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# On new generalized discrete $\mathbb{U}$-Bernoulli-Korobov-kind polynomials and some of their properties

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**Abstract**

The object of this study is to introduce the new generalized discrete U-Bernoulli-Korobov-kind polynomials. Additionally, we give several of its explicit representations, as well as relations with other families of polynomials. We state some properties for the $\Delta$ and $\nabla$ operators associated with this polynomial class. Finally, we focus our attention on the orthogonality relation and the three-term recurrence formula satisfied by these polynomials.

**Keywords:** Orthogonal polynomials, recurrence relations, Bernoulli polynomials, Bernoulli polynomials of the second kind, Cauchy numbers, the backward and forward difference operators, Pearson-type difference equation.

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## 1. Introduction

The study of generating functions and their various extensions leads to polynomials and numbers known for their exceptional and valuable properties, which have applications in some branches of mathematics, probability, engineering, and other scientific disciplines. Many mathematical physics issues can be solved analytically, thanks to the recent developments in generating functions theory [19, 20, 28, 29, 32, 33]. Numerous authors have been active in the study of degenerate numbers and polynomials, which has led to the discovery of some interesting results. We can find certain works related to degenerate Bernoulli polynomials, a study on the generalized degenerate form of 2D Appell polynomials using fractional operators, degenerate Stirling numbers of both types associated with hypergeometric degenerate numbers, $r$-Stirling degenerate numbers of the second type worked within the framework of a double-indexed sequence, a study of the $\lambda$-Stirling numbers of both types that are $\lambda$-analogues of the Stirling numbers, a version of the degenerate polylogarithm function that allows the construction and study of a new type of polynomials and degenerates Bernoulli numbers, and a study of the polynomials and completely degenerate numbers that arise naturally from the Volkenborn integral of the degenerate exponential function of $\mathbb{Z}_p$ (see, e.g., [2, 4, 6, 8–11, 14, 16–18, 23, 25–27, 31] for more details). Furthermore, in the literature, we

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find the use of a certain degenerate differential and degenerate difference operator to study the degenerate harmonic numbers and some properties of the degenerate Laguerre polynomials [7]. We also found a study on a new generalized family of degenerate three-variable Hermite-Appell polynomials defined using a fractional derivative [30]. The polynomials and Korobov numbers, some properties, identities, recurrence relations, connections with other polynomials, and some of their generalizations in different contexts have also been studied by several authors using umbral calculus [5, 15, 23]. The New U-Bernoulli, U-Euler, and U-Genocchi polynomials and their matrices have been introduced recently in [22] and they provide some generalizations and their relationship with the Riemann zeta function. On the other hand, in recent years, the investigations of discrete orthogonal polynomials have gained high attention for their applications to functional equations and differentials and their use to establish various analytic number theory properties (cf. [3, 5, 13, 15]).

This work is to introduce a novel family of polynomials, denominated as, new generalized discrete U-Bernoulli-Korobov-kind polynomials, with a parameter that outlines the advantages of techniques associated with the generating function. We will give some representative properties and we will show that these polynomials are orthogonal on $\mathbb{N}$ with respect to the inner product that will be studied. Here $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ will denote the sets of the numbers complex, real, positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{P}$ is the space of all polynomials in one variable with real coefficients, and $\log(z)$ denotes the principal value of the multi-valued logarithm function.

The outline of this paper is as follows. In Section 2, we provide well-known basic formulas and definitions that we shall need to use for the rest of the work. In Section 3, a new class of discrete polynomials is introduced using their generating function. We derive certain properties and explicit formulas for these polynomials. In Section 4, we study relations with the Korobov polynomials, the Stirling numbers of the first kind, the Daehee numbers, and their relations with the difference operators $\Delta$ and $\nabla$. Moreover, in Section 5, we establish that these new polynomials satisfy an orthogonality relationship. Finally, we study whether they satisfy the three-term recurrence relation.

## 2. Background and previous results

In this section, we recall some definitions and preliminary results, that will be used in this paper.

The classical Bernoulli polynomials, $B_n(x)$ are defined by employing the following generating function (see [24, 25]):

$$\left(\frac{z}{e^z - 1}\right) e^{zx} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{2.1}$$

For $x = 0$ in (2.1), we find the classical Bernoulli numbers $B_n := B_n(0) = B_n^{(0)}$ defined by the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{2.2}$$

The Bernoulli polynomials of the second kind, $b_n(x)$ in the variable $x$, are defined utilizing the generating function (see [23, p. 167, Eq. (1.2)]):

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{n=0}^{\infty} b_n(x) \frac{z^n}{n!}, \quad (|z| < 1). \tag{2.3}$$

At the point $x = 0$ in (2.3), $b_n := b_n(0)$ is called the Bernoulli numbers of the second kind (cf. [4, 15]), defined by the generating function:

$$\frac{z}{\log(1+z)} = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}, \quad (|z| < 1). \tag{2.4}$$

The Bernoulli polynomials of the second kind are also called Korobov polynomials of the first kind.

The Daehee polynomials $D_n(x)$ are defined by employing the generating function (see [4, 12, 16]):

$$\frac{\log(1+z)}{z}(1+z)^x = \sum_{n=0}^{\infty} D_n(x)\frac{z^n}{n!}, \quad (|z|<1). \tag{2.5}$$

If $x=0$, in (2.5), $D_n := D_n(0)$ denotes the so called Daehee numbers, defined by the generating function:

$$\frac{\log(1+z)}{z} = \sum_{n=0}^{\infty} D_n\frac{z^n}{n!}, \quad (|z|<1). \tag{2.6}$$

The falling factorial $x$ of order $n$, $\langle x\rangle$, is (see [16]):

$$\langle x\rangle_n = x(x-1)\cdots(x-n+1), \quad n \geqslant 1;\ \langle x\rangle_0 = 1. \tag{2.7}$$

The Stirling numbers of the first kind, $s(n,k)$, appear as the coefficients in the following generating function (see [25]):

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!}, \quad (|z|<1). \tag{2.8}$$

These numbers can also be given as (see [16, 25]).

$$\langle x\rangle_n = \sum_{k=0}^{n} s(n,k)x^n. \tag{2.9}$$

The U-Bernoulli numbers $M_n$ are defined by the following generating function [22]:

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} M_n\frac{z^n}{n!}, \quad |z|<2\pi. \tag{2.10}$$

Of the classical exponential function, is received

$$e^{-\alpha z}-1 = \sum_{m=0}^{\infty} \frac{(-\alpha)^{m+1}z^{m+1}}{(m+1)!}. \tag{2.11}$$

Let $f$ be some function of a real variable $x$. The backward and forward difference operators $\Delta$ and $\nabla$ respectively, are defined as (see [21]):

$$\nabla f(x) := f(x) - f(x-1), \tag{2.12}$$
$$\Delta f(x) := f(x+1) - f(x). \tag{2.13}$$

Further, for any real number $a$, we have

$$\Delta_a f(x) := f(x+a) - f(x). \tag{2.14}$$

If $a=1$ in (2.14), we obtain (2.13). The operators $\Delta$ and $\nabla$ also satisfied the following properties (see [21]):

$$\nabla f(x) = \Delta f(x) - \Delta\nabla f(x). \tag{2.15}$$
$$\nabla(f(x)g(x)) = f(x)\nabla g(x) + g(x-1)\nabla f(x). \tag{2.16}$$

For two arbitrary sequences $\{c_k\}_{k\geqslant 0}$ and $\{d_k\}_{k\geqslant 0}$, if $d_{-1}=0$, then applying summation by parts there holds (see [21]):

$$\sum_{k=0}^{\infty}(\Delta c_k)d_k = -\sum_{k=0}^{\infty} c_k\nabla d_k.$$

If we consider $\sigma$ a polynomial of degree$\leqslant$ 2 and $\tau$ a polynomial of degree$\leqslant$ 1, we have a first-order difference equation

$$\nabla\left[\sigma(x)\omega(x)\right]=\tau(x)\omega(x). \tag{2.17}$$

This equation is known as the Pearson equation. We note that in (2.17), the operator $\nabla$ is used for orthogonal polynomials on the lattice, and it is replaced by differentiation in the case of orthogonal polynomials on an interval of the real line.

## 3. New family of generalized discrete U-Bernoulli-Korobov-kind polynomials

In this section, a new class of discrete polynomials is introduced, which we denote by $\mathcal{P}_n(x;\alpha)$ and will we call generalized discrete U-Bernoulli-Korobov-kind polynomials, and study certain properties and explicit formulas that satisfy these new polynomials.

**Definition 3.1.** The new family of generalized discrete U-Bernoulli-Korobov-kind polynomials $\mathcal{P}_n(x;\alpha)$ of degree $n$ in the variable $x$, and parameter $\alpha\in\mathbb{R}-\{0\}$ are defined through the following generating function:

$$L(x,z;\alpha)=\left(\frac{z}{e^{-z\alpha}-1}\right)(1+z)^x=\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!},\quad\left(|z|<\frac{2\pi}{|\alpha|}\right). \tag{3.1}$$

From (3.1), we see that

$$L(x,z;\alpha)=\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^x=\frac{z\,e^{-\alpha}}{1-e^{\alpha z}}\left(\frac{d^{(x)}}{d\alpha^{(x)}}\,e^{\alpha(1+z)}\right). \tag{3.2}$$

By using (3.1), we can compute the first generalized discrete U-Bernoulli-Korobov-kind polynomials $\mathcal{P}_n(x;\alpha)$, as follows:

$$\begin{aligned}
\mathcal{P}_0(x;\alpha)&=-\frac{1}{\alpha},\\
\mathcal{P}_1(x;\alpha)&=-\frac{x}{\alpha}-\frac{1}{2},\\
\mathcal{P}_2(x;\alpha)&=-\frac{x^2}{\alpha}+\left(\frac{1-\alpha}{\alpha}\right)x-\frac{\alpha}{6},\\
\mathcal{P}_3(x;\alpha)&=\left(\frac{-1}{\alpha}\right)x^3+\frac{3(2-\alpha)}{2\alpha}x^2+\frac{(3\alpha-\alpha^2-4)}{2\alpha}x,\\
\mathcal{P}_4(x;\alpha)&=\left(\frac{-1}{\alpha}\right)x^4+\left(\frac{6-2\alpha}{\alpha}\right)x^3+\left(\frac{-\alpha^2+6\alpha-11}{\alpha}\right)x^2+\left(\frac{\alpha^2-4\alpha+6}{\alpha}\right)x+\frac{\alpha^3}{30},\\
\mathcal{P}_5(x;\alpha)&=\left(\frac{-1}{\alpha}\right)x^5+\left(\frac{20-5\alpha}{2\alpha}\right)x^4+\left(\frac{45\alpha-105-5\alpha^2}{3\alpha}\right)x^3+\left(\frac{10\alpha^2-55\alpha+100}{2\alpha}\right)x^2\\
&\quad+\left(\frac{\alpha^4+20\alpha^2+90\alpha-144}{6\alpha}\right)x.
\end{aligned}$$

For $x=0$, in (3.1) corresponds to the generating function of the generalized U-Bernoulli-Korobov-kind numbers $\mathcal{P}_n(\alpha)=\mathcal{P}_n:=\mathcal{P}(0;\alpha)$ given by

$$\frac{z}{e^{-z\alpha}-1}=\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!},\quad\left(|z|<\frac{2\pi}{|\alpha|}\right). \tag{3.3}$$

From (3.3), we get some of these numbers, as below:

$$\mathcal{P}_0(\alpha)=-\frac{1}{\alpha},\quad\mathcal{P}_1(\alpha)=-\frac{1}{2},\quad\mathcal{P}_2(\alpha)=-\frac{\alpha}{6},\quad\mathcal{P}_3(\alpha)=0,\quad\mathcal{P}_4(\alpha)=\frac{\alpha^3}{30},\quad\mathcal{P}_5(\alpha)=0.$$

By comparing (3.3) with (2.2) and (2.10), we have

$$\mathcal{P}_n(0,-1) := B_n, \qquad \mathcal{P}_n(0,1) := M_n.$$

Therefore, the generating function of $\mathcal{P}_n(\alpha)$ in (3.3) includes, as its special cases, the generating function of the Bernoulli numbers $B_n$ in (2.2) and the generating function of the U-Bernoulli numbers $M_n$ investigated in [22].

**Proposition 3.2.** *Let $\alpha \in \mathbb{R} - \{0\}$, and $\{\mathcal{P}_n(\alpha)\}_{n\geqslant 0}$ be a sequence of generalized U-Bernoulli-Korobov-kind numbers. Then, the following relationship is fulfilled:*

$$\sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \frac{\mathcal{P}_{n-k}(\alpha)}{n!} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

*Proof.* By using (3.3), we have

$$z = (e^{-\alpha z} - 1) \sum_{n=0}^{\infty} \mathcal{P}_n(\alpha) \frac{z^n}{n!}. \tag{3.4}$$

From (2.11) and (3.4), it follows that

$$z = \alpha z \left( \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \alpha^n}{(n+1)!} z^n \right) \left( \sum_{n=0}^{\infty} \mathcal{P}_n(\alpha) \frac{z^n}{n!} \right). \tag{3.5}$$

In (3.5), we obtain

$$\frac{1}{\alpha} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1} \alpha^k}{(k+1)!} \frac{\mathcal{P}_{n-k}(\alpha)}{(n-k)!} z^n,$$

and thus

$$1 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \mathcal{P}_{n-k}(\alpha) \frac{z^n}{n!}. \tag{3.6}$$

Comparing the coefficients in (3.6) completes the proof from Proposition 3.2. □

**Proposition 3.3.** *Let $\alpha \in \mathbb{R} - \{0\}$, and $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ be a sequence of generalized discrete U-Bernoulli-Korobov-kind polynomials. Then, the following relations hold:*

$$\mathcal{P}_n(x;\alpha) = \sum_{k=0}^{n} \langle n \rangle_k \binom{x}{k} \mathcal{P}_{n-k}(\alpha), \tag{3.7}$$

$$\mathcal{P}_n(x;\alpha) - \mathcal{P}_n(\alpha) = \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \langle x \rangle_{k+1} \mathcal{P}_{n-1-k}(\alpha), \tag{3.8}$$

*with $\langle x \rangle_k$ given in (2.7).*

*Proof.* From (3.1), (3.3), the Cauchy product rule, and the Newton binomial expansion, we can write

$$\begin{aligned} \sum_{k=0}^{n} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!} &= \left( \sum_{n=0}^{\infty} \mathcal{P}_n(\alpha) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \binom{x}{n} z^n \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{x}{k} \binom{n}{k} k! \mathcal{P}_{n-k}(\alpha) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \langle n \rangle_k \binom{x}{k} \mathcal{P}_{n-k}(\alpha) \frac{z^n}{n!}. \end{aligned} \tag{3.9}$$

As a result of (3.9), we obtain (3.7). Using (3.1), (3.3), the Newton binomial expansion, and the Cauchy product rule, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\mathcal{P}_n(x;\alpha)-\mathcal{P}_n(\alpha)\right]\frac{z^n}{n!} &= \frac{z}{e^{-\alpha z}-1}\left[(1+z)^x-1\right]\\ &= \left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n+1}z^{n+1}\right)\\ &= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}n\,k!\mathcal{P}_{n-1-k}(\alpha)\right)\frac{z^n}{n!}.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\left[\mathcal{P}_n(x;\alpha)-\mathcal{P}_n(\alpha)\right]\frac{z^n}{n!}=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{n}{k+1}\binom{n-1}{k}\langle x\rangle_{k+1}\mathcal{P}_{n-1-k}(\alpha)\frac{z^n}{n!}. \tag{3.10}$$

Because of (3.10), we get (3.8). Which completes the proof of the Proposition 3.3. □

**Proposition 3.4.** *For $n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}-\{0\}$, let $\{\mathcal{P}(x;\alpha)\}_{n\geqslant 0}$ be the sequence generalized discrete U-Bernoulli-Korobov-kind polynomials. Then the following summation formulas hold:*

$$\mathcal{P}_n(x+y;\alpha)=\sum_{k=0}^{n}\binom{n}{k}\langle x+y\rangle_k\,\mathcal{P}_{n-k}(\alpha), \tag{3.11}$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_n(x+y;\alpha)\mathcal{P}_{n-k}(\beta)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(x;\alpha)\mathcal{P}_k(y;\beta), \tag{3.12}$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_n(x+y;\alpha)\mathcal{P}_{n-k}(\alpha)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(x;\alpha)\mathcal{P}_k(y;\alpha), \tag{3.13}$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k(x;\alpha)\,\alpha^{n-k}=e^{-\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(\alpha)\left(\frac{d}{d\alpha}\right)^{(x)}e^{\alpha}\alpha^k, \tag{3.14}$$

$$\mathcal{P}_n(x;\alpha)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k(x;\alpha)\alpha^{n-k}+\sum_{k=0}^{n-1}n\,\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}. \tag{3.15}$$

*Proof.* The representation (3.11) follows from (2.7) and (3.1). On the other hand, because of (3.1) for $\alpha$, $\beta$, and $x,y\in\mathbb{Z}^+$, we have

$$\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^x=\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}, \tag{3.16}$$

$$\left(\frac{z}{e^{-\beta z}-1}\right)(1+z)^y=\sum_{n=0}^{\infty}\mathcal{P}_n(y;\beta)\frac{z^n}{n!}. \tag{3.17}$$

Multiplying member by member to (3.16) and (3.17), we deduce

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(\beta)\mathcal{P}_k(x+y;\alpha)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(x;\alpha)\mathcal{P}_k(y;\beta)\frac{z^n}{n!}. \tag{3.18}$$

Therefore, of (3.18), we derive (3.12). Similarly, we can obtain (3.13).

We now prove (3.14). By (3.1) and (3.2), we have

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\frac{z\,e^{-\alpha z}\,e^{-\alpha}}{e^{-\alpha z}-1}\right)\sum_{n=0}^{\infty}\left(\frac{d}{d\alpha}\right)^{(x)} e^{\alpha}\,\alpha^n\,\frac{z^n}{n!}.$$

Then,

$$\left(\sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\alpha^n\frac{z^n}{n!}\right) = e^{-\alpha}\left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{d}{d\alpha}\right)^{(x)} e^{\alpha}\alpha^n\frac{z^n}{n!}\right). \tag{3.19}$$

Because of (3.19), we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k(x;\alpha)\,\alpha^{n-k}\frac{z^n}{n!} = \sum_{n=0}^{\infty} e^{-\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(\alpha)\left(\frac{d}{d\alpha}\right)^{(x)} e^{\alpha}\alpha^k\frac{z^n}{n!}. \tag{3.20}$$

Therefore, from (3.20) holds (3.14). To prove (3.15), we see that multiplying (3.1) by $e^{\alpha z}$ leads to

$$z\left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n}z^n\right) = \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} - \left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\right).$$

Hence,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\langle x\rangle_k\binom{n}{k}\alpha^{n-k}\frac{z^{n+1}}{n!} = \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k(x;\alpha)\alpha^{n-k}\frac{z^n}{n!}. \tag{3.21}$$

From (3.21), we derive

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}\,n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left[\mathcal{P}_n(x;\alpha) - \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_k(x;\alpha)\alpha^{n-k}\right]\frac{z^n}{n!}. \tag{3.22}$$

Whence the formula (3.15) follows from (3.22), which completes the proof of the Proposition 3.4. □

**Theorem 3.5.** *For every $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}-\{0\}$, the generalized discrete U-Bernoulli-Korobov-kind polynomials satisfy*

$$(n-1)\mathcal{P}_n(x;\alpha) - n\,x\,\mathcal{P}_{n-1}(x-1;\alpha) = \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l(\alpha)^{l+1}\mathcal{P}_{n-j-l}(\alpha)\mathcal{P}_j(x;\alpha). \tag{3.23}$$

*Proof.* By differentiating both sides of (3.1) with respect to $z$, we get

$$\frac{(1+z)^x}{(e^{-\alpha z}-1)} + \frac{x\,z\,(z+1)^{x-1}}{(e^{-\alpha z}-1)} + \frac{\alpha z e^{-\alpha z}(1+z)^x}{(e^{-\alpha z}-1)^2} = \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)n\frac{z^{n-1}}{n!},$$

and from this follows

$$\frac{z(1+z)^x}{(e^{-\alpha z}-1)} + \frac{x\,z^2\,(z+1)^{x-1}}{(e^{-\alpha z}-1)} + \alpha e^{-\alpha z}\left(\frac{z}{e^{-\alpha z}-1}\right)\left(\frac{z(1+z)^x}{e^{-\alpha z}-1}\right) = \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)n\frac{z^n}{n!},$$

which gives

$$\sum_{n=0}^{\infty}x\mathcal{P}_n(x-1;\alpha)\frac{z^{n+1}}{n!} + \alpha\left(\sum_{n=0}^{\infty}(-\alpha)^n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) n \frac{z^n}{n!} - \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!}.$$

So,

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) n \frac{z^n}{n!} - \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!} - \sum_{n=0}^{\infty} n x \mathcal{P}_{n-1}(x-1;\alpha) \frac{z^n}{n!}$$
$$= \alpha \left( \sum_{n=0}^{\infty} (-\alpha)^n \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \mathcal{P}_n(\alpha) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!} \right),$$

therefore

$$\sum_{n=0}^{\infty} [n\mathcal{P}_n(x;\alpha) - \mathcal{P}_n(x;\alpha) - n x \mathcal{P}_{n-1}(x-1;\alpha)] \frac{z^n}{n!}$$
$$= \alpha \left( \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (-\alpha)^l \mathcal{P}_{n-l}(\alpha) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!} \right)$$
$$= \alpha \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-\alpha)^l \mathcal{P}_{n-j-l}(\alpha) \mathcal{P}_j(x;\alpha) \right) \frac{z^n}{n!}.$$

As a result of the above, expression (3.23) follows. This proves the Theorem 3.5. □

**Theorem 3.6.** *The following relations hold for the generalized discrete U-Bernoulli-Korobov-kind polynomials defined in* (3.1)*:*

$$\frac{\partial \mathcal{P}_n(x;\alpha)}{\partial x} = \sum_{k=0}^{n-1} (-1)^k n \binom{n-1}{k} \frac{k!}{k+1} \mathcal{P}_{n-k-1}(x;\alpha), \quad (n \in \mathbb{N}), \tag{3.24}$$

$$(n-1)\mathcal{P}_n(x;\alpha) - n\,\gamma(x,z) \frac{\partial}{\partial x} \mathcal{P}_{n-1}(x;\alpha) - n\,\psi(z;\alpha) \frac{\partial}{\partial x} \mathcal{P}_{n-1}(x;\alpha) = 0, \tag{3.25}$$

*where* $\alpha \in \mathbb{R} - \{0\}$, $z \in \mathbb{C} - \{0, -1\}$, *and* $n \in \mathbb{N}$, *with*

$$\gamma(x,z) = \frac{x}{(1+z)\log(1+z)} \quad \text{and} \quad \psi(z;\alpha) = \frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z} - 1)\log(1+z)}.$$

*Proof.* By differentiating (3.1) with respect to $x$, we have

$$\sum_{n=0}^{\infty} \frac{\partial \mathcal{P}_n(x;\alpha)}{\partial x} \frac{z^n}{n!} = \left( \sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} \right)$$
$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \mathcal{P}_{n-1-k}(x;\alpha)(-1)^k \binom{n-1}{k} \frac{k!}{(k+1)(n-1)!} z^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{\partial \mathcal{P}_n(x;\alpha)}{\partial x} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{n\,k!}{(k+1)} \mathcal{P}_{n-1-k}(x;\alpha) \frac{z^n}{n!}.$$

As a result of these computations, we obtain (3.24). To prove (3.25), we differentiate (3.1) concerning $z$ as follows:

$$\frac{\partial}{\partial z} L(x,z;\alpha) = \sum_{n=1}^{\infty} \mathcal{P}_n(x;\alpha) \frac{z^{n-1}}{(n-1)!},$$

and

$$\frac{\partial}{\partial z}L(x,z;\alpha)=\frac{(1+z)^{x}}{(e^{-\alpha z}-1)}+\left[\frac{z\,(1+z)^{x}}{(e^{-\alpha z}-1)}\right]\left[\frac{x}{(1+z)}\right]+\left[\frac{z\,(1+z)^{x}}{(e^{-\alpha z}-1)}\right]\left[\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)}\right]. \tag{3.26}$$

Furthermore, by differentiating (3.1) concerning $x$, we have

$$\frac{\partial}{\partial x}L(x,z;\alpha)=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathcal{P}_n(x,\alpha)\,\frac{z^n}{n!}, \tag{3.27}$$

$$\frac{\partial}{\partial x}L(x,z;\alpha)=\frac{z\,(1+z)^{x}\log(1+z)}{(e^{-\alpha z}-1)}. \tag{3.28}$$

Equation (3.26) yields

$$\begin{aligned}\frac{\partial L(x,z;\alpha)}{\partial z}-\frac{(1+z)^{x}}{(e^{-\alpha z}-1)}-\left[\frac{z\,(1+z)^{x}\log(1+z)}{(e^{-\alpha z}-1)}\right]\left[\frac{x}{(1+z)\log(1+z)}\right]&\\ -\left[\frac{z\,(1+z)^{x}\log(1+z)}{(e^{-\alpha z}-1)}\right]\left[\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]&=0.\end{aligned} \tag{3.29}$$

Combining (3.29) with (3.27) and (3.28), we can write

$$\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-\frac{(1+z)^{x}}{(e^{-\alpha z}-1)}=0. \tag{3.30}$$

Thus, from (3.30), we have

$$z\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{zx}{(1+z)\log(1+z)}+\frac{z\alpha e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-L(x,z;\alpha)=0. \tag{3.31}$$

Hence, from (3.27) and (3.31), and after simplifying, we can get

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\,n\,\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\,\frac{z^n}{n!}\\ &\quad-\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathcal{P}_{n-1}(x;\alpha)\frac{nz^n}{n!}=0,\end{aligned}$$

and consequently

$$\begin{aligned}n\mathcal{P}_n(x;\alpha)-\mathcal{P}_n(x;\alpha)-\left[\frac{n\,x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}\mathcal{P}_{n-1}(x;\alpha)&\\ -\left[\frac{n\,\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathcal{P}_{n-1}(x;\alpha)&=0.\end{aligned} \tag{3.32}$$

In (3.32), doing $\gamma(x,z)=\dfrac{x}{(1+z)\log(1+z)}$, $\psi(z;\alpha)=\dfrac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)(\log(1+z))}$ follows (3.25). Theorem 3.6 is proved. □

## 4. Some connection formulas for the polynomials $\mathcal{P}_n(x;\alpha)$ and the difference operators $\Delta$ and $\nabla$

Based on (3.1), we introduce here some interesting algebraic relations connecting the polynomials $\mathcal{P}_n(x;\alpha)$ and other families of polynomials. Also, we study its relation with the operators $\Delta$ and $\nabla$.

**Theorem 4.1.** *Given $\alpha \in \mathbb{R}-\{0\}$, and let $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ be a sequence of generalized discrete U-Bernoulli-Korobov-kind polynomials. Then, the following assertions hold:*

$$\mathcal{P}_n(x;\alpha) = \sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}x^k\mathcal{P}_{n-j}(\alpha)s(j,k), \quad \textit{with } s(n,k) \textit{ given in } (2.8), \tag{4.1}$$

$$\mathcal{P}_n(x;\alpha) = \sum_{q=0}^{n}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}\binom{n}{q}\binom{q}{l}\binom{n-q}{l}x^k\mathcal{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j, \tag{4.2}$$

*where $b_n$ and $D_n$ are given in (2.3) and (2.5), respectively, and*

$$\mathcal{P}_n(x;\alpha) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{n}{k}b_{n-k}(x)\mathcal{P}_{k-j}(\alpha)D_j, \tag{4.3}$$

*where $b_n(x)$ is defined in (2.3), and*

$$\mathcal{P}_n(x;\alpha) = \sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{l}\binom{l}{j}\binom{n-l}{q}b_jD_{l-j}s(q,k)\mathcal{P}_{n-l-q}(\alpha)x^k, \tag{4.4}$$

*where $b_n$ is defined in (2.4).*

*Proof.* The statement (4.1) follows from (2.8) and (3.1). By using (2.4), (2.6), (2.8), and (3.1), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(e^{x\log(1+z)}\right)\\
&= \left(z\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{x^k}{z}\frac{[\log(1+z)]^k}{k!}\right)\\
&= \left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}s(n,k)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\sum_{k=0}^{\infty}x^k\\
&= \sum_{n=0}^{\infty}\left[\sum_{q=0}^{n}\binom{n}{q}\binom{q}{l}\binom{n-q}{j}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}x^k\mathcal{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j\right]\frac{z^n}{n!},
\end{aligned}$$

which completes the proof of (4.2). Taking (2.3), (2.6) into account, and (3.1), we can find (4.3). To prove (4.4), we use (2.4) as well as (2.8), and (3.1). Then, we deduce

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}x^k\frac{[\log(1+z)]^k}{k!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{q}\binom{l}{j}\binom{n-q}{q}b_jD_{l-j}s(q,k)\mathcal{P}_{n-l-q}(\alpha)x^k\frac{z^n}{n!}.
\end{aligned}$$

From which assertion (4.4) follows. Theorem 4.1 is fully proven. □

**Theorem 4.2.** *Let $\alpha \in \mathbb{R}-\{0\}$ and $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ be the sequence of generalized discrete U-Bernoulli-Korobov-kind polynomials in the variable x. Then, the following relations hold:*

$$\Delta_a\mathcal{P}_n(x;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\,\mathcal{P}_{n-k}(x;\alpha) - \mathcal{P}_n(x;\alpha), \tag{4.5}$$

$$\Delta \mathcal{P}_n(x;\alpha) = n\mathcal{P}_{n-1}(x;\alpha), \tag{4.6}$$
$$\nabla \mathcal{P}_n(x;\alpha) = n\mathcal{P}_{n-1}(x-1;\alpha), \tag{4.7}$$
$$\Delta \mathcal{P}_n(x;\alpha) + n\Delta\mathcal{P}_{n-1}(x;\alpha) = n\mathcal{P}_{n-1}(x+1;\alpha), \tag{4.8}$$

*with $\nabla$ and $\Delta_a$ the operators given in (2.12) and (2.14), respectively.*

*Proof.* We see that from (2.14) and (3.1), it follows

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_a \mathcal{P}(x;\alpha)\frac{z^n}{n!} &= \frac{z}{e^{-\alpha z}-1}(1+z)^x(1+z)^a - \frac{z}{e^{-\alpha z}-1}(1+z)^x \\ &= \left(\sum_{n=0}^{\infty} \mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{a}{n}z^n\right) - \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}.\end{aligned} \tag{4.9}$$

Hence, in (4.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_a \mathcal{P}(x;\alpha)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{a}{k}\frac{n!}{(n-k)!}\mathcal{P}_{n-k}(x;\alpha) - \mathcal{P}_n(x;\alpha)\right)\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\,\mathcal{P}_{n-k}(x;\alpha) - \mathcal{P}_n(x;\alpha)\right)\frac{z^n}{n!},\end{aligned}$$

from which, (4.5) follows. For the case $a = 1$, we obtain (4.6). To prove (4.7), we see that for (2.12) and (3.1), we get

$$\sum_{n=0}^{\infty} \nabla\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\frac{z^2}{e^{-\alpha z}-1}\right)(1+z)^x\left(\frac{1}{1+z}\right),$$

and consequently

$$\sum_{n=1}^{\infty} \nabla\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\mathcal{P}_n(x-1;\alpha)\frac{z^{n+1}}{n!} = \sum_{n=1}^{\infty}\mathcal{P}_{n-1}(x-1;\alpha)\,n\,\frac{z^n}{n!},$$

from which, (4.7) follows. Taking (3.1) into account, as well as using the operator $\Delta$, we get the following expression:

$$\sum_{n=0}^{\infty} \Delta\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^{x+1}\frac{z}{1+z}.$$

Then, we see that

$$(1+z)\sum_{n=0}^{\infty} \Delta\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\mathcal{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}.$$

Thus, we have

$$\sum_{n=1}^{\infty}[\Delta\mathcal{P}_n(x;\alpha) + n\Delta\mathcal{P}_{n-1}(x;\alpha) - n\mathcal{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!} = 0,$$

and, as a consequence, (4.8) follows. Hence, Theorem 4.2 is proved. □

On the other hand, by using (4.7) and (4.8), we can see that the polynomials $\mathcal{P}_n(x;\alpha)$ satisfy (2.15) in such a way that

$$\nabla\mathcal{P}_n(x;\alpha) = \Delta\mathcal{P}_n(x;\alpha) - \Delta\nabla\mathcal{P}_n(x;\alpha). \tag{4.10}$$

**Proposition 4.3.** *For $\alpha \in \mathbb{R} - \{0\}$, let $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ be the sequence of generalized discrete U-Bernoulli-Korobov-kind polynomials in the variable $x$. Then, the following relations hold:*

$$\Delta(2\mathcal{P}_n(x;\alpha) + n\mathcal{P}_{n-1}(x;\alpha)) - 2\Delta\nabla\mathcal{P}_n(x;\alpha) = 2n\mathcal{P}_{n-1}(x;\alpha) + n(n-1)\mathcal{P}_{n-2}(x;\alpha), \tag{4.11}$$

$$\Delta\mathcal{P}_n(x;\alpha) - \Delta\nabla\mathcal{P}_n(x;\alpha) = n\mathcal{P}_{n-1}(x;\alpha) - n(n-1)\mathcal{P}_{n-2}(x-1;\alpha). \tag{4.12}$$

*Proof.* By using (3.1) and applying the operator $\Delta$, (4.8), and (4.10), it follows

$$\begin{aligned}
&\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^{x+1}(1+z) = (1+2z+z^2)\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\\
&\quad\Leftrightarrow \sum_{n}^{\infty}[\mathcal{P}_n(x+1;\alpha)-\mathcal{P}_n(x;\alpha)]\frac{z^n}{n!} = 2\sum_{n=1}^{\infty}n\mathcal{P}_{n-1}(x;\alpha)\frac{z^n}{n!} + \sum_{n=2}^{\infty}n(n-1)\mathcal{P}_{n-2}(x;\alpha)\frac{z^n}{n!}\\
&\quad\quad -\sum_{n=0}^{\infty}\mathcal{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}\\
&\quad\Leftrightarrow \sum_{n=1}^{\infty}\Delta\mathcal{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=1}^{\infty}[2n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x;\alpha)-n\mathcal{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!}\\
&\quad\Leftrightarrow n\mathcal{P}_{n-1}(x+1;\alpha) = 2n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x;\alpha)-\Delta\mathcal{P}_n(x;\alpha)\\
&\quad\Leftrightarrow 2\Delta\mathcal{P}_n(x;\alpha)+n\Delta\mathcal{P}_{n-1}(x;\alpha) = 2n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x;\alpha)\\
&\quad\Leftrightarrow 2\Delta\mathcal{P}_n(x;\alpha)-2\Delta\nabla\mathcal{P}_n(x;\alpha)+n\Delta\mathcal{P}_{n-1}(x;\alpha) = 2n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x;\alpha).
\end{aligned}$$

As a consequence of these computations, we obtain (4.11). To prove (4.12), we use (3.1) and the operator $\nabla$, (4.8), and (4.10), as follows:

$$\begin{aligned}
&(1+z)^2\sum_{n=0}^{\infty}\mathcal{P}_n(x-1;\alpha)\frac{z^n}{n!} = (1+z)\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}\\
&\quad\Leftrightarrow 2\sum_{n=0}^{\infty}\mathcal{P}_n(x-1;\alpha)\frac{z^{n+1}}{n!}+\sum_{n=0}^{\infty}\mathcal{P}_n(x-1;\alpha)\frac{z^{n+2}}{n!}\\
&\quad\quad = \sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^{n+1}}{n!}+\sum_{n=0}^{\infty}\mathcal{P}_n(x;\alpha)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathcal{P}_n(x-1;\alpha)\frac{z^n}{n!}\\
&\quad\Leftrightarrow 2n\mathcal{P}_{n-1}(x-1;\alpha)-n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x-1;\alpha) = \nabla\mathcal{P}_n(x;\alpha)\\
&\quad\Leftrightarrow 2n\mathcal{P}_{n-1}(x-1;\alpha) = n\mathcal{P}_{n-1}(x;\alpha)-n(n-1)\mathcal{P}_{n-2}(x-1;\alpha)+\nabla\mathcal{P}_n(x;\alpha)\\
&\quad\Leftrightarrow \Delta\,(2\mathcal{P}_n(x;\alpha)+n\,\mathcal{P}_{n-1}(x;\alpha)) = 2\Delta\nabla\mathcal{P}_n(x;\alpha) = 2\,n\mathcal{P}_{n-1}(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x;\alpha)\\
&\quad\Leftrightarrow \nabla\mathcal{P}_n(x;\alpha)+n(n-1)\mathcal{P}_{n-2}(x-1;\alpha)-n\mathcal{P}_{n-1}(x;\alpha) = 0,
\end{aligned}$$

from which, (4.12) follows. This proves the Proposition 4.3. □

## 5. Orthogonality of the generalized discrete U-Bernoulli-Korobov-kind polynomials

We define the discrete weight function $\omega^{\alpha}$ for U-Bernoulli-Korobov-kind polynomials as

$$\omega^{\alpha}(x;\beta) = \frac{(-\alpha)^x e^{\alpha}(1-e^{\alpha\beta})^2}{x!}, \tag{5.1}$$

with $x \in \mathbb{N}$, $\alpha < 0$, $z$, $\nu \in \mathbb{C}$, and $\lambda_1 \in \mathrm{Re}(z)$, $\sigma_1 \in \mathrm{Re}(\nu)$, $\beta = \lambda_1 = \sigma_1$. With this weight, we can consider on $\mathbb{P}$, the inner product $\langle f,\, g\rangle_{\omega^{\alpha}}$ as

$$\langle f,\, g\rangle_{\omega^{\alpha}} = \sum_{x=0}^{\infty} f(x)g(x)\omega^{\alpha}(x;\beta), \tag{5.2}$$

which has positive weights for every $\alpha < 0$. Note that of (2.17), the weight function $\omega^{\alpha}(x;\beta)$ satisfies the Pearson-type difference equation

$$\begin{aligned}\nabla\omega^{\alpha}(x;\beta) &= \omega^{\alpha}(x;\beta) - \omega^{\alpha}(x-1;\beta)\\ &= \left(1-\frac{x}{(-\alpha)}\right)\left(\frac{e^{\alpha}(-\alpha)^{x}(1-e^{\alpha\beta})^{2}}{x!}\right) = \left(\frac{\alpha+x}{\alpha}\right)\omega^{\alpha}(x;\beta).\end{aligned} \tag{5.3}$$

**Theorem 5.1.** *If $\alpha \in \mathbb{R}$, with $\alpha < 0$ and $m$, $n \in \mathbb{R}$, then, the generalized discrete U-Bernoulli -Korobov-kind polynomials satisfy the following orthogonality relation:*

$$\sum_{x=0}^{\infty} \mathcal{P}_m(x;\alpha)\mathcal{P}_n(x;\alpha)\,\omega^{\alpha}(x;\beta) = (-\alpha)^{n-1}\,n^2\,\Gamma(n)\delta_{mn}, \tag{5.4}$$

*where $\delta_{mn}$ denotes the Kronecker delta, $\Gamma$ is the gamma function, $\omega^{\alpha}(x;\beta)$ given in (5.1), and $|z|$, $|\nu| < \dfrac{2\pi}{|\alpha|}$.*

*Proof.* Using (3.1), (3.3), the Cauchy product property, and taking into account the binomial theorem, we can see that

$$L(x,z;\alpha) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k}\frac{\mathcal{P}_{n-k}(\alpha)}{(n-k)!}\,z^{n}. \tag{5.5}$$

Hence, from (5.5), it follows that

$$L(x,z;\alpha) = \sum_{n=0}^{\infty} L_n(x;\alpha)z^n, \tag{5.6}$$

we note that

$$L_n(x;\alpha) = \sum_{k=0}^{n}\binom{x}{k}\frac{\mathcal{P}_{n-k}(\alpha)}{(n-k)!}. \tag{5.7}$$

Therefore, by using (2.7) in (5.7), it follows that

$$L_n(x;\alpha) = \sum_{k=0}^{n}\frac{\langle x\rangle_k}{k!}\frac{\mathcal{P}_{n-k}(\alpha)}{(n-k)!}.$$

Likewise, we see that

$$L(x,\nu;\alpha) = \sum_{m=0}^{\infty} L_m(x;\alpha)\,\nu^{m}, \tag{5.8}$$

then we have

$$L_m(x;\alpha) = \sum_{k=0}^{m}\binom{x}{k}\frac{\mathcal{P}_{m-k}(\alpha)}{(m-k)!} = \sum_{k=0}^{m}\frac{\langle x\rangle_k}{k!}\frac{\mathcal{P}_{m-k}(\alpha)}{(m-k)!}.$$

Now, for any $k$, it follows from (5.6) and (5.8) that

$$\begin{aligned} L(x,z;\alpha)L(x,\nu;\alpha) &= \left[\frac{z\nu e^{\alpha z+\alpha\nu}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right][(1+z)(1+\nu)]^{x}\\ \Leftrightarrow (-\alpha)^{k}L(k,z;\alpha)L(k,\nu;\alpha) &= \left[\frac{z\nu e^{\alpha z+\alpha\nu}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right][-\alpha(1+z)(1+\nu)]^{k},\end{aligned}$$

which gives

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-\alpha)^k L(k,z;\alpha)L(k,\nu;\alpha)}{k!} &= \frac{z\nu e^{\alpha z+\alpha\nu}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\sum_{k=0}^{\infty}\frac{[-\alpha(1+z)(1+\nu)]^k}{k!}\\ &= \left[\frac{z\nu\, e^{-\alpha}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right] e^{-\alpha z\nu} = \sum_{n=0}^{\infty}\left[\frac{n\, e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right]\frac{z^n\nu^n}{n!}.\end{aligned} \tag{5.9}$$

On the other hand, because of (5.6) and (5.8), we also have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-\alpha)^k L(k,z;\alpha)L(k,\nu;\alpha)}{k!} &= \sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\sum_{n=0}^{\infty}L_n(k;\alpha)\,z^n\sum_{m=0}^{\infty}L_m(k;\alpha)\,\nu^m\\ &= \sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k;\alpha),L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!}\,z^n\nu^m.\end{aligned} \tag{5.10}$$

So, from (5.9) and (5.10) follows

$$\sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k;\alpha)L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!}\,z^n\nu^m = \sum_{n=0}^{\infty}\left[\frac{e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right]\frac{nz^n\nu^n}{n!}. \tag{5.11}$$

It is immediate that, from equation (5.11), we conclude

$$\sum_{k=0}^{\infty}L_m(k;\alpha)L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!} = \begin{cases}\dfrac{(-\alpha)^{n-1}\,n\,e^{-\alpha}}{n!}\left[\dfrac{1}{(1-e^{\alpha z})(1-e^{\alpha\nu})}\right], & \text{if } m=n,\\ 0, & \text{if } m\neq n.\end{cases} \tag{5.12}$$

Let us now take that $t_1 = e^{\alpha z}$, $t_2 = e^{\alpha\nu}$, and considering $z = \lambda_1 + i\lambda_2$, $\nu = \sigma_1 + i\sigma_2$. It follows that

$$t_1 = e^{\alpha z} = e^{\alpha\lambda_1}e^{i\,\alpha\lambda_2} \quad\text{and}\quad t_2 = e^{\alpha\nu} = e^{\alpha\sigma_1}e^{i\,\alpha\sigma_2},$$

so we see that $|t_1| = e^{\alpha\lambda_1}$, $|t_2| = e^{\alpha\sigma_1}$ with $|\lambda_1|$, $|\lambda_2|$, $|\sigma_1|$, $|\sigma_2| < \dfrac{2\pi}{|\alpha|}$. Thus, we get

$$(1-e^{\alpha z})(1-e^{\alpha\nu}) = (1-|t_1|e^{i\,\alpha\lambda_2})(1-|t_2|e^{i\,\alpha\sigma_2}). \tag{5.13}$$

If now we establish $\lambda_2$, $\sigma_2 \to 0$, and $\beta = \lambda_1 = \sigma_1$, then we can be write (5.13) as

$$(1-e^{\alpha z})(1-e^{\alpha\nu}) = (1-e^{\alpha\beta})^2.$$

Therefore, in (5.12), we find

$$\sum_{x=0}^{\infty}\frac{\mathcal{P}_m(x;\alpha)}{m!}\frac{\mathcal{P}_n(x;\alpha)}{n!}\omega^{\alpha}(x;\beta) = \frac{n\,(-\alpha)^{n-1}}{n!}\delta_{mn} \Leftrightarrow \sum_{x=0}^{\infty}\mathcal{P}_m(x;\alpha)\mathcal{P}_n(x;\alpha)\omega^{\alpha}(x;\beta) = \frac{n\,(-\alpha)^{n-1}}{n!}\delta_{mn},$$

And, as a consequence (5.4) follows, so Theorem 5.1 is proved. □

Under the assumption of Theorem 5.1, we can then obtain a three-term recurrence relation that the sequence $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ satisfies.

**Theorem 5.2.** *Let $\alpha < 0$ and $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$ be a sequence of generalized discrete U-Bernoulli-Korobov-kind polynomials that are orthogonal on $\mathbb{N}$ for the inner product (5.2). Then, we have the following three-term recurrence relation:*

$$x\mathcal{P}_{n-1}(x;\alpha) = \gamma_n\mathcal{P}_n(x;\alpha) + \xi_n\mathcal{P}_{n-1}(x;\alpha) + \lambda_n\mathcal{P}_{n-2}(x;\alpha), \quad n > 2, \tag{5.14}$$

*with*

$$\gamma_n = \frac{n\,\alpha}{2}\xi_n = \left[(s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6}\right]\lambda_n = \frac{(-\alpha)(n-1)^3}{n(n-3)^2}\frac{\Gamma(n-1)}{\Gamma(n-2)},$$

*and $s(n,k)$ given in (2.8).*

*Proof.* To prove (5.14), we first expand the polynomial $x\mathcal{P}_{n-1}(x;\alpha)$, which is of degree $n$ in terms of $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$:

$$x\mathcal{P}_{n-1}(x;\alpha)=\sum_{k=0}^{n}\frac{a(k,n-1)}{g_k(\alpha)}\mathcal{P}_n(x;\alpha), \tag{5.15}$$

with $\alpha<0$ is a fixed parameter, $n,x\in\mathbb{N}$. From the orthogonality of $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$, we obtain

$$\frac{a(k,n-1)}{g_k(\alpha)}=\frac{\langle x\mathcal{P}_{n-1}(x;\alpha),\mathcal{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathcal{P}_k(x;\alpha),\mathcal{P}_k(x;\alpha)\rangle_{\omega^\alpha}}=\frac{\langle \mathcal{P}_{n-1}(x;\alpha),x\mathcal{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathcal{P}_k(x;\alpha),\mathcal{P}_k(x;\alpha)\rangle_{\omega^\alpha}}.$$

As $x\mathcal{P}_k(x;\alpha)$ is a polynomial of degree $k+1$, by orthogonality $a(k,n-1)=0$ for $k<n-2$ and therefore (5.15) can be written in the form

$$x\mathcal{P}_{n-1}(x;\alpha)=\frac{a(n,n-1)}{g_n(\alpha)}\mathcal{P}_n(x;\alpha)+\frac{a(n-1,n-1)}{g_{n-1}(\alpha)}\mathcal{P}_{n-1}(x;\alpha)+\frac{a(n-2,n-1)}{g_{n-2}(\alpha)}\mathcal{P}_{n-2}(x;\alpha). \tag{5.16}$$

On the other hand, taking (2.9), (3.7) into account, and (5.16), we can obtain:

$$\begin{aligned}\mathcal{P}_n(x;\alpha)&=\mathcal{P}_0(\alpha)x^n+\left(\mathcal{P}_0(\alpha)s(n,n-1)+\mathcal{P}_1(\alpha)\binom{n}{n-1}\right)x^{n-1}\\&+\left(\mathcal{P}_0(\alpha)s(n,n-2)+\mathcal{P}_1(\alpha)\binom{n}{n-1}s(n,n-2)+\mathcal{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-2}+\cdots,\end{aligned} \tag{5.17}$$

also

$$\mathcal{P}_{n-1}(x;\alpha)=\mathcal{P}_1(\alpha)\binom{n}{n-1}x^{n-1}+\left(\mathcal{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1}+\mathcal{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-2}+\cdots, \tag{5.18}$$

in an analogous way

$$\mathcal{P}_{n-2}(x;\alpha)=\mathcal{P}_1(\alpha)\binom{n-1}{n-2}x^{n-2}+\left(\mathcal{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2}+\mathcal{P}_2(\alpha)\binom{n-1}{n-3}\right)x^{n-3}+\cdots, \tag{5.19}$$

we can find

$$x\mathcal{P}_{n-1}(x;\alpha)=\mathcal{P}_1(\alpha)\binom{n}{n-1}x^{n}+\left(\mathcal{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1}+\mathcal{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-1}+\cdots,$$

moreover

$$x\mathcal{P}_{n-2}(x;\alpha)=\mathcal{P}_1(\alpha)\binom{n-1}{n-2}x^{n-1}+\left(\mathcal{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2}+\mathcal{P}_2(\alpha)\binom{n-1}{n-3}\right)x^{n-2}+\cdots,$$

also, we can write $x\mathcal{P}_{n-2}(x;\alpha)$ in terms of $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$, we have

$$\begin{aligned}x\mathcal{P}_{n-2}(x;\alpha)&=\sum_{k=0}^{n-1}\frac{a(k,n-2)}{g_k(\alpha)}\mathcal{P}_k(x;\alpha)\\&=\frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathcal{P}_{n-1}(x;\alpha)+\frac{a(n-2,n-2)}{g_{n-2}(\alpha)}\mathcal{P}_{n-2}(x;\alpha)+\frac{a(n-3,n-2)}{g_{n-3}(\alpha)}\mathcal{P}_{n-3}(x;\alpha).\end{aligned} \tag{5.20}$$

Now, from (5.19) and (5.20), we deduce:

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)}=\binom{n-1}{n}.$$

So,

$$x\mathcal{P}_{n-2}(x;\alpha) = \binom{n-1}{n}\mathcal{P}_{n-1}(x;\alpha) + P(x). \tag{5.21}$$

From (5.16) and (5.21), it is seen that

$$\frac{a(n-2,n-1)}{g_{n-2}(\alpha)} = \frac{\langle \mathcal{P}_{n-1}(x;\alpha), x\mathcal{P}_{n-2}(x;\alpha)\rangle_{\omega^\alpha}}{g_{n-2}(\alpha)} = \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}. \tag{5.22}$$

By using (5.16), (5.17), and (5.18), we obtain

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{\mathcal{P}_1(\alpha)}{\mathcal{P}_0(\alpha)}\binom{n}{n-1}. \tag{5.23}$$

Now the substitution of (5.22) and (5.23) into (5.16) gives

$$x\mathcal{P}_{n-1}(x;\alpha) = \frac{\mathcal{P}_1(\alpha)}{\mathcal{P}_0(\alpha)}\binom{n}{n-1}\mathcal{P}_n(x;\alpha) + \frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathcal{P}_{n-1}(x;\alpha) + \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}\mathcal{P}_{n-2}(x;\alpha). \tag{5.24}$$

Comparing the coefficients of the highest terms on the left-hand and right-hand sides of (5.24), we have

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)} = (s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6}. \tag{5.25}$$

Because of Theorem 5.1, it follows

$$\frac{a(n-2,n-1)}{g_{n-2}(\alpha)} = \frac{(-\alpha)(n-1)^3}{n(n-3)^2}\frac{\Gamma(n-1)}{\Gamma(n-2)}, \tag{5.26}$$

finally from (5.23), we have

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{n\alpha}{2}. \tag{5.27}$$

Using (5.25), (5.26), and (5.27) into (5.24) follows (5.14), so Theorem 5.2 is proved. □

By using the orthogonality of the polynomials $\mathcal{P}_n(x;\alpha)$, we give the following relation.

**Proposition 5.3.** *The generalized discrete U-Bernoulli-Korobov-kind polynomials, which are orthogonal with respect to the inner product (5.2), fulfill the relation*

$$\Delta\mathcal{P}_n(x;\alpha) = J^{\alpha}_{k,n}\mathcal{P}_{n-1}(x;\alpha), \tag{5.28}$$

*where $J^{\alpha}_{k,n}$ are the Fourier coefficients.*

*Proof.* If we write the polynomial $\Delta\mathcal{P}_n(x;\alpha)$ in terms of $\{\mathcal{P}_n(x;\alpha)\}_{n\geqslant 0}$, we have

$$\mathcal{P}_n(x+1;\alpha) - \mathcal{P}_n(x;\alpha) = \sum_{k=0}^{n-1} J^{\alpha}_{k,n}\mathcal{P}_k(x;\alpha),$$

besides, for $0 \leqslant k \leqslant n-1$,

$$J^{\alpha}_{k,n} = \frac{\langle \Delta\mathcal{P}_n, \mathcal{P}_k\rangle_{\omega^\alpha}}{\langle \mathcal{P}_k, \mathcal{P}_k\rangle_{\omega^\alpha}}.$$

Hence, by (2.7) and (2.16), we have

$$\langle \mathcal{P}_k, \mathcal{P}_k\rangle_{\omega^\alpha} J^{\alpha}_{k,n} = \langle \Delta\mathcal{P}_n, \mathcal{P}_k\rangle_{\omega^\alpha}$$

$$
\begin{aligned}
&= \sum_{g=0}^{\infty} \Delta \mathcal{P}_n(g;\alpha)\mathcal{P}_k(g;\alpha)\omega^{\alpha}(g;\beta) \\
&= -\sum_{g=0}^{\infty} \mathcal{P}_n(g;\alpha)\nabla(\omega^{\alpha}(g;\beta)\mathcal{P}_k(g;\alpha)) \\
&= -\sum_{g=0}^{\infty} \mathcal{P}_n(g;\alpha)\omega^{\alpha}(g;\beta)\nabla\mathcal{P}_k(g;\alpha) - \sum_{g=0}^{\infty} \mathcal{P}_n(g;\alpha)\mathcal{P}_k(g-1;\alpha)\nabla\omega^{\alpha}(g;\beta),
\end{aligned}
$$

from which, by orthogonality, the first sum is zero since $\nabla\mathcal{P}_k$ is of degree $k < n+1$. For the second sum let us consider (5.3),

$$
\langle \mathcal{P}_k, \mathcal{P}_k\rangle_{\omega^{\alpha}} J^{\alpha}_{k,n} = -\frac{1}{\alpha}\sum_{g=0}^{\infty} \mathcal{P}_n(g;\alpha)\mathcal{P}_k(g-1;\alpha)(\alpha+g)\omega^{\alpha}(g;\beta),
$$

if we use orthogonality again, only $J^{\alpha}_{n-1,n}$ can be non-zero, and as a consequence, (5.28) follows, so Proposition 5.3 is proved. □

### Author's contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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