t. \tag{3.2}$$

Proof. By hypothesis, we employ inequality (3.1) on the interval $[a, \frac{a+b}{2}]$, then

$$\int_a^{\frac{a+b}{2}} \left|(F(t))^{\lambda_1}\left(F^{\Delta}(t)\right)^{\lambda_2} \Delta t\right| \leqslant \frac{(\lambda_2)^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \int_a^{\frac{a+b}{2}} \left|F^{\Delta}(t)\right|^{\lambda_1+\lambda_2} \Delta t.$$

Let $s = t + \dfrac{b-a}{2}$. Then

$$\int_{\frac{a+b}{2}}^b \left|(F(t))^{\lambda_1}\left(F^{\Delta}(t)\right)^{\lambda_2} \Delta t\right| \leqslant \frac{(\lambda_2)^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \int_{\frac{a+b}{2}}^b \left|F^{\Delta}(t)\right|^{\lambda_1+\lambda_2} \Delta t,$$

and the result (3.2) follows. □

Corollary 3.6. *If $\mathbb{T} = \mathbb{R}$ in Theorem 3.5, then*

$$\int_a^b \left|(F(t))^{\lambda_1}\left(F'(t)\right)^{\lambda_2}\right| dt \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \int_a^b |F'(t)|^{\lambda_1+\lambda_2} dt,$$

which is inequality (3.2) *in Theorem 3.3 of* [26].

Corollary 3.7. *If $\mathbb{T} = \mathbb{Z}$ in Theorem 3.5, then*

$$\sum_{t=a}^{b-1} \left|(F(t))^{\lambda_1}(\Delta F(t))^{\lambda_2}\right| \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \sum_{t=a}^{b-1} |\Delta F(t)|^{\lambda_1+\lambda_2}.$$

Corollary 3.8. *If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 3.5, then*

$$\sum_{t=(\log_q a)}^{\log_q b-1} q^t \left|\left(F(q^t)\right)^{\lambda_1}\left(F^{\Delta}(q^t)\right)^{\lambda_2}\right| \leqslant \frac{\lambda_2^{\frac{\lambda_2}{\lambda_1+\lambda_2}}(b-a)^{\lambda_1}}{(\lambda_1+\lambda_2)2^{\lambda_1}} \sum_{t=(\log_q a)}^{\log_q b-1} q^t |F^{\Delta}(q^t)|^{\lambda_1+\lambda_2}.$$

Theorem 3.9. *Let $\mathbb{T}$ be a time scale with $\tau \in \mathbb{T}$, $p(t)$ be positive and rd-continuous on $[a,\tau]_{\mathbb{T}}$ with $\int_a^\tau \frac{\Delta t}{p(t)} \leqslant \infty$, $r(t)$ be positive, bounded and non-increasing on $[a,\tau]_{\mathbb{T}}$. Farther, $F_1, F_2 \in C_{rd}([a,\tau]_{\mathbb{T}}, \mathbb{R})$, and $F_1(a) = F_2(a) = 0$. Then the following inequality holds:*

$$\int_a^\tau r(t)\left[|F_1(t)F_2^{\Delta}(t)| + |F_1^{\Delta}(t)F_2(t)|\right] \Delta t \leqslant \frac{1}{2}\left[\int_a^\tau \frac{1}{p(t)}\Delta t\right]\left[\int_a^\tau p(t)r(t)[|F_1^{\Delta}(t)|^2 + |F_2^{\Delta}(t)|^2]\Delta t\right]. \tag{3.3}$$

Proof. For $i = 1, 2$, let

$$G_i(t) = \int_a^t \sqrt{r(s)}|F_i^{\Delta}(s)|\Delta s, \quad t \in [a,\tau]_{\mathbb{T}}.$$

So that

$$G_i^{\Delta}(t) = \sqrt{r(t)}|F_i^{\Delta}(t)| \geqslant 0,$$

and

$$|F_i(t)| = |F_i(t) - F_i(a)| = \left|\int_a^t F_i^{\Delta}(s)\Delta s\right| \leqslant \int_a^t |F_i^{\Delta}(s)|\Delta s \leqslant \int_a^t \frac{\sqrt{r(s)}}{\sqrt{r(t)}}|F_i^{\Delta}(s)|\Delta s \leqslant \frac{G_i(t)}{\sqrt{r(t)}}.$$

Thus, we have

$$\begin{aligned}\int_a^\tau r(t)\left[|F_1(t)F_2^\Delta(t))|+|F_1^\Delta(t)F_2(t)|\right]\Delta t &\leqslant \int_a^\tau \left[G_1(t)G_2^\Delta(t)+G_1^\Delta(t)G_2(t)\right]\Delta t\\ &\leqslant \int_a^\tau \left[G_1(t)G_2^\Delta(t)+G_1^\Delta(t)G_2^\sigma(t)\right]\Delta t\\ &\leqslant G_1(\tau)G_2(\tau)\leqslant \frac{1}{2}\left[G_1^2(\tau)+G_2^2(\tau)\right],\end{aligned} \tag{3.4}$$

from the definition of $G_i(t)$ and the Cauchy-Schwarz inequality (2.2) involving time scale, we have

$$G_i^2(\tau)=\left[\int_a^\tau \frac{1}{\sqrt{p(t)}}\sqrt{p(t)r(t)}|F_i^\Delta(t)|\Delta t\right]^2 \leqslant \int_a^\tau \frac{1}{p(t)}\Delta t\int_a^\tau p(t)r(t)|F_i^\Delta(t)|^2\Delta t. \tag{3.5}$$

The inequality (3.3) now follows immediately from (3.4) and the inequalities analogous to (3.5) for $G_i^2(\tau)$, $i=1,2$. □

Remark 3.10. Taking $p(t)=r(t)=1$ in Theorem 3.9, we get

$$\int_a^\tau \left[|F_1(t)F_2^\Delta(t)|+|F_1^\Delta(t)F_2(t)|\right]\Delta t\leqslant \frac{1}{2}(\tau-a)\left[\int_a^\tau [|F_1^\Delta(t)|^2+|F_2^\Delta(t)|^2]\Delta t\right].$$

Corollary 3.11. *When $\mathbb{T}=\mathbb{R}$ in Theorem 3.9, then the inequality (3.3) reduces to the following inequality*

$$\int_a^\tau r(t)\left[|F_1(t)F_2'(t)|+|F_1'(t)F_2(t)|\right]dt\leqslant \frac{1}{2}\left[\int_a^\tau \frac{1}{p(t)}dt\right]\left[\int_a^\tau p(t)r(t)[|F_1'(t)|^2+|F_2'(t)|^2]dt\right],$$

which is inequality in Theorem 2.14.1 of [3].

Corollary 3.12. *If $\mathbb{T}=\mathbb{Z}$ in Theorem 3.9, then*

$$\sum_{t=a}^{\tau-1} r(t)\left[|F_1(t)\Delta F_2(t)|+|\left(\Delta F_1(t)\right)F_2(t)|\right]\leqslant \frac{1}{2}\left[\sum_{t=a}^{\tau-1}\frac{1}{p(t)}\right]\left[\sum_{t=a}^{\tau-1}p(t)r(t)[|\Delta F_1(t)|^2+|\Delta F_2(t)|^2]\right].$$

Corollary 3.13. *If $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ in Theorem 3.9, then*

$$\begin{aligned}&\sum_{t=\log_q a}^{\log_q \tau-1} q^t r(q^t)\left[|F_1(q^t)F_2^\Delta(q^t)|+|\left(F_1^\Delta(q^t)\right)F_2(q^t)|\right]\\ &\leqslant \frac{1}{2}\left[(q-1)\sum_{t=\log_q a}^{\log_q \tau-1} q^t\frac{1}{p(q^t)}\right]\left[\sum_{t=\log_q a}^{\log_q \tau-1} q^t p(q^t)r(q^t)[|F_1^\Delta(q^t)|^2+|F_2^\Delta(q^t)|^2]\right].\end{aligned}$$

Theorem 3.14. *Let $\mathbb{T}$ be a time scale. For delta differentiable $F,G:[a,b]_{\mathbb{T}}\to\mathbb{R}$, and $F,G\in C_{rd}$, $\lambda_1,\lambda_2\geqslant 1$,*

(1) *if $F(a)=G(a)=0$, then we have*

$$\begin{aligned}&\int_a^b \left(\left|(F(t))^{\lambda_1}(G^\Delta(t))^{\lambda_2}\right|+\left|(G(t))^{\lambda_1}(F^\Delta(t))^{\lambda_2}\right|\right)\Delta t\\ &\leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}}\int_a^b\left(\frac{|F^\Delta(t)|^{p\lambda_1}}{p}+\frac{|F^\Delta(t)|^{q\lambda_2}}{q}+\frac{|G^\Delta(t)|^{p\lambda_1}}{p}+\frac{|G^\Delta(t)|^{q\lambda_2}}{q}\right)\Delta t;\end{aligned} \tag{3.6}$$

(2) *if* $F(a) = F(b) = G(a) = G(b) = 0$, *then we have*

$$\begin{aligned}&\int_a^b \left(\left|(F(t))^{\lambda_1}(G^\Delta(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(F^\Delta(t))^{\lambda_2}\right|\right)\Delta t \\ &\quad\leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1}(2p)^{\frac{1}{p}}}\int_a^b \left(\frac{|F^\Delta(t)|^{p\lambda_1}}{p} + \frac{|F^\Delta(t)|^{q\lambda_2}}{q} + \frac{|G^\Delta(t)|^{p\lambda_1}}{p} + \frac{|G^\Delta(t)|^{q\lambda_2}}{q}\right)\Delta t.\end{aligned} \tag{3.7}$$

Proof.

(1) If $\lambda_1 = \lambda_2 = 1$, from Remark 3.10, we have

$$\int_a^b \left(|F(t)G^\Delta(t)| + |G(t)F^\Delta(t)|\right)\Delta t \leqslant \frac{b-a}{2}\int_a^b \left(|F^\Delta(t)|^2 + |G^\Delta(t)|^2\right)\Delta t.$$

If $\lambda_1 \geqslant 2$, $\lambda_2 \geqslant 2$, $F(a) = G(a) = 0$, then we have

$$|F(t)| \leqslant \int_a^t |F^\Delta(x)|\Delta x \leqslant \int_a^t (\Delta x)^{\frac{p\lambda_1 - 1}{p\lambda_1}} \left(\int_a^t |F^\Delta(x)|^{p\lambda_1}\Delta x\right)^{\frac{1}{p\lambda_1}}.$$

It follows that

$$\begin{aligned}\int_a^b |F(t)|^{p\lambda_1}\Delta t \leqslant \int_a^b |F(t)|^{p\lambda_1}\Delta t &\leqslant \int_a^b (t-a)^{p\lambda - 1}\int_a^t |F^\Delta(x)|^{p\lambda_1}\Delta x \\ &\leqslant \frac{(b-a)^{p\lambda_1}}{p\lambda_1}\int_a^b |F^\Delta(x)|^{p\lambda_1}\Delta x \leqslant \frac{(b-a)^{p\lambda_1}}{2p}\int_a^b |F^\Delta(t)|^{p\lambda_1}\Delta t.\end{aligned}$$

Thanks to Hölder's inequality and the elementary inequalities $\alpha, \beta \geqslant 0$, $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \leqslant \frac{\alpha}{p} + \frac{\beta}{q}$, we have

$$\begin{aligned}\int_a^b |(F(t))^{\lambda_1}(G^\Delta(t))^{\lambda_2}|\Delta t &\leqslant \left\{\int_a^b \left|F(t)^{p\lambda_1}\Delta t\right|\right\}^{\frac{1}{p}} \left\{\int_a^b \left|G^\Delta(t)^{q\lambda_2}\Delta t\right|\right\}^{\frac{1}{q}} \\ &\leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}}\left(\int_a^b |F^\Delta(t)|^{p\lambda_1}\Delta t\right)^{\frac{1}{p}}\left(\int_a^b \left|G^\Delta(t)^{q\lambda_2}\Delta t\right|\right)^{\frac{1}{q}} \\ &\leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}}\int_a^b \left[\frac{|F^\Delta(t)|^{p\lambda_1}}{p} + \frac{|G^\Delta(t)|^{q\lambda_2}}{q}\right]\Delta t.\end{aligned}$$

Similarly, we have

$$\int_a^b |(G(t))^{\lambda_1}(F^\Delta(t))^{\lambda_2}|\Delta t \leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}}\int_a^b \left[\frac{|G^\Delta(t)|^{p\lambda_1}}{p} + \frac{|F^\Delta(t)|^{q\lambda_2}}{q}\right]\Delta t.$$

Then the result (3.6) follows.

(2) If $F(a) = F(b) = G(a) = G(b) = 0$, we employ inequality (3.6) on the interval $[a, \frac{a+b}{2}]$, then

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} \left[|(F(t))^{\lambda_1}(G^\Delta(t))^{\lambda_2}| + |(G(t))^{\lambda_1}(F^\Delta(t))^{\lambda_2}|\right]\Delta t \\ &\quad\leqslant \frac{(\frac{b-a}{2})^{\lambda_1}}{(2p)^{\frac{1}{p}}}\int_a^{\frac{a+b}{2}} \left(\frac{|F^\Delta(t)|^{p\lambda_1}}{p} + \frac{|F^\Delta(t)|^{q\lambda_2}}{q} + \frac{|G^\Delta(t)|^{p\lambda_1}}{p} + \frac{|G^\Delta(t)|^{q\lambda_2}}{q}\right)\Delta t.\end{aligned}$$

Let $s = t + \frac{b-a}{2}$. Then

$$\int_{\frac{a+b}{2}}^{b} \left[|(F(t))^{\lambda_1}(G^{\Delta}(t))^{\lambda_2}| + |(G(t))^{\lambda_1}(F^{\Delta}(t))^{\lambda_2}|\right] \Delta t \leqslant \frac{(\frac{b-a}{2})^{\lambda_1}}{(2p)^{\frac{1}{p}}} \int_{\frac{a+b}{2}}^{b} \left(\frac{|F^{\Delta}(t)|^{p\lambda_1}}{p} + \frac{|F^{\Delta}(t)|^{q\lambda_2}}{q} + \frac{|G^{\Delta}(t)|^{p\lambda_1}}{p} + \frac{|G^{\Delta}(t)|^{q\lambda_2}}{q} \right) \Delta t,$$

and the result (3.7) follows. □

Corollary 3.15. *When $\mathbb{T} = \mathbb{R}$ in Theorem 3.14, we get*

(1) *if $F(a) = G(a) = 0$, then we have*

$$\int_{a}^{b} \left(\left|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(F'(t))^{\lambda_2}\right| \right) dt \leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}} \int_{a}^{b} \left(\frac{|F'(t)|^{p\lambda_1}}{p} + \frac{|F'(t)|^{q\lambda_2}}{q} + \frac{|G'(t)|^{p\lambda_1}}{p} + \frac{|G'(t)|^{q\lambda_2}}{q} \right) dt, \tag{3.8}$$

when $p = q = 2$ in (3.8), we obtain

$$\int_{a}^{b} \left[|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}| + |(G(t))^{\lambda_1}(F'(t))^{\lambda_2}|\right] dt \leqslant \frac{(b-a)^{\lambda_1}}{4} \left[\int_{a}^{b} \left(|F'(t)|^{2\lambda_1} + |F'(t)|^{2\lambda_2} + |G'(t)|^{2\lambda_1} + |G'(t)|^{2\lambda_2}\right) dt \right],$$

which is inequality (3.8) *in* [26];

(2) *if $F(a) = F(b) = G(a) = G(b) = 0$, then we have*

$$\int_{a}^{b} \left(\left|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(F'(t))^{\lambda_2}\right| \right) dt \leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1}(2p)^{\frac{1}{p}}} \int_{a}^{b} \left(\frac{|F'(t)|^{p\lambda_1}}{p} + \frac{|F'(t)|^{q\lambda_2}}{q} + \frac{|G'(t)|^{p\lambda_1}}{p} + \frac{|G'(t)|^{q\lambda_2}}{q} \right) dt, \tag{3.9}$$

when $p = q = 2$ in (3.9), we obtain

$$\int_{a}^{b} \left[|(F(t))^{\lambda_1}(G'(t))^{\lambda_2}| + |(G(t))^{\lambda_1}(F'(t))^{\lambda_2}|\right] dt \leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1+2}} \left[\int_{a}^{b} \left(|F'(t)|^{2\lambda_1} + |F'(t)|^{2\lambda_2} + |G'(t)|^{2\lambda_1} + |G'(t)|^{2\lambda_2}\right) dt \right],$$

which is inequality (3.9) *in* [26].

Corollary 3.16. *When $\mathbb{T} = \mathbb{Z}$ in Theorem 3.14, then*

(1) *if $F(a) = G(a) = 0$, then we have*

$$\sum_{t=a}^{b-1} \left(\left|(F(t))^{\lambda_1}(\Delta G(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(\Delta F(t))^{\lambda_2}\right| \right) \leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}} \sum_{t=a}^{b-1} \left(\frac{|\Delta F(t)|^{p\lambda_1}}{p} + \frac{|\Delta F(t)|^{q\lambda_2}}{q} + \frac{|\Delta G(t)|^{p\lambda_1}}{p} + \frac{|\Delta G(t)|^{q\lambda_2}}{q} \right);$$

(2) *if* $F(a) = F(b) = G(a) = G(b) = 0$, *then we have*

$$\sum_{t=a}^{b-1} \left(\left|(F(t))^{\lambda_1}(\Delta G(t))^{\lambda_2}\right| + \left|(G(t))^{\lambda_1}(\Delta F(t))^{\lambda_2}\right|\right) \leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1}(2p)^{\frac{1}{p}}} \sum_{t=a}^{b-1} \left(\frac{|\Delta F(t)|^{p\lambda_1}}{p} + \frac{|\Delta F(t)|^{q\lambda_2}}{q} + \frac{|\Delta G(t)|^{p\lambda_1}}{p} + \frac{|\Delta G(t)|^{q\lambda_2}}{q}\right).$$

Corollary 3.17. *When* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem 3.14, then*

(1) *if* $F(a) = G(a) = 0$, *then we have*

$$\sum_{t=\log_q a}^{\log_q b-1} q^t \left(\left|(F(q^t))^{\lambda_1}(G^{\Delta}(q^t))^{\lambda_2}\right| + \left|(G(q^t))^{\lambda_1}(F^{\Delta}(q^t))^{\lambda_2}\right|\right) \leqslant \frac{(b-a)^{\lambda_1}}{(2p)^{\frac{1}{p}}} \sum_{t=\log_q a}^{\log_q b-1} q^t \left(\frac{|F^{\Delta}(q^t)|^{p\lambda_1}}{p} + \frac{|F^{\Delta}(q^t)|^{q\lambda_2}}{q} + \frac{|G^{\Delta}(q^t)|^{p\lambda_1}}{p} + \frac{|G^{\Delta}(q^t)|^{q\lambda_2}}{q}\right);$$

(2) *if* $F(a) = F(b) = G(a) = G(b) = 0$, *then we have*

$$\sum_{t=\log_q a}^{\log_q b-1} q^t \left(\left|(F(q^t))^{\lambda_1}(G^{\Delta}(q^t))^{\lambda_2}\right| + \left|(G(q^t))^{\lambda_1}(F^{\Delta}(q^t))^{\lambda_2}\right|\right) \leqslant \frac{(b-a)^{\lambda_1}}{2^{\lambda_1}(2p)^{\frac{1}{p}}} \sum_{t=\log_q a}^{\log_q b-1} q^t \left(\frac{|F^{\Delta}(q^t)|^{p\lambda_1}}{p} + \frac{|F^{\Delta}(q^t)|^{q\lambda_2}}{q} + \frac{|G^{\Delta}(q^t)|^{p\lambda_1}}{p} + \frac{|G^{\Delta}(q^t)|^{q\lambda_2}}{q}\right).$$

4. Opial type inequalities concerning higher order derivatives

Lemma 4.1. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$, $F \in C^k_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, $F^{\Delta^i}(a) = 0$, $i \in [0, k-1]$ *and* $\lambda_1, \lambda_2 \geqslant 0$. *Then*

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{\Delta^k}(t))^{\lambda_2}\right| \Delta t \leqslant H\,(c_0\lambda_2)^{c_o\lambda_2} \left\{\int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}, \tag{4.1}$$

where

$$H = \left(\int_a^b \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}}\,\Delta s\right\}^{\frac{(1-c_o)}{c_o}} \Delta t\right)^{c_o\lambda_1}, \quad c_o = \frac{1}{\lambda_1+\lambda_2}.$$

Proof. By hypothesis, we have

$$F(t) = \int_a^t h_{k-1}(t,s)F^{\Delta^k}(s)\Delta s, \qquad t \in [a, b]_{\mathbb{T}}.$$

Letting $c_o = \frac{1}{\lambda_1+\lambda_2}$, and using Hölder's inequality with indices $p = \frac{1}{1-c_o}$, $q = \frac{1}{c_o}$, we obtain

$$|F(t)| \leqslant \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}}\,\Delta s\right\}^{1-c_o} \left\{\int_a^t |F^{\Delta^k}(s)|^{\lambda_1+\lambda_2}\Delta s\right\}^{c_o}.$$

Then, we get

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{\Delta^k}(t))^{\lambda_2}\right| \Delta t$$
$$\leqslant \int_a^b \left[\left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s\right\}^{\lambda_1(1-c_o)} \left|F^{\Delta^k}(t)\right|^{\lambda_2} \left\{\int_a^t \left|F^{\Delta^k}(s)\right|^{\lambda_1+\lambda_2} \Delta s\right\}^{c_o\lambda_1}\right] \Delta t.$$

Consequently, applying Hölder's inequality again with indices $p = \frac{1}{\lambda_1 c_o}$, $q = \frac{1}{\lambda_2 c_o}$, we have

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{\Delta^k}(t))^{\lambda_2}\right| \Delta t$$
$$\leqslant \left(\int_a^b \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s\right\}^{\frac{(1-c_o)}{c_o}} \Delta t\right)^{c_o\lambda_1} \left(\int_a^b \left|F^{\Delta^k}(t)\right|^{\lambda_1+\lambda_1} \left\{\int_a^t \left|F^{\Delta^k}(s)\right|^{\lambda_1+\lambda_1} \Delta s\right\}^{\frac{\lambda_1}{\lambda_2}} \Delta t\right)^{c_o\lambda_2}$$
$$\leqslant H\left[\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)\left\{\int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}^{\frac{\lambda_1+\lambda_2}{\lambda_2}}\right]^{c_o\lambda_2}$$
$$\leqslant H\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{c_o\lambda_2}\left\{\int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\} \leqslant H\,(c_o\lambda_2)^{c_o\lambda_2}\left\{\int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}.$$

□

Corollary 4.2. *When $\mathbb{T} = \mathbb{R}$ in Lemma 4.1, $h_{k-1}(t,s) = \frac{(t-s)^{k-1}}{(k-1)!}$, so that*

$$H = \left(\int_a^b \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} ds\right\}^{\frac{(1-c_o)}{c_o}} dt\right)^{c_o\lambda_1}$$
$$= \left(\int_a^b \left\{\int_a^t \left|\frac{(t-s)^{k-1}}{(k-1)!}\right|^{\frac{1}{1-c_o}} ds\right\}^{\frac{(1-c_o)}{c_o}} dt\right)^{c_o\lambda_1}$$
$$= \left(\frac{1}{(k-1)!}\left(\frac{1-c_o}{k-c_o}\right)^{1-c_o}\right)^{\lambda_1}\left(\frac{c_o}{k}\right)^{c_o\lambda_1}(b-a)^{k\lambda_1},$$

then the inequality (4.1) reduces to the following inequality

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{(k)}(t))^{\lambda_2}\right| dt \leqslant C(b-a)^{\lambda_1 k}\int_a^b |F^{(k)}(t)|^{\lambda_1+\lambda_2}dt, \tag{4.2}$$

where

$$C = c_o\lambda_2^{\lambda_2 c_o}(k!)^{-\lambda_1}\left(\frac{k(1-c_o)}{k-c_o}\right)^{\lambda_1(1-c_o)},$$

which is inequality (4.1) *in* [26].

Corollary 4.3. *If $\mathbb{T} = \mathbb{Z}$ in Lemma 4.1, $h_{k-1}(t,s) = \binom{t-s}{k-1}$, then*

$$\sum_{t=a}^{b-1}\left|(F(t))^{\lambda_1}(\Delta^k F(t))^{\lambda_2}\right| \leqslant Q\,(c_o\lambda_2)^{c_o\lambda_2}\left\{\sum_{t=a}^{b-1}|\Delta^k F(t)|^{\lambda_1+\lambda_2}\right\},$$

where

$$Q = \left(\sum_{t=a}^{b-1} \left\{ \sum_{s=a}^{t-1} \left| \binom{t-s}{k-1} \right|^{\frac{1}{1-c_o}} \right\}^{\frac{(1-c_o)}{c_o}} \right)^{c_o \lambda_1},$$

and c_o *is defined by* (4.1).

Corollary 4.4. *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Lemma* 4.1, $h_{k-1}(t,s) = \prod_{\nu=0}^{k-2} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}$, *then*

$$\sum_{t=\log_q a}^{\log_q b-1} q^t \left| (F(q^t))^{\lambda_1} (F^{\Delta^k}(q^t))^{\lambda_2} \right| \leqslant \left((q-1) \sum_{t=\log_q a}^{\log_q b-1} q^t Y^{\frac{(1-c_o)}{c_o}}(q^t) \right)^{c_o \lambda_1} (c_0 \lambda_2)^{c_o \lambda_2} \left\{ \sum_{t=\log_q a}^{\log_q b-1} q^t |F^{\Delta^k}(q^t)|^{\lambda_1+\lambda_2} \right\}, \tag{4.3}$$

where

$$Y(t) = \left\{ (q-1) \sum_{s=\log_q a}^{\log_q t-1} q^s \left| \prod_{\nu=0}^{k-2} \frac{t - q^\nu q^s}{\sum_{\mu=0}^{\nu} q^\mu} \right|^{\frac{1}{1-c_o}} \right\},$$

and c_o *is defined by* (4.1).

Lemma 4.5. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$, $F \in C^k_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, $F^{(i)}(a) = F^{(i)}(b) = 0$, $i \in [0, k-1]$, *let* $K = \left[\int_a^{\frac{a+b}{2}} \left\{ \int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s \right\}^{\frac{(1-c_o)}{c_o}} \Delta t \right]^{c_o \lambda_1} = \left[\int_{\frac{a+b}{2}}^b \left\{ \int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s \right\}^{\frac{(1-c_o)}{c_o}} \Delta t \right]^{c_o \lambda_1}$ *and* λ_1, $\lambda_2 \geqslant 1$. *Then*

$$\int_a^b |(F(t))^{\lambda_1} (F^{\Delta^k}(t))^{\lambda_2}| \Delta t \leqslant K \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{c_o \lambda_2} \left\{ \int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2} \Delta t \right\}, \tag{4.4}$$

where c_o *is defined by* (4.1).

Proof. By hypothesis, we employ inequality (4.1) on interval $[a, \frac{a+b}{2}]$. Then

$$\int_a^{\frac{a+b}{2}} |(F(t))^{\lambda_1} (F^{\Delta^k}(t))^{\lambda_2}| \Delta t \leqslant \left(\int_a^{\frac{a+b}{2}} \left\{ \int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s \right\}^{\frac{(1-c_o)}{c_o}} \Delta t \right)^{c_o \lambda_1} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{c_o \lambda_2} \left\{ \int_a^{\frac{a+b}{2}} |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2} \Delta t \right\}.$$

Let $s = t + \frac{b-a}{2}$. Then

$$\int_{\frac{a+b}{2}}^b |(F(t))^{\lambda_1} (F^{\Delta^k}(t))^{\lambda_2}| \Delta t \leqslant \left(\int_{\frac{a+b}{2}}^b \left\{ \int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s \right\}^{\frac{(1-c_o)}{c_o}} \Delta t \right)^{c_o \lambda_1} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{c_o \lambda_2} \left\{ \int_{\frac{a+b}{2}}^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2} \Delta t \right\},$$

and the result follows by

$$\int_a^{\frac{a+b}{2}} |(F(t))^{\lambda_1} (F^{\Delta^k}(t))^{\lambda_2}| \Delta t + \int_{\frac{a+b}{2}}^b |(F(t))^{\lambda_1} (F^{\Delta^k}(t))^{\lambda_2}| \Delta t$$

$$\leqslant \left(\int_a^{\frac{a+b}{2}} \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s\right\}^{\frac{(1-c_o)}{c_o}} \Delta t\right)^{c_o\lambda_1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{c_o\lambda_2} \left\{\int_a^{\frac{a+b}{2}} |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}$$

$$+ \left(\int_{\frac{a+b}{2}}^{b} \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} \Delta s\right\}^{\frac{(1-c_o)}{c_o}} \Delta t\right)^{c_o\lambda_1} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{c_o\lambda_2} \left\{\int_{\frac{a+b}{2}}^{b} |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}$$

$$\leqslant K\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{c_o\lambda_2} \left\{\int_a^b |F^{\Delta^k}(t)|^{\lambda_1+\lambda_2}\Delta t\right\}.$$

□

Corollary 4.6. *When* $\mathbb{T}=\mathbb{R}$ *in Lemma 4.5,* $h_{k-1}(t,s)=\frac{(t-s)^{k-1}}{(k-1)!}$*, so that*

$$K=\left(\int_a^{\frac{a+b}{2}} \left\{\int_a^t |h_{k-1}(t,s)|^{\frac{1}{1-c_o}} ds\right\}^{\frac{(1-c_o)}{c_o}} dt\right)^{c_o\lambda_1}$$

$$=\left(\int_a^{\frac{a+b}{2}} \left\{\int_a^t \left(\frac{(t-s)^{k-1}}{(k-1)!}\right)^{\frac{1}{1-c_o}} ds\right\}^{\frac{(1-c_o)}{c_o}} dt\right)^{c_o\lambda_1}$$

$$=\frac{1}{(k-1)!}\left(\frac{1-c_o}{k-c_o}\right)^{1-c_o}\left(\frac{c_o}{k}\right)^{c_o\lambda_1}\left(\frac{b-a}{2}\right)^{k\lambda_1},$$

then inequality (4.4) becomes

$$\int_a^b \left|(F(t))^{\lambda_1}(F^{(k)}(t))^{\lambda_2}\right| dt \leqslant C\left(\frac{b-a}{2}\right)^{\lambda_1 k}\int_a^b |F^{(k)}(t)|^{\lambda_1+\lambda_2}dt,$$

where C *is defined in (4.2), which is inequality* (4.2) *in* [26].

Corollary 4.7. *If* $\mathbb{T}=\mathbb{Z}$ *in Lemma 4.5,* $h_{k-1}(t,s)=\binom{t-s}{k-1}$*, then*

$$\sum_{t=a}^{b-1}\left|(F(t))^{\lambda_1}(\Delta^k F(t))^{\lambda_2}\right| \leqslant A\,(c_0\lambda_2)^{c_o\lambda_2}\left\{\sum_{t=a}^{b-1}|\Delta^k F(t)|^{\lambda_1+\lambda_2}\right\},$$

where

$$A=\left(\sum_{t=a}^{\frac{b-a}{2}-1}\left\{\sum_{s=a}^{t-1}\left|\binom{t-s}{k-1}\right|^{\frac{1}{1-c_o}}\right\}^{\frac{(1-c_o)}{c_o}}\right)^{c_o\lambda_1},$$

and c_o *is defined by (4.1).*

Corollary 4.8. *If* $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ *in Lemma 4.5,* $h_{k-1}(t,s)=\prod_{\nu=0}^{k-2}\frac{t-q^{\nu}s}{\sum_{\mu=0}^{\nu}q^{\mu}}$*, then*

$$\sum_{t=\log_q a}^{\log_q b-1} q^t\left|(F(q^t))^{\lambda_1}(F^{\Delta^k}(q^t))^{\lambda_2}\right|$$

$$\leqslant\left((q-1)\sum_{t=\log_q a}^{\log_q \frac{b-a}{2}-1} q^t Y^{\frac{(1-c_o)}{c_o}}(q^t)\right)^{c_o\lambda_1}(c_0\lambda_2)^{c_o\lambda_2}\left\{\sum_{t=\log_q a}^{\log_q b-1} q^t|F^{\Delta^k}(q^t)|^{\lambda_1+\lambda_2}\right\},$$

where $Y(t)$ *is defined by (4.3) and* c_o *is defined by (4.1).*

Theorem 4.9. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$, $F \in C^{n-1}_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *be such that* $F^{\Delta^i}(a) = 0$, $0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. *Further,* $F^{\Delta^{n-1}} \in C_{rd}$ *and* $\int_a^b |F^{\Delta^n}(t)|^q \Delta t < \infty$. *Then the following inequality holds*

$$\int_a^b |F(t)F^{\Delta^n}(t)|\Delta t \leqslant A_n \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[\int_a^b |F^{\Delta^n}(t)|^q \Delta t\right]^{\frac{2}{q}}, \tag{4.5}$$

where

$$A_n = \left[\int_a^b \left(\int_a^t |h_{n-1}(t,s)|^p \,\Delta s\right) \Delta t\right]^{\frac{1}{p}}.$$

Proof. In view of the assumptions on $F(t)$, for any $t \in [a, b]_{\mathbb{T}}$, we have

$$F(t) = \int_a^t h_{n-1}(t,s) F^{\Delta^n}(s)\Delta s. \tag{4.6}$$

Multiplying (4.6) by $F^{\Delta^n}(t)$ and using Hölder's inequality with indices p, q we obtain

$$|F(t)F^{\Delta^n}(t)|\Delta t \leqslant |F^{\Delta^n}(t)| \left(\int_a^t |h_{n-1}(t,s)|^p \,\Delta s\right)^{\frac{1}{p}} \left(\int_a^t |F^{\Delta^n}(s)|^q \Delta s\right)^{\frac{1}{q}}. \tag{4.7}$$

Thus, integrating (4.7) from a to b and applying Hölder's inequality to the right side again, we obtain

$$\begin{aligned}
\int_a^b |F(t)F^{\Delta^n}(t)|\Delta t &\leqslant \left[\int_a^b \left(\int_a^t |h_{n-1}(t,s)|^p \,\Delta s\right) \Delta t\right]^{\frac{1}{p}} \left[\int_a^b |F^{\Delta^n}(t)|^q \left(\int_a^t |F^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right]^{\frac{1}{q}} \\
&\leqslant \left[\int_a^b \left(\int_a^t |h_{n-1}(t,s)|^p \,\Delta s\right) \Delta t\right]^{\frac{1}{p}} \left[\int_a^b \frac{1}{2}\left[\left(\int_a^t |F^{\Delta^n}(s)|^q \Delta s\right)^2\right]^{\Delta} \Delta t\right]^{\frac{1}{q}} \\
&\leqslant \left[\int_a^b \left(\int_a^t |h_{n-1}(t,s)|^p \,\Delta s\right) \Delta t\right]^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[\int_a^b |F^{\Delta^n}(t)|^q \Delta t\right]^{\frac{2}{q}}.
\end{aligned}$$

□

Corollary 4.10. *When* $\mathbb{T} = \mathbb{R}$ *in Theorem 4.9,* $h_{n-1}(t,s) = \frac{(t-s)^{n-1}}{(n-1)!}$, *so that*

$$\begin{aligned}
A_n &= \left(\int_a^b \left\{\int_a^t |h_{n-1}(t,s)|^p \,ds\right\} dt\right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left\{\int_a^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right)^p ds\right\} dt\right)^{\frac{1}{p}} \\
&= \frac{1}{(n-1)!} \left(\int_a^b \left[\frac{(t-a)^{(n-1)p+1}}{(n-1)p+1}\right] dt\right)^{\frac{1}{p}} \\
&= \frac{1}{(n-1)!} \left(\frac{(b-a)^{(n-1)+\frac{2}{p}}}{([(n-1)p+1][(n-1)p+2])^{\frac{1}{p}}}\right),
\end{aligned}$$

then the inequality (4.5) *becomes*

$$\int_a^b |F(t)F^{(n)}(t)|dt \leqslant \frac{1}{(n-1)!} \left(\frac{(b-a)^{(n-1)+\frac{2}{p}}}{([(n-1)p+1][(n-1)p+2])^{\frac{1}{p}}}\right) \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[\int_a^b |F^{(n)}(t)|^q dt\right]^{\frac{2}{q}}. \tag{4.8}$$

Corollary 4.11. *When* $p = q = 2$, $a = 0$ *in* (4.8), *we get*

$$\int_0^b \left|F(t)F^{(n)}(t)\right| dt \leqslant \frac{b^n}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}}\left[\int_0^b |F^{(n)}(t)|^2 dt\right],$$

which is the inequality in Theorem 3.3.1 of [3].

Corollary 4.12. *When* $\mathbb{T} = \mathbb{Z}$ *in Theorem* 4.9, $h_{n-1}(t,s) = \binom{t-s}{n-1}$, *then*

$$\sum_{t=a}^{b-1} |F(t)\Delta^n F(t)| \leqslant \left[\sum_{t=a}^{b-1}\left(\sum_{s=a}^{t-1}\left|\binom{t-s}{n-1}\right|^p\right)\right]^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\sum_{t=a}^{b-1}|\Delta^n F(t)|^q\right]^{\frac{2}{q}}.$$

Corollary 4.13. *When* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 4.9, $h_{n-1}(t,s) = \prod_{\nu=0}^{n-2}\frac{t-q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}$, *then*

$$\begin{aligned}&\sum_{t=\log_q a}^{\log_q b-1} q^t|F(q^t)F^{\Delta^n}(q^t)| \\ &\leqslant (q-1)^{\frac{2}{q}-1}\left((q-1)\sum_{t=\log_q a}^{\log_q b-1} q^t B_n(q^t)\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\sum_{t=\log_q a}^{\log_q b-1} q^t|F^{\Delta^n}(q^t)|^q\right]^{\frac{2}{q}},\end{aligned} \tag{4.9}$$

where

$$B_n(t) = \left((q-1)\sum_{s=\log_q a}^{\log_q t-1} q^s\left|\prod_{\nu=0}^{n-2}\frac{t-q^\nu q^s}{\sum_{\mu=0}^{\nu} q^\mu}\right|^p\right).$$

Theorem 4.14. *Let* $\mathbb{T}$ *be a time scale with* $a, b \in \mathbb{T}$, *for* $j = 1, 2$, $F_j(t) \in C_{rd}^{n-1}([a,b]_{\mathbb{T}}, \mathbb{R})$, *such that* $F_j^{\Delta^i}(a) = 0$, $0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. *Further, let* $F_j^{\Delta^{n-1}}(t)$ *be rd-continuous, and* $\int_a^b |F_j^{\Delta^n}(t)|^q \Delta t \leqslant \infty$. *Then the following inequality holds:*

$$\int_a^b \left[|F_1(t)F_2^{\Delta^n}(t)| + |F_1^{\Delta^n}(t)F_2(t)|\right]\Delta t \leqslant \frac{2^{\frac{q-1}{q}}A_n}{q}\int_a^b\left[|F_1^{\Delta^n}(t)|^q + |F_2^{\Delta^n}(t)|^q\right]\Delta t, \tag{4.10}$$

where A_n *is defined by* (4.5).

Proof. Following the proof of Theorem 4.9 it is straight forward to obtain

$$\int_a^b |F_1(t)F_2^{\Delta^n}(t)|\Delta t \leqslant A_n\left[\int_a^b |F_2^{\Delta^n}(t)|^q\left(\int_a^t |F_1^{\Delta^n}(s)|^q\Delta s\right)\Delta t\right]^{\frac{1}{q}},$$

and

$$\int_a^b |F_1^{\Delta^n}(t)F_2(t)|\Delta t \leqslant A_n\left[\int_a^b |F_1^{\Delta^n}(t)|^q\left(\int_a^t |F_2^{\Delta^n}(s)|^q\Delta s\right)\Delta t\right]^{\frac{1}{q}}.$$

Thus in view of the elementary inequalities $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \leqslant \frac{\alpha}{p} + \frac{\beta}{q}$, $\alpha, \beta \geqslant 0$, $0 \leqslant \frac{1}{p} \leqslant 1$, $\alpha^{\frac{1}{q}} + \beta^{\frac{1}{q}} \leqslant 2^{\frac{q-1}{q}}(\alpha+\beta)^{\frac{1}{q}}$, we have

$$\int_a^b \left[|F_1(t)F_2^{\Delta^n}(t)| + |F_1^{\Delta^n}(t)F_2(t)|\right]\Delta t$$

$$\leqslant A_n \left[\left[\int_a^b |F_2^{\Delta^n}(t)|^q \left(\int_a^t |F_1^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right]^{\frac{1}{q}} + \left[\int_a^b |F_1^{\Delta^n}(t)|^q \left(\int_a^t |F_2^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right]^{\frac{1}{q}}\right]$$

$$\leqslant 2^{\frac{q-1}{q}} A_n \left[\left[\int_a^b |F_2^{\Delta^n}(t)|^q \left(\int_a^t |F_1^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right] + \left[\int_a^b |F_1^{\Delta^n}(t)|^q \left(\int_a^t |F_2^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right]\right]^{\frac{1}{q}}$$

$$\leqslant 2^{\frac{q-1}{q}} A_n \left[\left[\int_a^b |F_2^{\Delta^n}(t)|^q \left(\int_a^t |F_1^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right] + \left[\int_a^b |F_1^{\Delta^n}(t)|^q \left(\int_a^{\sigma(t)} |F_2^{\Delta^n}(s)|^q \Delta s\right) \Delta t\right]\right]^{\frac{1}{q}}$$

$$\leqslant 2^{\frac{q-1}{q}} A_n \left[\int_a^b \left[\left(\int_a^t |F_1^{\Delta^n}(s)|^q \Delta s\right)\left(\int_a^t |F_2^{\Delta^n}(s)|^q \Delta s\right)\right]^{\Delta} \Delta t\right]^{\frac{1}{q}}$$

$$\leqslant 2^{\frac{q-1}{q}} A_n \left[\left(\int_a^b |F_1^{\Delta^n}(t)|^q \Delta t\right)\left(\int_a^b |F_2^{\Delta^n}(t)|^q \Delta t\right)\right]^{\frac{1}{q}}$$

$$\leqslant \frac{2^{\frac{q-1}{q}} A_n}{q} \int_a^b \left[|F_1^{\Delta^n}(t)|^q + |F_2^{\Delta^n}(t)|^q\right] \Delta t.$$

□

Corollary 4.15. *If $\mathbb{T} = \mathbb{R}$ in Theorem 4.14, $h_{n-1}(t,s) = \frac{(t-s)^{n-1}}{(n-1)!}$ as Corollary 4.10, then the inequality (4.10) becomes*

$$\int_a^b \left[|F_1(t)F_2^{(n)}(t)| + |F_1^{(n)}(t)F_2(t)|\right] dt \leqslant \frac{2^{\frac{q-1}{q}}}{q(n-1)!}\left(\frac{(b-a)^{(n-1)+\frac{2}{p}}}{([(n-1)p+1][(n-1)p+2])^{\frac{1}{p}}}\right)\int_a^b \left[|F_1^{(n)}(t)|^q + |F_2^{(n)}(t)|^q\right] dt. \tag{4.11}$$

Corollary 4.16. *When $p = q = 2$ in (4.11), we get*

$$\int_a^b \left[\left|F_1(t)F_2^{(n)}(t)\right| + \left|F_1^{(n)}(t)F_2(t)\right|\right] dt \leqslant \frac{(b-a)^n}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}} \int_a^b \left(|F_1^{(n)}(t)|^2 + |F_2^{(n)}(t)|^2\right) dt,$$

which is the inequality in Lemma 4.9 of [26].

Corollary 4.17. *When $p = q = 2$, $a = 0$ in (4.11), we get*

$$\int_0^b \left[\left|F_1(t)F_2^{(n)}(t)\right| + \left|F_1^{(n)}(t)F_2(t)\right|\right] dt \leqslant \frac{b^n}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}} \int_0^b \left(|F_1^{(n)}(t)|^2 + |F_2^{(n)}(t)|^2\right) dt,$$

which is the inequality in Theorem 3.8.1 of [3].

Corollary 4.18. *If $\mathbb{T} = \mathbb{Z}$ in Theorem 4.14, $h_{n-1}(t,s) = \binom{t-s}{n-1}$, then*

$$\sum_{t=a}^{b-1} |F_1(t)\Delta^n F_2(t)| + |(\Delta^n F_1(t))F_2(t)| \leqslant \frac{2^{\frac{q-1}{q}}}{q}\left[\sum_{t=a}^{b-1}\left(\sum_{s=a}^{t-1}\left|\binom{t-s}{n-1}\right|^p\right)\right]^{\frac{1}{p}} \sum_{t=a}^{b-1}\left[|\Delta^n F_1(t)|^q + |\Delta^n F_2(t)|^q\right].$$

Corollary 4.19. *If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 4.14, $h_{n-1}(t,s) = \prod_{\nu=0}^{n-2} \frac{t-q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}$, then*

$$\sum_{t=\log_q a}^{\log_q b-1} q^t \left(|F_1(q^t)F_2^{\Delta^n}(q^t)| + |(F_1^{\Delta^n}(q^t))F_2(q^t)|\right)$$

$$\leqslant \frac{2^{\frac{q-1}{q}}}{q}\left((q-1)\sum_{t=\log_q a}^{\log_q b-1} q^t B_n(q^t)\right)^{\frac{1}{p}} \sum_{t=\log_q a}^{\log_q b-1} q^t \left(|F_1^{\Delta^n}(q^t)|^q + |F_2^{\Delta^n}(q^t)|^q\right),$$

where $B_n(t)$ *is defined by* (4.9).

5. Conclusion

In this paper we have proved some new generalizations of dynamic Opial type inequalities on time scales. These inequalities have certain conditions that have not been studied before. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus.

Authors' contributions

Y. A. A. Elsaid and A. A. El-Deeb wrote the main manuscript text and conceptualization and A. A. S. Zaghrout did formal analysis and investigation. All authors read and reviewed the manuscript.

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