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On Hermite-Hadamard and Ostrowski type inequalities for strongly convex functions via quantum calculus with applications

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Abstract

In this study, we use q_a- and q^b-integrals to prove Hermite-Hadamard and Ostrowski type inequalities for strongly convex functions. The relationship between the results and comparable results in the related literature is also discussed in this study. Furthermore, this study also presents how the newly established inequalities can be utilized in special means, including arithmetic mean and logarithmic ones.

Keywords: Ostrowski type inequalities, Hermite-Hadamard type inequalities, strongly convex functions, quantum calculus, q_a-integral, q^b-integral.

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1. Introduction

Recently, many researchers have been actively involved with quantum calculus (q-calculus) with no limits since it was introduced by Euler (1707-1783), who created Newton's infinite series. In 1910, Jackson [40, 41] extended the principle of Euler by providing the definition of the q-derivative and q-integral of a continuous function on the interval $(0, \infty)$ mainly focusing on obtaining the q-analogues of mathematical objects which was recaptured by taking $q \to 1$. Recently, the q-calculus has had many applications in various mathematical and physical landscapes, including theories of numbers, combinatorics, orthogonal polynomials, fundamental hypergeometric functions, theories of quantum, etc. Many researchers have paid more attention to consider it an in-corporative conjunction between mathematics and physics. For ones who are interested in the new development of quantum calculus, please refer to [2, 3, 14, 17, 30, 34–39, 48, 59, 67] and Kac and Cheung [43] for newly applications of the q-calculus and inequalities in the q-calculus theories.

In 2013, Tariboon and Ntouyas [71, 72] proposed the q-calculus of a continuous function on finite intervals and tested some of its qualifications, which are called q_a-calculus. There are significant integral inequalities such as Hermite-Hadamard type inequalities, Simpson type inequalities, Ostrowski

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type inequalities, Fejér type inequalities, Newton type inequalities, Hanh integral inequalities, Hermite-Hadamard-like inequalities, see [8, 15, 22, 24, 29, 33, 42, 44, 50, 61, 62, 73–76] for more details and references therein.

In 2020, Bermudo et al. [19] newly presented the q-calculus of a continuous function on finite intervals known as q^b-calculus. Many integral inequalities have been investigated using quantum integrals of functions of various types. For example, in [5, 6, 11, 12, 19, 20, 22, 42, 52, 63, 64, 66], the authors employed quantum calculus to attest inequalities of Hermite-Hadamard type integral together with their left-right estimates for convex and coordinated convex functions. In [51], Noor et al. proposed a generalized type of inequalities of quantum Hermite-Hadamard integral. In the generalized quasi-convex functions, Nwaeze et al. tested the definite parameterized inequalities of the quantum integral in [53]. Khan et al. tested the inequalities of quantum Hermite-Hadamard integral by using the green function [1]. Asawasamrit et al. [16] also tested the inequalities of Hermite-Hadamard and Ostrowski type for s-convex functions in the second sense using q-calculus. Many researchers continuously developed quantum calculus, such as inequalities of quantum Simpson's and quantum Newton's types for convex and coordinated convex functions, as shown in [4, 7, 24, 28, 73].

Many mathematicians have paid great attention to many studies of various types of integral inequalities focused on mathematics in terms of pure and applied ones. As the mathematical discoveries of Ostrowski [54], it was developed based on the classic integral inequality as presented below.

Theorem 1.1 ([54])**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be the interval, be a differentiable mapping in I° (the interior of the interval I), where $f' \in L[\alpha, \beta]$ and $\alpha, \beta \in I$ with $\alpha < \beta$. If $|f'| \leqslant \mu$, then the following inequality holds:*

$$\left| f(\varphi) - \frac{1}{\beta - \alpha} \int_\alpha^\beta f(\psi) d\psi \right| \leqslant \frac{\mu \left((\varphi - \alpha)^2 + (\beta - \varphi)^2 \right)}{2(\beta - \alpha)}. \tag{1.1}$$

This inequality is generally recognized as the inequality of the Ostrowski. Some examples that can be generalized, improved, and extended, the inequality of (1.1) are shown in [10, 55] and the references therein.

The inequality of Ostrowski has generally been investigated in many mathematics fields, including numerical analysis and probability. Numerous researchers paid more attention to the extensions and generalizations of the inequality of Ostrowski for the bounded variation, convex, coordinated convex functions, monotonic, Lipschitzian, etc. Moreover, the results related to the inequality of Ostrowski were presented as in [9, 18, 23, 27, 31, 32, 45, 46, 56–58, 68].

For a convex function f on $[\alpha, \beta]$, inequality of the Hermite-Hadamard inequality. This inequality was published by Hermite in 1883 and independently, by Hadamard in 1893, as presented below:

$$f\left(\frac{\alpha + \beta}{2}\right) \leqslant \frac{1}{(\beta - \alpha)} \int_\alpha^\beta f(\varphi) d\varphi \leqslant \frac{f(\alpha) + f(\beta)}{2}. \tag{1.2}$$

This inequality estimates the mean value of a convex function f, and it is important to note that it also refines the Jensen inequality. The interested reader is referred to [25, 26, 65, 77, 78] and references therein for more information and other extensions of Hermite-Hadamard inequality.

Polyak introduced the strongly convex functions in [60], which is important in theories of mathematical programming and application of mathematical models since their qualifications and utilizations can be seen in many studies as in [13, 47, 49, 69].

Definition 1.2 ([60])**.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be called strongly convex function with modulus $\nu > 0$, if

$$f(t\varphi + (1-t)\omega) \leqslant tf(\varphi) + (1-t)f(\omega) - \nu t(1-t)(\varphi - \omega)^2$$

for all $\varphi, \omega \in I$ and $t \in [0, 1]$.

The following is the structure of this paper. A brief overview of the concepts of q-calculus, as well as some related works, is given in Section 2. In Section 3, we show the relationship between the results presented here and comparable results in the literature by proving the inequalities of quantum Hermite-Hadamard for strongly convex functions. The inequalities of quantum Ostrowski type for strongly convex functions are presented in Section 4. Some applications to special means are given in Section 5. Section 6 concludes with some recommendations for future studies.

2. Preliminaries

This section presents the employed basic concepts of the q-calculus. Let $0 < q < 1$ be a constant thoroughly. Set the notation:

$$[\zeta]_q := \frac{1-q^{\zeta}}{1-q} = 1 + q + q^2 + \cdots + q^{\zeta-1}, \quad \zeta \in \mathbb{N},$$

which means the q-number of ζ, see [43] for more details.

Definition 2.1. If $f : [a, b] \to \mathbb{R}$ is a continuous function. The q_a-derivative of f at $\varphi \in [a, b]$ is, consequently, defined by the following expression

$$_aD_q f(\varphi) = \frac{f(\varphi) - f(q\varphi + (1-q)a)}{(1-q)(\varphi - a)}, \quad \varphi \neq a. \tag{2.1}$$

If $\varphi = a$, we define

$$_aD_q f(a) = \lim_{\varphi \to a} {}_aD_q f(\varphi),$$

whether it exists and is finite, see [43, 71] for more details. If we take $a = 0$ in (2.1), then we have $_0D_q f(\varphi) = D_q f(\varphi)$, which can be decreased to

$$D_q f(\varphi) = \frac{f(\varphi) - f(q\varphi)}{(1-q)\varphi}, \quad \varphi \neq 0,$$

which means the q-Jackson derivative, see [40, 41] for more details.

Example 2.2. Define function $f : [a, b] \to \mathbb{R}$ by $f(t) = \delta t^2 + \kappa$, where δ, κ are the constants and $t \in [a, b]$. Using Definition 2.1, then we obtain

$$_aD_q(\delta t^2 + \kappa) = \frac{(\delta t^2 + \kappa) - \left(\delta(qt + (1-q)a)^2 + \kappa\right)}{(1-q)(t-a)} = \frac{\delta(t^2 - a^2) + \delta q(t-a)^2}{(t-a)} = \delta(t+a) + \delta q(t-a).$$

Definition 2.3. If $f : [a, b] \to \mathbb{R}$ is a continuous function, then, the q^b-derivative of f at $\varphi \in [a, b]$ is defined by

$$^bD_q f(\varphi) = \frac{f(q\varphi + (1-q)b) - f(\varphi)}{(1-q)(b - \varphi)}, \quad \varphi \neq b. \tag{2.2}$$

If $\varphi = b$, we define

$$^bD_q f(b) = \lim_{\varphi \to b} {}^bD_q f(\varphi),$$

whether it exists and is finite, see [19] for more details.

Example 2.4. Define function $f : [a, b] \to \mathbb{R}$ by $f(t) = \delta t^2 + \kappa$, where δ, κ are the constants and $t \in [a, b]$. Using Definition 2.3, then we obtain

$$^bD_q(\delta t^2 + \kappa) = \frac{\left(\delta(qt + (1-q)b)^2 + \kappa\right) - (\delta t^2 + \kappa)}{(1-q)(t-b)} = \frac{\delta(b^2 - t^2) - \delta q(b-t)^2}{(b-t)} = \delta(b+t) - \delta q(b-t).$$

Definition 2.5. If $f : [a, b] \to \mathbb{R}$ is a continuous function, then, the q_a-integral on $\varphi \in [a, b]$ is defined by

$$\int_a^{\varphi} f(t)_a d_q t = (1-q)(\varphi - a) \sum_{n=0}^{\infty} q^n f(q^n \varphi + (1-q^n)a). \tag{2.3}$$

Note that, if $a = 0$, then (2.3) reduces to

$$\int_0^{\varphi} f(t)_0 d_q t = \int_0^{\varphi} f(t) d_q t = (1-q)\varphi \sum_{n=0}^{\infty} q^n f(q^n \varphi),$$

which means the q_a-integral, see [43, 71] for more details.

Example 2.6. Define function $f : [a, b] \to \mathbb{R}$ by $f(t) = \xi t^2$, where ξ is a constant and $\psi \in [a, b]$. Using Definition 2.5, then we obtain

$$\begin{aligned}\int_a^{\psi} f(t)_a d_q t = \int_a^{\psi} \xi t_a^2 d_q t &= \xi(1-q)(\psi - a) \sum_{n=0}^{\infty} q^n \left(q^n \psi + (1-q^n)a\right) \\ &= \xi(\psi - a)\left(\frac{(\psi-a)^2}{1+q+q^2} + \frac{2a(\psi-a)}{1+q} + a^2\right) \\ &= \xi(\psi - a)\left(\frac{(\psi-a)^2}{[3]_q} + \frac{2a(\psi-a)}{[2]_q} + a^2\right).\end{aligned}$$

Definition 2.7. If $f : [a, b] \to \mathbb{R}$ is a continuous function, the q^b-integral on $\varphi \in [a, b]$ is, consequently, defined by

$$\int_{\varphi}^{b} f(t)^b d_q t = (1-q)(b - \varphi) \sum_{n=0}^{\infty} q^n f(q^n \varphi + (1-q^n)b). \tag{2.4}$$

Note that, if $\varphi = 0$, then (2.4) reduces to

$$\int_0^{b} f(t)^b d_q t = (1-q)b \sum_{n=0}^{\infty} q^n f((1-q^n)b),$$

which means the q^b-integral, see [19] for more details.

Example 2.8. Set a function $f : [a, b] \to \mathbb{R}$ by $f(t) = \xi t^2$, where ξ is a constant and $\psi \in [a, b]$. Using Definition 2.7, then we obtain

$$\begin{aligned}\int_{\psi}^{b} f(t)^b d_q t = \int_{\psi}^{b} \xi t^{2\ b} d_q t &= \xi(1-q)(b-\psi) \sum_{n=0}^{\infty} q^n \left(q^n \psi + (1-q^n)b\right) \\ &= \xi(b-\psi)\left(\frac{(b-\psi)^2}{1+q+q^2} - \frac{2b(b-\psi)}{1+q} + b^2\right) \\ &= \xi(b-\psi)\left(\frac{(b-\psi)^2}{[3]_q} + \frac{2b(b-\psi)}{[2]_q} + b^2\right).\end{aligned}$$

In 2020, Bermudo et al. [19] developed the quantum Hermite-Hadamard type inequality as follows.

Theorem 2.9. *If $f : [a, b] \to \mathbb{R}$ is the convex function, where $\varphi \in [a, b]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)}\left|\int_a^b f(\varphi)_a d_q \varphi + \int_a^b f(\varphi)^b d_q \varphi\right| \leqslant \frac{1}{2}\left(f(a) + f(b)\right). \tag{2.5}$$

In 2021, Budak et al. [21] studied the Ostrowski type inequalities by using the concepts of q-calculus as follows.

Theorem 2.10. *Let a function* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$*, such as* ${}^{b}D_q f$ *and* ${}_{a}D_q f$ *be two continuous and integrable functions on* $[a, b]$*. If* $|{}^{b}D_q f|, |{}_{a}D_q f| \leqslant \mu$*, where* $\varphi, \psi \in [a, b]$*, then the following inequality holds:*

$$\left| f(\varphi) - \frac{1}{b-a}\left(\int_a^{\varphi} f(\psi)\,{}_a d_q \psi + \int_{\varphi}^{b} f(\psi)\,{}^b d_q \psi\right)\right| \leqslant \frac{q\mu}{(b-a)}\left(\frac{(\varphi-a)^2+(b-\varphi)^2}{[2]_q}\right). \tag{2.6}$$

3. The Hermite-Hadamard type inequalities

In this section, we employed the concepts of q-calculus to prove the Hermite-Hadamard type inequalities for strongly convex functions.

Theorem 3.1. *Let* $f : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ *be strongly convex function on* $[a, b]$ *with respect to* $\nu > 0$*, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)}\left(\int_a^b f(\varphi)\,{}_a d_q \varphi + \int_a^b f(\varphi)\,{}^b d_q \varphi\right) - \frac{\nu(b-a)^2}{4}\left(\frac{q^3-2q^2+2q-1}{[2]_q[3]_q}\right) \tag{3.1}$$

$$\leqslant \frac{f(a)+f(b)}{2} - \frac{\nu q^2 (b-a)^2}{[2]_q[3]_q}. \tag{3.2}$$

Proof. Since f is strongly convex function on $\mathbb{R}^+$ for all $\varphi, \omega \in \mathbb{R}^+, \nu > 0$ and $t \in [0, 1]$, then we obtain

$$f(t\varphi + (1-t)\omega) \leqslant tf(\varphi) + (1-t)f(\omega) - \nu t(1-t)(\varphi-\omega)^2, \tag{3.3}$$

and

$$2f\left(\frac{\varphi+\omega}{2}\right) \leqslant f(\varphi) + f(\omega) - \frac{\nu(\varphi-\omega)^2}{2}. \tag{3.4}$$

By putting $\varphi = tb + (1-t)a$ and $\omega = ta + (1-t)b$ in (3.4), we get

$$2f\left(\frac{\varphi+\omega}{2}\right) \leqslant f(tb + (1-t)a) + f(ta + (1-t)b) - \frac{\nu\left(4t^2-4t+1\right)(b-a)^2}{2}. \tag{3.5}$$

From Definitions 2.5 and 2.7, we get

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)}\left(\int_a^b f(\varphi)\,{}_a d_q \varphi + \int_a^b f(\varphi)\,{}^b d_q \varphi\right) - \frac{\nu(b-a)^2}{4}\left(\frac{q^3-2q^2+2q-1}{[2]_q[3]_q}\right), \tag{3.6}$$

and the inequality of (3.1) holds. For the inequality of (3.2), we employed the strongly convexity and we obtain

$$f(tb + (1-t)a) \leqslant tf(b) + (1-t)f(a) - \nu t(1-t)(b-a)^2, \tag{3.7}$$

and

$$f(ta + (1-t)b) \leqslant tf(a) + (1-t)f(b) - \nu t(1-t)(b-a)^2. \tag{3.8}$$

To add (3.7) and (3.8), from Definitions 2.5 and 2.7, then we have

$$\frac{1}{2(b-a)}\left(\int_a^b f(\varphi)\,{}_a d_q \varphi + \int_a^b f(\varphi)\,{}^b d_q \varphi\right) - \frac{\nu(b-a)^2}{4}\left(\frac{q^3-2q^2+2q-1}{[2]_q[3]_q}\right)$$
$$\leqslant \frac{f(a)+f(b)}{2} - \frac{\nu q^2(b-a)^2}{[2]_q[3]_q}.$$

This completes the proof. □

Remark 3.2. If we take the limit as $q \to 1^-$ in Theorem 3.1, then we obtain

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{(b-a)}\int_a^b f(\varphi)d\varphi + \frac{\nu(b-a)^2}{12} \leqslant \frac{f(a)+f(b)}{2} - \frac{\nu(b-a)^2}{6}. \tag{3.9}$$

Remark 3.3. In Remark 3.2, if we take $\nu \to 0^+$, then the inequality (3.9) becomes the inequality (1.2).

4. The Ostrowski's type inequalities

In this section, we investigate Ostrowski's type inequalities for strongly convex functions by using the concepts of q-calculus. We employ the following lemma to investigate the new results.

Lemma 4.1 ([21]). *Let $f : \mathbb{R} \to \mathbb{R}$ be function. If $|^bD_q f|$ and $|_aD_q f|$ are two continuous and integrable functions on $[a, b]$, then, for all $\varphi \in [a, b]$, the following inequality holds:*

$$\begin{aligned} f(\varphi) - \frac{1}{b-a}\left|\int_a^\varphi f(t)\,{}_ad_qt + \int_\varphi^b f(t)\,{}^bd_qt\right| &= \frac{q(\varphi-a)^2}{b-a}\int_0^1 t\,{}_aD_qf(t\varphi+(1-t)a)\,{}_0d_qt \\ &\quad - \frac{q(b-\varphi)^2}{b-a}\int_0^1 t\,{}^bD_qf(t\varphi+(1-t)b)\,{}_0d_qt. \end{aligned}$$

Theorem 4.2. *Let $f : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function. If $|_aD_q f|$ and $|^bD_q f|$ are strongly convex functions on $[a, b]$ with respect to $\nu > 0$, then, for all $\varphi \in [a, b]$, the following inequality holds:*

$$\begin{aligned} &\left|f(\varphi) - \frac{1}{b-a}\left(\int_a^\varphi f(t)\,{}_ad_qt + \int_\varphi^b f(t)\,{}^bd_qt\right)\right| \\ &\leqslant \frac{q(\varphi-a)^2}{(b-a)}\left(\frac{|{}_aD_qf(\varphi)|}{[3]_q} + \frac{q^2|{}_aD_qf(a)|}{[2]_q[3]_q} - \frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right) \\ &\quad + \frac{q(b-\varphi)^2}{(b-a)}\left(\frac{|{}^bD_qf(\varphi)|}{[3]_q} + \frac{q^2|{}^bD_qf(b)|}{[2]_q[3]_q} - \frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right). \end{aligned} \tag{4.1}$$

Proof. According to Lemma 4.1 and the property of modulus, then we obtain

$$\begin{aligned} \left|f(\varphi) - \frac{1}{b-a}\left(\int_a^\varphi f(t)\,{}_ad_qt + \int_\varphi^b f(t)\,{}^bd_qt\right)\right| &\leqslant \frac{q(\varphi-a)^2}{b-a}\int_0^1 t|{}_aD_qf(t\varphi+(1-t)a)|\,{}_0d_qt \\ &\quad + \frac{q(b-\varphi)^2}{b-a}\int_0^1 t|{}^bD_qf(t\varphi+(1-t)b)|\,{}_0d_qt. \end{aligned} \tag{4.2}$$

Since $|_aD_q f|$ and $|^bD_q f|$ are strongly convex functions on $[a, b]$, consequently

$$\begin{aligned} \int_0^1 t\,|{}_aD_qf(t\varphi+(1-t)a)|\,{}_0d_qt &\leqslant \int_0^1 \left(t^2|{}_aD_qf(\varphi)| + t(1-t)|{}_aD_qf(a)| - \nu(t^2-t^3)(\varphi-a)^2\right)\,{}_0d_qt \\ &= \frac{|{}_aD_qf(\varphi)|}{[3]_q} + \frac{q^2|{}_aD_qf(a)|}{[2]_q[3]_q} - \frac{\nu(\varphi-a)^2}{[3]_q[4]_q} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \int_0^1 t\,\left|{}^bD_qf(t\varphi+(1-t)b)\right|\,{}_0d_qt &\leqslant \int_0^1 \left(t^2|{}^bD_qf(\varphi)| + t(1-t)|{}^bD_qf(b)| - \nu(t^2-t^3)(b-\varphi)^2\right)\,{}_0d_qt \\ &= \frac{|{}^bD_qf(\varphi)|}{[3]_q} + \frac{q^2|{}^bD_qf(b)|}{[2]_q[3]_q} - \frac{\nu(b-\varphi)^2}{[3]_q[4]_q}. \end{aligned} \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.2), then we obtain the inequality (4.1) as required. □

Remark 4.3. If we take $\nu \to 0^+$ in Theorem 4.2, then we obtain the following inequality:

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_a d_q t+\int_{\varphi}^{b} f(t)\,{}^b d_q t\right)\right|\\&\leqslant \frac{q}{(b-a)[2]_q[3]_q}\left((\varphi-a)^2\left([2]_q|{}_aD_q f(\varphi)|+q^2|{}_aD_q f(a)|\right)\right)\\&+\left((b-a)^2\left([2]_q|{}^bD_q f(\varphi)|+q^2|{}^bD_q f(b)|\right)\right),\end{aligned}$$

which is proposed by Budak et al. in [21].

Corollary 4.4. *If we suppose $|{}_aD_q f|$ and $|{}^bD_q f| \leqslant \mu$ in Theorem 4.2, then we have following inequality:*

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_a d_q t+\int_{\varphi}^{b} f(t)\,{}^b d_q t\right)\right|\\&\leqslant \frac{q(\varphi-a)^2}{(b-a)}\left(\frac{\mu}{[2]_q}-\frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right)+\frac{q(b-\varphi)^2}{(b-a)}\left(\frac{\mu}{[2]_q}-\frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right).\end{aligned} \tag{4.5}$$

Remark 4.5. If we take $\nu \to 0^+$ in Corollary 4.4, then we have the inequality (2.6).

Remark 4.6. If we take the limit as $q \to 1^-$ in Corollary 4.4, it is reduced to [70, Theorem 2.2].

Theorem 4.7. *Let $f : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function. If $|{}_aD_q f|^r$ and $|{}^bD_q f|^r, r \geqslant 0$ are strongly convex functions on $[a, b]$ with respect to $\nu > 0$, then, for all $\varphi \in [a, b]$, the following inequality holds:*

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_a d_q t+\int_{\varphi}^{b} f(t)\,{}^b d_q t\right)\right|\\&\leqslant \frac{q}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left((\varphi-a)^2\left(\frac{|{}_aD_q f(\varphi)|^r}{[3]_q}+\frac{q^2|{}_aD_q f(a)|^r}{[2]_q[3]_q}-\frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right.\\&\left.+(b-\varphi)^2\left(\frac{|{}^bD_q f(\varphi)|^r}{[3]_q}+\frac{q^2|{}^bD_q f(b)|^r}{[2]_q[3]_q}-\frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right).\end{aligned} \tag{4.6}$$

Proof. By Lemma 4.1, $|{}_aD_q f|^r$ and $|{}^bD_q f|^r, r \geqslant 0$ are strongly convex functions on $[a, b]$ and using properties of the modulus and power mean inequality, then we have

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_a d_q t+\int_{\varphi}^{b} f(t)\,{}^b d_q t\right)\right|\\&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t|{}_aD_q f(t\varphi+(1-t)a)|d_q t\right)+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t|{}^bD_q f(t\varphi+(1-t)b)|d_q t\right)\\&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t|{}_aD_q f(t\varphi+(1-t)a)|^r d_q t\right)^{\frac{1}{r}}\\&+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t|{}^bD_q f(t\varphi+(1-t)b)|^r d_q t\right)^{\frac{1}{r}}\\&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 (t^2|{}_aD_q f(\varphi)|^r+(t-t^2)|{}_aD_q f(a)|^r-\nu(t^2-t^3)(\varphi-a)^2)d_q t\right)^{\frac{1}{r}}\end{aligned}$$

$$+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 (t^2|_a D_q f(\varphi)|^r+(t-t^2)|_a D_q f(b)|^r-\nu(t^2-t^3)(b-\varphi)^2) d_q t\right)^{\frac{1}{r}}$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{|_a D_q f(\varphi)|^r}{[3]_q}+\frac{q^2|_a D_q f(a)|^r}{[2]_q[3]_q}-\frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{|^b D_q f(\varphi)|^r}{[3]_q}+\frac{q^2|^b D_q f(b)|^r}{[2]_q[3]_q}-\frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}$$
$$=\frac{q}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left((\varphi-a)^2\left(\frac{|_a D_q f(\varphi)|^r}{[3]_q}+\frac{q^2|_a D_q f(a)|^r}{[2]_q[3]_q}-\frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right.$$
$$\left.+(b-\varphi)^2\left(\frac{|^b D_q f(\varphi)|^r}{[3]_q}+\frac{q^2|^b D_q f(b)|^r}{[2]_q[3]_q}-\frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right).$$

This completes the proof. □

Remark 4.8. If we take $\nu \to 0^+$ in Theorem 4.7, then the following inequality holds:

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_a d_q t+\int_{\varphi}^b f(t)\,^b d_q t\right)\right|$$
$$\leqslant \frac{q}{(b-a)[2]_q}\left((\varphi-a)^2\left(\frac{[2]_q|_a D_q f(\varphi)|^r+q^2|_a D_q f(a)|^r}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right.$$
$$\left.+(b-a)^2\left(\frac{[2]_q|^b D_q f(\varphi)|^r+q^2|^b D_q f(b)|^r}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right),$$

which is proposed by Budak et al. in [21].

Corollary 4.9. *If* $|_a D_q f|, |^b D_q f| \leqslant \mu$ *in Theorem 4.7, then the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_a d_q t+\int_{\varphi}^b f(t)\,^b d_q t\right)\right|$$
$$\leqslant \frac{q}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left((\varphi-a)^2\left(\frac{\mu^r}{[3]_q}+\frac{q^2\mu^r}{[2]_q[3]_q}-\frac{\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right.$$
$$\left.+(b-\varphi)^2\left(\frac{\mu^r}{[3]_q}+\frac{q^2\mu^r}{[2]_q[3]_q}-\frac{\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\right).$$

Remark 4.10. If we take the limit as $q \to 1^-$ in Corollary 4.9, then we obtain

$$\left|f(\varphi)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{(\varphi-a)^2}{2(b-a)}\left(\mu^r-\frac{\nu(\varphi-a)^2}{6}\right)^{\frac{1}{r}}+\frac{(b-\varphi)^2}{2(b-a)}\left(\mu^r-\frac{\nu(b-\varphi)^2}{6}\right)^{\frac{1}{r}}, \tag{4.7}$$

which appeared in [70, Theorem 2.8].

Remark 4.11. If we take $\nu \to 0^+$ in inequality (4.7), then we obtain inequality (1.1).

Theorem 4.12. *Let* $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ *be a differentiable function. If* $|{}_aD_qf|^r$ *and* $|{}^bD_qf|^r, r\geqslant 0$ *are strongly convex functions on* $[a,b]$ *with respect to* $\nu>0$*, then the following inequality holds:*

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_ad_qt+\int_{\varphi}^{b} f(t)\,{}^bd_qt\right)\right|\\&\leqslant\frac{q}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left((\varphi-a)^2\left(\frac{|{}_aD_qf(\varphi)|^r}{[2]_q}+\frac{q|{}_aD_qf(a)|^r}{[2]_q}-\frac{q^2\nu(\varphi-a)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right.\\&\quad\left.+(b-\varphi)^2\left(\frac{|{}^bD_qf(\varphi)|^r}{[2]_q}+\frac{q|{}^bD_qf(b)|^r}{[2]_q}-\frac{q^2\nu(b-\varphi)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right)\end{aligned}\tag{4.8}$$

for all $\varphi\in[a,b], s,r>0$ *and* $\frac{1}{s}+\frac{1}{r}=1$.

Proof. From Lemma 4.1, using properties of the modulus, Hölder's inequality, and the fact that $|{}_aD_qf|^r$ and $|{}^bD_qf|^r$ are strongly convex functions, then we obtain

$$\begin{aligned}&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_ad_qt+\int_{\varphi}^{b} f(t)\,{}^bd_qt\right)\right|\\
&\leqslant\frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t|{}_aD_qf(t\varphi+(1-t)a)|d_qt\right)+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t|{}^bD_qf(t\varphi+(1-t)b)|d_qt\right)\\
&\leqslant\frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1|{}_aD_qf(t\varphi+(1-t)a)|^rd_qt\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1|{}_aD_qf(t\varphi+(1-t)b)|^rd_qt\right)^{\frac{1}{r}}\\
&\leqslant\frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1(t|{}_aD_qf(\varphi)|^r+(1-t)|{}_aD_qf(a)|^r-\nu(t-t^2)(\varphi-a)^2)d_qt\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1(t|{}_aD_qf(\varphi)|^r+(1-t)|{}_aD_qf(b)|^r-\nu(t-t^2)(b-\varphi)^2)d_qt\right)^{\frac{1}{r}}\\
&\leqslant\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left(\frac{|{}_aD_qf(\varphi)|^r}{[2]_q}+|{}_aD_qf(a)|^r-\frac{|{}_aD_qf(a)|^r}{[2]_q}-\frac{\nu(\varphi-a)^2}{[2]_q}+\frac{c(\varphi-a)^2}{[3]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left(\frac{|{}^bD_qf(\varphi)|^r}{[2]_q}+|{}^bD_qf(b)|^r-\frac{|{}^bD_qf(b)|^r}{[2]_q}-\frac{\nu(b-\varphi)^2}{[2]_q}+\frac{c(b-\varphi)^2}{[3]_q}\right)^{\frac{1}{r}}\\
&\leqslant\frac{q}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left((\varphi-a)^2\left(\frac{|{}_aD_qf(\varphi)|^r}{[2]_q}+\frac{q|{}_aD_qf(a)|^r}{[2]_q}-\frac{q^2\nu(\varphi-a)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right.\\
&\quad\left.+(b-\varphi)^2\left(\frac{|{}^bD_qf(\varphi)|^r}{[2]_q}+\frac{q|{}^bD_qf(b)|^r}{[2]_q}-\frac{q^2\nu(b-\varphi)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right).\end{aligned}$$

Therefore, the proof is completed. □

Remark 4.13. If we take $\nu\to 0^+$ in inequality (4.8), then the following inequality holds:

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_ad_qt+\int_{\varphi}^{b} f(t)\,{}^bd_qt\right)\right|$$

$$\leqslant \frac{q}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left((\varphi-a)^2\left(\frac{|_aD_qf(\varphi)|^r+q|_aD_qf(a)|^r}{[2]_q}\right)^{\frac{1}{r}}\right.$$
$$\left.+(b-\varphi)^2\left(\frac{|^bD_qf(\varphi)|^r+q|^bD_qf(b)|^r}{[2]_q}\right)^{\frac{1}{r}}\right),$$

which is proposed by Budak et al. in [21].

Corollary 4.14. *If we set $|_aD_qf|, |^bD_qf| \leqslant \mu$ in Theorem 4.12, then the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_ad_qt+\int_{\varphi}^{b} f(t)\,^bd_qt\right)\right| \tag{4.9}$$
$$\leqslant \frac{q}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left\{(\varphi-a)^2\left(\mu^r+\frac{q^2\nu(\varphi-a)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}+(b-\varphi)^2\left(\mu^r+\frac{q^2\nu(b-\varphi)^2}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right\}.$$

Remark 4.15. If we take $\nu\to 0^+$ in Corollary 4.14, then the following inequality holds:

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_ad_qt+\int_{\varphi}^{b} f(t)\,^bd_qt\right)\right| \leqslant \frac{q\mu}{b-a}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left((\varphi-a)^2+(b-\varphi)^2\right),$$

which is proposed by Budak et al. in [21].

Theorem 4.16. *Let $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ be a differentiable function. If $|_aD_qf|^r$ and $|^bD_qf|^r$, $r\geqslant 0$ are strongly convex functions on $[a,b]$ with respect to $\nu>0$, then, for all $\varphi\in[a,b]$, the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_ad_qt+\int_{\varphi}^{b} f(t)\,^bd_qt\right)\right|$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^3|_aD_qf(x)|^r}{[3]_q[4]_q}+\frac{(q^2+q^5)|_aD_qf(a)|^r}{[2]_q[3]_q[4]_q}+\frac{(q^3+q^7)\nu(\varphi-a)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{|_aD_qf(\varphi)|^r}{[4]_q}+\frac{q^3|_aD_qf(a)|^r}{[3]_q[4]_q}-\frac{q^4\nu(\varphi-a)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}} \tag{4.10}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^3|^bD_qf(\varphi)|^r}{[3]_q[4]_q}+\frac{(q^2+q^5)|^bD_qf(b)|^r}{[2]_q[3]_q[4]_q}+\frac{(q^3+q^7)\nu(b-\varphi)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{|^bD_qf(\varphi)|^r}{[4]_q}+\frac{q^3|^bD_qf(b)|^r}{[3]_q[4]_q}-\frac{q^4\nu(b-\varphi)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}}.$$

Proof. By Lemma 4.1, using properties of the modulus and power mean inequality and the fact that $|_aD_qf|^r$ and $|^bD_qf|^r$ are strongly convex functions, then we obtain

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,_ad_qt+\int_{\varphi}^{b} f(t)\,^bd_qt\right)\right|$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t|_aD_qf(t\varphi+(1-t)a)|d_qt\right)+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t|^bD_qf(t\varphi+(1-t)b)|d_qt\right)$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t(1-t)d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1 t(1-t)|_aD_qf(t\varphi+(1-t)a)|^rd_qt\right)^{\frac{1}{r}}$$

$$+\frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t^2 d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^2 |{}_aD_q f(t\varphi+(1-t)a)|^r d_q t\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t(1-t) d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t(1-t) |{}^bD_q f(t\varphi+(1-t)b)|^r d_q t\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t^2 d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^2 |{}^bD_q f(t\varphi+(1-t)a)|^r d_q t\right)^{\frac{1}{r}}$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^3|{}_aD_q f(\varphi)|^r}{[3]_q[4]_q}+\frac{(q^2+q^5)|{}_aD_q f(a)|^r}{[2]_q[3]_q[4]_q}+\frac{(q^3+q^7)\nu(\varphi-a)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{|{}_aD_q f(\varphi)|^r}{[4]_q}+\frac{q^3|{}_aD_q f(a)|^r}{[3]_q[4]_q}-\frac{q^4\nu(\varphi-a)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^3|{}^bD_q f(\varphi)|^r}{[3]_q[4]_q}+\frac{(q^2+q^5)|{}^bD_q f(b)|^r}{[2]_q[3]_q[4]_q}+\frac{(q^3+q^7)\nu(b-\varphi)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{|{}^bD_q f(\varphi)|^r}{[4]_q}+\frac{q^3|{}^bD_q f(b)|^r}{[3]_q[4]_q}-\frac{q^4\nu(b-\varphi)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Corollary 4.17. *If we suppose $|{}_aD_q f|, |{}^bD_q f| \leqslant \mu$ in Theorem 4.16, then the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^\varphi f(t)\,{}_ad_q t+\int_\varphi^b f(t)\,{}^bd_q t\right)\right|$$
$$\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^2\mu^r}{[2]_q[3]_q}+\frac{(q^3+q^7)\nu(\varphi-a)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{\mu^r}{[3]_q}+\frac{q^3c(\varphi-a)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{q^2}{[2]_q[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{q^2\mu^r}{[2]_q[3]_q}+\frac{(q^3+q^7)\nu(b-\varphi)^2}{[3]_q[4]_q[5]_q}\right)^{\frac{1}{r}}$$
$$+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{r}}\left(\frac{\mu^r}{[3]_q}+\frac{q^3c(b-\varphi)^2}{[4]_q[5]_q}\right)^{\frac{1}{r}}.$$

Remark 4.18. If we take $\nu \to 0^+$ in Corollary 4.17, then the following inequality holds:

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^\varphi f(t)\,{}_ad_q t+\int_\varphi^b f(t)\,{}^bd_q t\right)\right| \leqslant \frac{q\mu}{(b-a)}\left(\frac{(\varphi-a)^2+(b-\varphi)^2}{[2]_q}\right), \tag{4.11}$$

which appeared in inequality (2.6).

Remark 4.19. If we take $q \to 1^-$ in inequality (4.11), then we obtain the inequality (1.1).

Theorem 4.20. *Let $f : [a, b] \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function. If $|{}_aD_q f|^r$ and $|{}^bD_q f|^r, r \geqslant 0$ are strongly convex functions on $[a, b]$ with respect to $\nu > 0$, then the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^\varphi f(t)\,{}_ad_q t+\int_\varphi^b f(t)\,{}^bd_q t\right)\right|$$

$$
\begin{aligned}
&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q^2|{}_aD_qf(\varphi)|^r}{[2]_q[3]_q}+\frac{(q+q^3)|{}_aD_qf(a)|^r}{[2]_q[3]_q}+\frac{(q^2+q^3)\nu(\varphi-a)^2}{[2]_q[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{|{}_aD_qf(\varphi)|^r}{[3]_q}+\frac{q^2|{}_aD_qf(a)|^r}{[2]_q[3]_q}-\frac{q^3\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q^2|{}^bD_qf(\varphi)|^r}{[2]_q[3]_q}+\frac{(q+q^3)|{}^bD_qf(b)|^r}{[2]_q[3]_q}+\frac{(q^2+q^3)\nu(b-\varphi)^2}{[2]_q[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{|{}^bD_qf(\varphi)|^r}{[3]_q}+\frac{q^2|{}^bD_qf(b)|^r}{[2]_q[3]_q}-\frac{q^3\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}
\end{aligned}
$$

for all $\varphi \in [a, b]$ *and* $\frac{1}{s}+\frac{1}{r}=1$.

Proof. By Lemma 4.1, using properties of the modulus, Hölder's inequality, and the fact that $|{}_aD_qf|^r$ and $|{}^bD_qf|^r$ are strongly convex functions, then we obtain

$$
\begin{aligned}
&\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_ad_qt+\int_{\varphi}^{b} f(t)\,{}^bd_qt\right)\right|\\
&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t|{}_aD_qf(t\varphi+(1-t)a)|d_qt\right)+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t|{}^bD_qf(t\varphi+(1-t)b)|d_qt\right)\\
&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\int_0^1(1-t)t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1 t(1-t)|{}_aD_qf(t\varphi+(1-t)a)|^r d_qt\right)^{\frac{1}{r}}\\
&\quad+\frac{q(\varphi-a)^2}{b-a}\left(\int_0^1 t^{s+1}d_qt\right)^{\frac{1}{s}}\left(\int_0^1 t|{}_aD_qf(t\varphi+(1-t)a)|^r d_qt\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1(1-t)t^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1(1-t)|{}^bD_qf(t\varphi+(1-t)b)|^s d_qt\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\int_0^1 t^{s+1}d_qt\right)^{\frac{1}{s}}\left(\int_0^1 t|{}^bD_qf(t\varphi+(1-t)a)|^s d_qt\right)^{\frac{1}{r}}\\
&\leqslant \frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q^2|{}_aD_qf(\varphi)|^r}{[2]_q[3]_q}+\frac{(q+q^3)|{}_aD_qf(a)|^r}{[2]_q[3]_q}+\frac{(q^2+q^3)\nu(\varphi-a)^2}{[2]_q[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(\varphi-a)^2}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{|{}_aD_qf(\varphi)|^r}{[3]_q}+\frac{q^2|{}_aD_qf(a)|^r}{[2]_q[3]_q}-\frac{q^3\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q^2|{}^bD_qf(\varphi)|^r}{[2]_q[3]_q}+\frac{(q+q^3)|{}^bD_qf(b)|^r}{[2]_q[3]_q}+\frac{(q^2+q^3)\nu(b-\varphi)^2}{[2]_q[3]_q[4]_q}\right)^{\frac{1}{r}}\\
&\quad+\frac{q(b-\varphi)^2}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{|{}^bD_qf(\varphi)|^r}{[3]_q}+\frac{q^2|{}^bD_qf(b)|^r}{[2]_q[3]_q}-\frac{q^3\nu(b-\varphi)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

Corollary 4.21. *If we assume* $|{}_aD_qf|, |{}^bD_qf| \leqslant \mu$ *in Theorem* 4.20, *then the following inequality holds:*

$$
\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_ad_qt+\int_{\varphi}^{b} f(t)\,{}^bd_qt\right)\right|
$$

$$\leqslant \frac{q}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q\mu^r}{[2]_q}+\frac{(q^2+q^3)\nu(\varphi-a)^2}{[2]_q[3]_q[4]_q}\right)^{\frac{1}{r}}((\varphi-a)^2+(b-\varphi)^2)$$
$$+\frac{q}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{\mu^r}{[2]_q}-\frac{q^3\nu(\varphi-a)^2}{[3]_q[4]_q}\right)^{\frac{1}{r}}((\varphi-a)^2+(b-\varphi)^2).$$

Corollary 4.22. *If we take $\nu\to 0^+$ in Corollary 4.21, then the following inequality holds:*

$$\left|f(\varphi)-\frac{1}{b-a}\left(\int_a^{\varphi} f(t)\,{}_a d_q t+\int_{\varphi}^{b} f(t)\,{}^b d_q t\right)\right|$$
$$\leqslant \frac{q}{b-a}\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{q\mu^r}{[2]_q}\right)^{\frac{1}{r}}((\varphi-a)^2+(b-\varphi)^2)$$
$$+\frac{q}{b-a}\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\left(\frac{\mu^r}{[2]_q}\right)^{\frac{1}{r}}((\varphi-a)^2+(b-\varphi)^2)$$
$$=\frac{q\mu\left((\varphi-a)^2+(b-\varphi)^2\right)}{(b-a)}\left(\frac{1}{[2]_q}\right)^{\frac{1}{r}}\left(\left(\frac{1}{[s+1]_q}-\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}+\left(\frac{1}{[s+2]_q}\right)^{\frac{1}{s}}\right).$$

5. Applications to special means

In this section, we present the applications of special means to illustrate the results in Theorems 4.2, 4.7, and 4.12, and Corollaries 4.4 and 4.14. To set the arbitrary positive numbers $\vartheta_1,\vartheta_2(\vartheta_1\neq\vartheta_2)$, we consider the means as follows:

1. The arithmetic mean: $\mathcal{A}=\mathcal{A}(\vartheta_1,\vartheta_2)=\frac{\vartheta_1+\vartheta_2}{2}$.
2. The logarithmic mean: $L_P^P=L_P^P(\vartheta_1,\vartheta_2)=\frac{\vartheta_2^{P+1}-\vartheta_1^{P+1}}{(P+1)(\vartheta_1-\vartheta_2)}$.

Proposition 5.1. *For $0<a<b$ and $0<q<1$, the following inequality holds:*

$$\left|\frac{1}{s+1}(\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\vartheta_1,\vartheta_2))\right|\leqslant\frac{q(b-a)}{4}\left(\frac{1}{[3]_q}\left(L_S^S(q\frac{a+b}{2}+(1-q)a,\frac{a+b}{2})+L_S^S(q\frac{a+b}{2}+(1-q)b,\frac{a+b}{2})\right)+\frac{q^2(a^s+b^s)}{[2]_q[3]_q}-\frac{(b-a)^2}{2[3]_q[4]_q}\right), \tag{5.1}$$

where

$$\vartheta_1=(1-q)\sum_{n=0}^{\infty}q^n\left(q^n\frac{a+b}{2}+(1-q^n)a\right)^{s+1}\quad\text{and}\quad\vartheta_2=(1-q)\sum_{n=0}^{\infty}q^n\left(q^n\frac{a+b}{2}+(1-q^n)b\right)^{s+1}.$$

Proof. Substituting $\varphi=\frac{a+b}{2}$ for $f(\varphi)=\frac{\varphi^{s+1}}{s+1}$, where $\varphi>0$ and $s\in(0,1)$ in the inequality (4.1) of Theorem 4.2, we obtain inequality (5.1) as required. □

Proposition 5.2. *For $0<a<b$ and $0<q<1$, the following inequality holds:*

$$\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\vartheta_1,\vartheta_2)\right]\right|\leqslant\frac{q(b-a)}{2}\left(\frac{\mu}{[2]_q}-\frac{\nu(b-a)^2}{4[3]_q[4]_q}\right). \tag{5.2}$$

Proof. The inequality (4.5) in Corollary 4.4 with $\varphi=\frac{a+b}{2}$ for $f(\varphi)=\frac{\varphi^{s+1}}{s+1}$, where $\varphi>0$ and $s\in(0,1)$ reduces to the inequality (5.2) as required. □

Proposition 5.3. *For* $0 < a < b$ *and* $0 < q < 1$, *the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\vartheta_1,\vartheta_2)\right]\right|\\ &\leqslant \frac{q(b-a)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left\{\left(\frac{|L_S^S(q\frac{a+b}{2}+(1-q)a,\frac{a+b}{2})|^r}{[3]_q}+\frac{q^2|a^s|^r}{[2]_q[3]_q}-\frac{\nu(b-a)^2}{4[3]_q[4]_q}\right)^{\frac{1}{r}}\right.\\ &\left.+\left(\frac{|L_S^S(q\frac{a+b}{2}+(1-q)b,\frac{a+b}{2})|^r}{[3]_q}+\frac{q^2|b^s|^r}{[2]_q[3]_q}-\frac{\nu(b-a)^2}{4[3]_q[4]_q}\right)^{\frac{1}{r}}\right\}.\end{aligned} \tag{5.3}$$

Proof. By substituting $\varphi = \frac{a+b}{2}$ for $f(\varphi) = \frac{\varphi^{s+1}}{s+1}$, where $\varphi > 0$ and $s \in (0,1)$ in inequality (4.6) of Theorem 4.7, we obtain the inequality (5.3) as required. □

Proposition 5.4. *For* $0 < a < b$ *and* $0 < q < 1$, *the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\vartheta_1,\vartheta_2)\right]\right|\\ &\leqslant \frac{q(b-a)}{4}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left\{\left(\frac{|L_S^S(q\frac{a+b}{2}+(1-q)a,\frac{a+b}{2})|^r}{[2]_q}+\frac{q|a^s|^r}{[2]_q}-\frac{q^2\nu(b-a)^2}{4[2]_q[3]_q}\right)^{\frac{1}{r}}\right.\\ &\left.+\left(\frac{|L_S^S(q\frac{a+b}{2}+(1-q)b,\frac{a+b}{2})|^r}{[2]_q}+\frac{q|b^s|^r}{[2]_q}-\frac{q^2\nu(b-a)^2}{4[2]_q[3]_q}\right)^{\frac{1}{r}}\right\}.\end{aligned} \tag{5.4}$$

Proof. By substituting $\varphi = \frac{a+b}{2}$ for $f(\varphi) = \frac{\varphi^{s+1}}{s+1}$, where $\varphi > 0$ and $s \in (0,1)$ in the inequality (4.8) of Theorem 4.12, we obtain the inequality (5.4) as required. □

Proposition 5.5. *For* $0 < a < b$ *and* $0 < q < 1$, *the following inequality holds:*

$$\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\vartheta_1,\vartheta_2)\right]\right| \leqslant \frac{q(b-a)}{2}\left(\frac{1}{[s+1]_q}\right)^{\frac{1}{s}}\left(\mu^r+\frac{q^2\nu(b-a)^2}{4[2]_q[3]_q}\right)^{\frac{1}{r}}. \tag{5.5}$$

Proof. By substituting $\varphi = \frac{a+b}{2}$ for $f(\varphi) = \frac{\varphi^{s+1}}{s+1}$, where $\varphi > 0$ and $s \in (0,1)$ in the inequality (4.9) of Corollary 4.14, we obtain the inequality (5.5) as required. □

6. Conclusions

In this paper, we applied the concepts of the newly defined ${}_aD_q, {}^bD_q$-derivatives and q_a, q^b-integrals to investigate some new quantum Hermite-Hadamard and Ostrowski type inequalities for strongly convex functions. It was found that the obtained results have the potential for generalizing the existing comparable results in the literature. This paper is expected to stimulate researchers in the field of coordinated strongly convex functions for further study.

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