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# Existence and oscillation results for the Hybrid generalized pantograph Hilfer fractional difference equation

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## Abstract

The objective of this study is to analyze the hybrid generalized pantograph Hilfer fractional difference equation's existence, uniqueness and oscillatory behaviour. Our technique, in contrast to previous approaches in the literature, is based on certain newly defined features of discrete fractional calculus and a few mathematical inequalities. The hybrid fixed point theorem has been used to investigate the existence of solutions, and the Banach contraction theorem has been used to show that the solution is unique. Furthermore, a set of adequate requirements is deduced to guarantee oscillation in the solutions of the hybrid generalized pantograph Hilfer fractional difference equation. We provide two numerical simulations at the end of the article to demonstrate the effects of the main results.

**Keywords:** Oscillation, hybrid, pantograph, Hilfer fractional difference operator.

**2020 MSC:** 26A33, 39A12, 39A21.



## 1. Introduction

Mathematicians and scientists have known about fractional calculus for over three centuries. But only in the last few decades, these notions have been utilized in several disciplines such as engineering, science, economics, and so on [11, 40]. The qualitative characteristics, such as oscillation, stability, controllability, asymptotic behavior, and so on, are of great interest to scientists and researchers [4, 5, 10, 14, 17, 23, 24, 33–37, 42]. The understanding of fractional differential equations has come a long way. However, there hasn't been much advancement in the concepts of fractional difference equations.

In dynamical models, oscillation and delay effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [27, 28]. Oscillation is a crucial aspect of applied mathematics and can be created or removed by the inclusion of nonlinearity, delay or a stochastic term. The oscillation of differential and difference equations opens up a wide range of practical applications, including torsional oscillations, periodic oscillations, voltage-controlled neuron theories and harmonic oscillation with damping; see, e.g., the papers [3, 12, 13, 16, 29,

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30] for more details. In [19], Grace et al. introduced and examined the oscillation of fractional differential equations. In the past twenty years, researchers have been continuing this work due to the growth in the ideology of fractional calculus. For examples, see [2, 8, 39] and the references cited therein.

In line with this, several researchers have reported many findings on the fractional difference equations [1, 3, 6, 25, 31, 32]. Only a few fractional difference operators, such as the Caputo operator and the Riemann-Liouville operator, are used extensively to model real-world phenomena, despite the existence of various fractional order difference operators such as Hirota's operator, Atsushi's operator, Grunwald Letnikov's operator. In [31, 32] , Marian et al. produced few oscillation findings for the fractional difference equations

$$\begin{cases} \Delta^{\rho}\varpi(\kappa)+\Upsilon_1(\kappa,\varpi(\kappa+\rho)) = \mathfrak{w}(\kappa)+\Upsilon_2(\kappa,\varpi(\kappa+\rho)),\\ \Delta^{\rho-1}\varpi(\kappa)|_{\kappa=0} = \Upsilon_0,\end{cases}$$

and

$$\begin{cases} \Delta^{\rho}\varpi(\kappa)+\Upsilon_1(\kappa+\rho,\varpi(\kappa+\rho)) = \mathfrak{w}(\kappa)+\Upsilon_2(\kappa+\rho,\varpi(\kappa+\rho)),\\ \Delta^{\rho-1}\varpi(\kappa)|_{\kappa=0} = \Upsilon_0,\end{cases}$$

respectively, where $\kappa \in \mathbb{N}_{\alpha}$, $\Delta^{\rho}$ symbolizes the Riemann-Liouville fractional difference operator of order $\rho$ and $0 < \rho \leqslant 1$. They used

($\mathscr{A}$**.1**) $\varpi\Upsilon_i(\kappa,\varpi) > 0 (i = 1,2)$, $\varpi \neq 0$, $\kappa \geqslant \alpha$;

($\mathscr{A}$**.2**) $|\Upsilon_1(\kappa,\varpi)| \geqslant |\mathfrak{p}_1(\kappa)||\varpi|^{\iota_1}$ and $|\Upsilon_2(\kappa,\varpi)| \leqslant |\mathfrak{p}_2(\kappa)||\kappa|^{\iota_2}$, $\varpi \neq 0$, $\kappa \geqslant \alpha$, where $\mathfrak{p}_1,\mathfrak{p}_2 \in \mathcal{C}([\alpha,\infty),\mathbb{R}^+)$ and $\iota_1,\iota_2 > 0$ are real numbers,

by taking $\alpha = 0$ to obtain the oscillation results.

In [25], Kisalar et al. obtained oscillation theorems for the higher-order nonlinear fractional difference equations

$$\begin{cases} (\Delta^{\rho}\varpi)(\kappa)+\Upsilon_1(\kappa,\varpi(\kappa+\rho)) = \mathfrak{w}(\kappa)+\Upsilon_2(\kappa,\varpi(\kappa+\rho)),\\ (\Delta^{-(\mathfrak{k}-\rho)}\varpi)(\kappa)|_{\kappa=\alpha} = \Upsilon_{\mathfrak{k}} \in \mathbb{R}, \qquad \mathfrak{k}=1,2,\dots,\mathfrak{m},\end{cases}$$

and

$$\begin{cases} (\Delta^{\rho}_{*}\varpi)(\kappa)+\Upsilon_1(\kappa,\varpi(\kappa+\rho)) = \mathfrak{w}(\kappa)+\Upsilon_2(\kappa,\varpi(\kappa+\rho)),\\ (\Delta^{(\mathfrak{k})}\varpi)(\kappa)|_{\kappa=\alpha} = \tilde{\Upsilon}_{\mathfrak{k}} \in \mathbb{R}, \quad \mathfrak{k}=0,1,2,\dots,\mathfrak{m}-1,\end{cases}$$

involving the Riemann-Liouville operator $\Delta^{\rho}$ and the Caputo operator $\Delta^{\rho}_{*}$, of arbitrary order $\rho$, such that $\mathfrak{m}-1 < \rho \leqslant \mathfrak{m}$ for some $\mathfrak{m} \in \mathbb{N}$, under the assumptions ($\mathscr{A}$**.1**) and ($\mathscr{A}$**.2**). They also gave oscillation results using ($\mathscr{A}$**.1**) and

($\mathscr{A}$**.3**) $|\Upsilon_1(\kappa,\varpi)| \leqslant |\mathfrak{p}_1(\kappa)||\varpi|^{\iota_1}$ and $|\Upsilon_2(\kappa,\varpi)| \geqslant |\mathfrak{p}_2(\kappa)||\varpi|^{\iota_2}$, $\varpi \neq 0$, $\kappa \geqslant \alpha$, where $\mathfrak{p}_1,\mathfrak{p}_2 \in \mathcal{C}([\alpha,\infty),\mathbb{R}^+)$ and $\iota_1,\iota_2 > 0$ are real numbers.

for the above two fractional difference equations. The Hilfer fractional difference operator, which generalizes the Riemann-Liouville and the Caputo fractional difference operators, was derived by Haider et al. [21] in 2020. In [41], Uzun studied the oscillatory behavior for higher order nonlinear Hilfer fractional difference equation

$$\begin{cases} (\Delta^{\rho,\omega}\varpi)(\kappa)+\Upsilon_1\big(\kappa,\varpi(\kappa+\rho)\big) = \mathfrak{w}(\kappa)+\Upsilon_2\big(\kappa,\varpi(\kappa+\rho)\big), & \kappa \in \mathbb{N}_{\alpha+\mathfrak{n}-\rho},\\ \Delta_{\alpha}^{\mathfrak{k}-(\mathfrak{n}-\gamma)}\varpi(\kappa)|_{\kappa=\alpha+\mathfrak{n}-\gamma} = \Upsilon_{\mathfrak{k}}, & \mathfrak{k}=0,1,2,\dots,\mathfrak{n},\end{cases}$$

of order $\rho$ and type $\omega$ such that $\mathfrak{n}-1 < \rho \leqslant \mathfrak{n}$ for some $\mathfrak{n} \in \mathbb{N}$ and $0 \leqslant \omega \leqslant 1$. They obtained some oscillation results using ($\mathscr{A}$**.3**) and

($\mathscr{A}$**.4**) $\frac{\Upsilon_i(\kappa,\varpi)}{\varpi} > 0$ $(i = 1,2)$, $\varpi \neq 0$, $\kappa \geqslant \alpha$.

Further, in nonlinear analysis, perturbation techniques are effective tools for examining several features of the solution of nonlinear dynamical systems. They can be used to explain, predict and show the nonlinear effects brought on by vibrating systems. Often, perturbed nonlinear equations are approached using hybrid fixed point theory. Hybrid difference equations are nonlinear difference equations that have a perturbation of the equation combining multiplication or division by a term (quadratic perturbation). In recent years, academicians have become increasingly interested in hybrid equations of fractional order since they encompass a variety of dynamic systems. Using the Hilfer operator, Shammakh et al. [38] presented the stability characteristics associated with the nonlinear fractional order generalized pantograph equation with discrete time of the form

$$\begin{cases} \Delta_\alpha^{\rho,\omega}\mathfrak{z}(\kappa) = \mathfrak{w}\big(\kappa+\rho-1, \varpi(\kappa+\rho-1), \varpi(\mathfrak{p}((\kappa+\rho-1))\big), \\ \Delta_\alpha^{-(1-\gamma)}\mathfrak{z}((\kappa)|_{\kappa=\alpha+1-\gamma} = \mathfrak{z}_1, \end{cases}$$

where $\gamma = \rho+\omega-\rho\omega$ with $0<\rho\leqslant 1$, $0\leqslant\omega\leqslant 1$ and $\mathfrak{z}((\kappa) = \frac{\varpi(\kappa)}{\varpi(\kappa,\varpi(\kappa),\varpi(\mathfrak{q}(\kappa)))}$. As far as we are aware, the oscillatory behavior of hybrid generalized pantograph Hilfer fractional difference equations has not been studied yet. Inspired by the preceding, we study the existence, uniqueness, and oscillatory behavior of the following hybrid generalized pantograph Hilfer fractional difference problem (HGPHFDP)

$$\begin{cases} \Delta_\alpha^{\rho,\omega}\mathfrak{z}(\kappa) + \Upsilon_1\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big) = \mathfrak{w}(\kappa) + \Upsilon_2\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), & \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big), \\ \Delta_\alpha^{\mathfrak{k}-(\mathfrak{n}-\gamma)}\mathfrak{z}((\kappa)|_{\kappa=\alpha+\mathfrak{n}-\gamma} = \mathfrak{z}_{\mathfrak{k}}, & \mathfrak{k} = 0,1,2,\ldots,\mathfrak{n}-1, \end{cases} \tag{1.1}$$

where $\mathfrak{n}-1<\rho\leqslant\mathfrak{n}$, $\mathfrak{n}\in\mathbb{N}_1$, $0\leqslant\omega\leqslant 1$, $\kappa\in\mathbb{N}_{\alpha+\mathfrak{n}-\rho}$, $\gamma=\rho+\mathfrak{n}\omega-\rho\omega$, and $\mathfrak{z}((\kappa) = \frac{\varpi(\kappa)}{\varpi(\kappa,\varpi(\kappa),\varpi(\mathfrak{q}(\kappa)))}$. Here $\Delta_\alpha^{\rho,\omega}$ is the Hilfer Type fractional difference operator of order $\rho$ and type $\omega$. Also, $\Upsilon_1$, $\Upsilon_2 : \mathbb{N}_{\alpha+\mathfrak{n}-\rho}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\mathfrak{w}:\mathbb{N}_{\alpha+\mathfrak{n}-\rho}\to\mathbb{R}$, $\mathfrak{F}:\mathbb{N}_\alpha\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $\mathfrak{p}:\mathbb{N}_{\alpha+\mathfrak{n}-\rho}\to[0,1]\cap\mathbb{N}_{\alpha+\mathfrak{n}-\rho}$, and $\mathfrak{q}:\mathbb{N}_\alpha\to[0,1]\cap\mathbb{N}_\alpha$ are continuous functions. The article is structured as follows. Section 2 offers crucial lemmas and definitions. Section 3 provides the existence and uniqueness results of the HGPHFDP (1.1). The oscillation results are exhibited in Section 4, and the main results are illustrated by numerical examples in Section 5.

## 2. Preliminaries

In the subsequent section, we present some preliminary discrete fractional calculus observations that will be applied to the main findings.

**Definition 2.1** ([9])**.** Let $\varpi:\mathbb{N}_\alpha\to\mathbb{R}$ and $\rho>0$. Then the $\rho^{\text{th}}$ fractional sum of $\varpi$ is defined by

$$\Delta_\alpha^{-\rho}\varpi(\kappa) := \sum_{\vartheta=\alpha}^{\kappa-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\varpi(\vartheta)$$

for $\kappa\in\mathbb{N}_{\alpha+\rho}$, where $\vartheta^{(\rho)} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\rho)}$ and $h_\rho(\vartheta,\tau) = \frac{(\vartheta-\tau)^{(\rho)}}{\Gamma(\rho+1)}$.

**Definition 2.2** ([9])**.** Let $\varpi:\mathbb{N}_\alpha\to\mathbb{R}$ and $\lceil\rho\rceil=\mathfrak{n}$. Then the $\rho^{\text{th}}$ Riemann-Liouville fractional difference of $\varpi$ is defined by

$$\Delta_\alpha^{\rho}\varpi(\kappa) := \Delta^{\mathfrak{n}}\Delta_\alpha^{-(\mathfrak{n}-\rho)}\varpi(\kappa), \quad \kappa\in\mathbb{N}_{\alpha+\mathfrak{n}-\rho}.$$

**Definition 2.3** ([7])**.** Let $\varpi:\mathbb{N}_\alpha\to\mathbb{R}$ and $\lceil\rho\rceil=\mathfrak{n}$. Then the $\rho^{\text{th}}$ Caputo fractional difference of $\varpi$ is defined by

$$\Delta_\alpha^{\rho}\varpi(\kappa) := \Delta_\alpha^{-(\mathfrak{n}-\rho)}\Delta^{\mathfrak{n}}\varpi(\kappa), \quad \kappa\in\mathbb{N}_{\alpha+\mathfrak{n}-\rho}.$$

**Definition 2.4** ([21]). Let $\varpi : \mathbb{N}_\alpha \to \mathbb{R}$ and $\lceil \rho \rceil = \mathfrak{n}$. Then the Hilfer like fractional difference of order $\rho$ and type $0 \leqslant \omega \leqslant 1$ of function $\varpi$ is defined by

$$\Delta_\alpha^{\rho,\omega}\varpi(\kappa) := \Delta_{\alpha+(1-\omega)(\mathfrak{n}-\rho)}^{-\omega(\mathfrak{n}-\rho)}\Delta^{\mathfrak{n}}\Delta_\alpha^{-(1-\omega)(\mathfrak{n}-\rho)}\varpi(\kappa), \quad \kappa \in \mathbb{N}_{\alpha+\mathfrak{n}-\rho}.$$

The special cases Riemann-Liouville fractional difference and Caputo fractional difference will be obtained by putting $\omega = 0$ and $\omega = 1$, respectively.

**Theorem 2.5** ([15, Hybrid fixed point theorem]). *Let $\mathcal{W}$ be the nonempty, closed, bounded, and convex subset of Banach algebra $\mathfrak{Q}$. Let the operators be $\mathcal{P}_1 : \mathfrak{Q} \to \mathfrak{Q}$, $\mathcal{P}_2 : \mathcal{W} \to \mathfrak{Q}$ such that*

(i) *$\mathcal{P}_1$ is Lipschitz continuous with constant $\mathfrak{M}_1$;*

(ii) *$\mathcal{P}_2$ is completely continuous on $\mathcal{W}$;*

(iii) $\varpi = \mathcal{P}_1\varpi\mathcal{P}_2\mathfrak{v} \implies \varpi \in \mathcal{W}, \forall \mathfrak{v} \in \mathcal{W}$;

(iv) *$\mathfrak{M}_1\mathfrak{B} < 1$, where $\mathfrak{B} = \|\mathcal{P}_2(\mathcal{W})\|$.*

*Then $\mathcal{P}_1\varpi\mathcal{P}_2\varpi = \varpi$ has a solution.*

**Theorem 2.6** ([20, Arzela-Ascoli's theorem]). *Let $(\mathfrak{X}, \mathfrak{d})$ be a compact space. A subset $\mathcal{F}$ of the vector space of all real, continuous functions on $\mathfrak{X}$, $\mathcal{C}(\mathfrak{X})$ is relatively compact if and only if, $\mathcal{F}$ is equibounded and equicontinuous.*

**Theorem 2.7** ([26, Banach contraction mapping principle]). *In a complete metric space, a contraction mapping has precisely one fixed point.*

**Lemma 2.8** ([22, Young's inequality]).

(i) *If $\chi$ and $\zeta$ are nonnegative, $\mathfrak{u} > 1$, and $\frac{1}{\mathfrak{u}} + \frac{1}{\mathfrak{v}} = 1$, then $\chi\zeta \leqslant \frac{1}{\mathfrak{u}}\chi^{\mathfrak{u}} + \frac{1}{\mathfrak{v}}\zeta^{\mathfrak{v}}$, where equality holds if and only if $\zeta = \chi^{\mathfrak{u}-1}$.*

(ii) *If $\chi$ and $\zeta$ are nonnegative, $0 < \mathfrak{u} < 1$, and $\frac{1}{\mathfrak{u}} + \frac{1}{\mathfrak{v}} = 1$, then $\chi\zeta \geqslant \frac{1}{\mathfrak{u}}\chi^{\mathfrak{u}} + \frac{1}{\mathfrak{v}}\zeta^{\mathfrak{v}}$, where equality holds if and only if $\zeta = \chi^{\mathfrak{u}-1}$.*

**Lemma 2.9** ([18]). *Let $\varpi : \mathbb{N}_\alpha \to \mathbb{R}$, $\mathfrak{l} \in \mathbb{N}_0$, $\mathfrak{a} - 1 < \rho_1 < \mathfrak{a}$, and $\mathfrak{b} - 1 < \rho_2 \leqslant \mathfrak{b}$. Then*

(i) $\Delta_{\alpha+\rho_2}^{-\rho_1}\Delta_\alpha^{-\rho_2}\varpi(\kappa) = \Delta_\alpha^{-\rho_1-\rho_2}\varpi(\kappa) = \Delta_{\alpha+\rho_1}^{-\rho_2}\Delta_\alpha^{-\rho_1}\varpi(\kappa)$, *for* $\kappa \in \mathbb{N}_{\alpha+\rho_1+\rho_2}$;

(ii) $\Delta^{\mathfrak{l}}\Delta_\alpha^{-\rho_1}\varpi(\kappa) = \Delta_\alpha^{\mathfrak{l}-\rho_1}\varpi(\kappa)$, *for* $\kappa \in \mathbb{N}_{\alpha+\rho_1}$;

(iii) $\Delta_{\alpha+\mathfrak{b}-\rho_2}^{-\rho_1}\Delta_\alpha^{\rho_2}\varpi(\kappa) = \Delta_\alpha^{\rho_2-\rho_1}\varpi(\kappa) - \sum_{\mathfrak{k}=0}^{\mathfrak{b}-1} h_{\rho_1-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{b}-\rho_1)\Delta_\alpha^{\mathfrak{k}-(\mathfrak{b}-\rho_2)}\varpi(\alpha+\mathfrak{b}-\rho_2)$, *for* $\kappa \in \mathbb{N}_{\alpha+\mathfrak{b}-\rho_2+\rho_1}$.

**Lemma 2.10.** *The HGPHFDP (1.1) has a unique solution*

$$\varpi(\kappa) = \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1} h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}} + \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta)$$
$$+ \Upsilon_2\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big) - \Upsilon_1\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big)\big]\Bigg],$$

*for* $x \in \mathbb{N}_{\alpha+\mathfrak{n}}$.

*Proof.* Applying $\Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}$ on both sides of (1.1), we obtain

$$\begin{aligned}\Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}\Delta_{\alpha}^{\rho,\omega}\mathfrak{z}(\kappa) = \Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}\big[\mathfrak{w}(\kappa) + \Upsilon_2\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big)\\ - \Upsilon_1\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big)\big].\end{aligned} \tag{2.1}$$

Using Definition 2.4 and Lemma 2.9, the left hand side of (2.1) can be written as

$$\begin{aligned}\Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}\Delta_{\alpha}^{\rho,\omega}\mathfrak{z}(\kappa) &= \Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}\big[\Delta_{\alpha+(1-\omega)(\mathfrak{n}-\rho)}^{-\omega(\mathfrak{n}-\rho)}\Delta^{\mathfrak{n}}\Delta_{\alpha}^{-(1-\omega)(\mathfrak{n}-\rho)}\mathfrak{z}(\kappa)\big]\\ &= \Delta_{\alpha+\mathfrak{n}-\rho-\omega(\mathfrak{n}-\rho)}^{-\rho-\omega(\mathfrak{n}-\rho)}\big[\Delta^{\mathfrak{n}}\Delta_{\alpha}^{-(1-\omega)(\mathfrak{n}-\rho)}\mathfrak{z}(\kappa)\big]\\ &= \Delta_{\alpha+\mathfrak{n}-(\rho+\mathfrak{n}\omega-\rho\omega)}^{-(\rho+\mathfrak{n}\omega-\rho\omega)}\Delta_{\alpha}^{\rho+\mathfrak{n}\omega-\rho\omega}\mathfrak{z}(\kappa) = \mathfrak{z}((\kappa) - \sum_{\mathfrak{k}=0}^{\mathfrak{n}-1}h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}}.\end{aligned}$$

Therefore, equation (2.1) becomes

$$\begin{aligned}\mathfrak{z}(\kappa) = &\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1}h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}} + \Delta_{\alpha+\mathfrak{n}-\rho}^{-\rho}\big[\mathfrak{w}(\kappa) + \Upsilon_2\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big)\\ &- \Upsilon_1\big(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\kappa+\rho-\mathfrak{n}))\big)\big].\end{aligned}$$

This implies

$$\begin{aligned}\varpi(\kappa) = \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[&\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1}h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, a+\mathfrak{n}-\gamma)z_{\mathfrak{k}} + \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\kappa-\rho}h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta)\\ &+ \Upsilon_2\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big) - \Upsilon_1\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big)\big]\Bigg].\end{aligned}$$

Hence, the proof is complete. □

## 3. Existence and uniqueness result

In this section, we study the existence and uniqueness of the solution for the HGPHFDP (1.1) using the Hybrid fixed point theorem and Banach contraction principle respectively. Let $\mathfrak{Q}$ be the Banach space of all functions $\varpi$ such that $\varpi$ satisfies (1.1) with norm $\|\varpi\| = \sup_{\kappa\in\mathfrak{D}}|\varpi(\kappa)|$, where $\mathfrak{D} = [\alpha+\mathfrak{n}-1, \mathfrak{d}] \cap \mathbb{N}_{\alpha}$, $\alpha \in \mathbb{R}$, $\mathfrak{d} \in \mathbb{N}$. Indicate the operator $\mathcal{T} : \mathfrak{Q} \to \mathfrak{Q}$ by

$$\begin{aligned}\mathcal{T}\varpi(\kappa) = \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[&\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1}h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{k} + \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\kappa-\rho}h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta)\\ &+ \Upsilon_2\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big) - \Upsilon_1\big(\vartheta, \varpi(\vartheta+\rho-\mathfrak{n}), \varpi(\mathfrak{p}(\vartheta+\rho-\mathfrak{n}))\big)\big]\Bigg]\end{aligned} \tag{3.1}$$

for $\kappa \in \mathfrak{D}$. For all $\varpi, \varpi^* \in \mathfrak{Q}$ and $\kappa \in \mathfrak{D}$, we make the following assumptions.

($\mathscr{B}$.1) There exists $\mathfrak{M}_1$ in $\mathbb{R}^+$ s.t. $\|\mathfrak{F}\big(\kappa, \varpi, \varpi(\mathfrak{q})\big) - \mathfrak{F}\big(\kappa, \varpi^*, \varpi^*(\mathfrak{q})\big)\| \leqslant \mathfrak{M}_1\|\varpi - \varpi^*\|$.

($\mathscr{B}$.2) There exists $\mathfrak{w}_1 \in \mathbb{R}^+$ s.t. $|\mathfrak{w}(\kappa)| \leqslant \mathfrak{w}_1$.

($\mathscr{B}$.3) There exists $\mathfrak{m} \in \mathbb{R}^+$ s.t. $|\Upsilon_2\big(\kappa, \varpi, \varpi(\mathfrak{p})\big) - \Upsilon_1\big(\kappa, \varpi, \varpi(\mathfrak{p})\big)| \leqslant \mathfrak{m}$.

($\mathscr{B}$.4) There exists $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{R}^+$ s.t.

$$\begin{aligned}\|\Upsilon_1\big(\kappa, \varpi, \varpi(\mathfrak{p})\big) - \Upsilon_1\big(\kappa, \varpi^*, \varpi^*(\mathfrak{p})\big)\| \leqslant \mathfrak{m}_1\|\varpi - \varpi^*\|,\\ \|\Upsilon_2\big(\kappa, \varpi, \varpi(\mathfrak{p})\big) - \Upsilon_2\big(\kappa, \varpi^*, \varpi^*(\mathfrak{p})\big)\| \leqslant \mathfrak{m}_2\|\varpi - \varpi^*\|.\end{aligned}$$

**Theorem 3.1** (Existence). *Assume that* $(\mathscr{B}.1)$-$(\mathscr{B}.4)$ *hold. Eventually, a solution* $\varpi(\kappa)$ *to the HGPHFDP* (1.1) *will exist provided*

$$\sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\mathfrak{d}, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + (\mathfrak{w}_1 + m) \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \left|h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))\right| < \frac{1}{\mathfrak{M}_1},$$

*for* $\kappa \in \mathfrak{D}$.

*Proof.* Take $\mathfrak{M}_0 = \max_{\kappa\in\mathfrak{D}} |\mathfrak{F}(\kappa, 0, 0)|$ and

$$\mathfrak{A} = \sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\mathfrak{d}, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + (\mathfrak{w}_1 + m) \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \left|h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))\right|.$$

Now, let $\mathcal{W} = \{\varpi \in \mathfrak{Q} : \|\varpi\| \leqslant \Omega\}$, where $\Omega$ is a real number such that $\Omega \geqslant \frac{\mathfrak{M}_0\mathfrak{A}}{1-\mathfrak{M}_1\mathfrak{A}}$. Clearly $\mathcal{W}$ is non-empty, closed, bounded, and convex. Define $\mathcal{P}_1 : \mathfrak{Q} \to \mathfrak{Q}$ and $\mathcal{P}_2 : \mathcal{W} \to \mathfrak{Q}$ by

$$\begin{aligned}
\mathcal{P}_1\varpi(\kappa) &= \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big),\\
\mathcal{P}_2\varpi(\kappa) &= \sum_{\mathfrak{k}=0}^{n-1} h_{\gamma-n+\mathfrak{k}}(\kappa, \alpha+n-\gamma)\mathfrak{z}_{\mathfrak{k}} + \sum_{\vartheta=\alpha+n-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta)\\
&\quad + \Upsilon_2\big(\vartheta, \varpi(\vartheta+\rho-n), \varpi(\mathfrak{p}(\vartheta+\rho-n))\big) - \Upsilon_1\big(\vartheta, \varpi(\vartheta+\rho-n), \varpi(\mathfrak{p}(\vartheta+\rho-n))\big)\big],
\end{aligned}$$

for $\kappa \in \mathfrak{D}$. In the following steps, we will illustrate that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ will satisfy the criteria of Theorem 2.5. Now,

$$|\mathcal{P}_1\varpi(\kappa) - \mathcal{P}_1\varpi^*(\kappa)| = |\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big) - \mathfrak{F}\big(\kappa, \varpi^*(\kappa), \varpi^*(\mathfrak{q}(\kappa))\big)|.$$

From $(\mathscr{B}.1)$, we get $\|\mathcal{P}_1\varpi - \mathcal{P}_1\varpi^*\| \leqslant \mathfrak{M}_1\|\varpi - \varpi^*\|$. This implies that $\mathcal{P}_1$ is Lipschitz continuous with constant $\mathfrak{M}_1$. Next, for given $\epsilon_1 > 0$ and for every $\Upsilon_1, \Upsilon_2 \in \mathcal{W}$,

$$\begin{aligned}
|\mathcal{P}_2\Upsilon_1(\kappa) - \mathcal{P}_2\Upsilon_2(\kappa)| \leqslant &\sum_{\vartheta=\alpha+n-\rho}^{\kappa-\rho} |h_{\rho-1}(\kappa, \sigma(\vartheta))|\big[|\Upsilon_2\big(\vartheta, \Upsilon_1(\vartheta+\rho-n), \Upsilon_1(\mathfrak{p}(\vartheta+\rho-n))\big)\\
&- \Upsilon_2\big(\vartheta, \Upsilon_2(\vartheta+\rho-n), \Upsilon_2(\mathfrak{p}(\vartheta+\rho-n))\big)| + |\Upsilon_1\big(\vartheta, \Upsilon_1(\vartheta+\rho-n), \Upsilon_1(\mathfrak{p}(\vartheta+\rho-n))\big)\\
&- \Upsilon_1\big(\vartheta, \Upsilon_2(\vartheta+\rho-n), \Upsilon_2(\mathfrak{p}(\vartheta+\rho-n))\big)|\big].
\end{aligned}$$

This implies $\|\mathcal{P}_2\Upsilon_1 - \mathcal{P}_2\Upsilon_2\| \leqslant \mathfrak{B}\|\Upsilon_1 - \Upsilon_2\| < \epsilon_1$, whenever $\|\Upsilon_1 - \Upsilon_2\| < \epsilon_1/\mathfrak{B}$, where

$$\mathfrak{B} = \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} |h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))||\mathfrak{m}_1 + \mathfrak{m}_2|.$$

Therefore, $\mathcal{P}_2$ is continuous on $\mathcal{W}$. For $\varpi \in \mathcal{W}$,

$$\begin{aligned}
|\mathcal{P}_2\varpi(\kappa)| &\leqslant \sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\kappa, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + (\mathfrak{w}_1 + m) \sum_{\vartheta=\alpha+n-\rho}^{\kappa-\rho} \left|h_{\rho-1}(\kappa, \sigma(\vartheta))\right|\\
&\leqslant \sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\mathfrak{d}, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + (\mathfrak{w}_1 + m) \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \left|h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))\right|.
\end{aligned}$$

This implies $\|\mathcal{P}_2\varpi\| \leqslant \mathfrak{A}$, and hence $\mathcal{P}_2$ is equibounded. Now, for any $\varpi \in \mathcal{W}$, $\mathcal{P}_2(\varpi) \in \mathcal{P}_2(\mathcal{W})$. Let $\epsilon_2 > 0$ and let $\kappa_1, \kappa_2 \in \mathfrak{D}$ such that $\kappa_1 < \kappa_2$. Then,

$$\begin{aligned}|\mathcal{P}_2\varpi(\kappa_1) - \mathcal{P}_2\varpi(\kappa_2)| \leqslant &\sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\kappa_1, \alpha+n-\gamma) - h_{\gamma-n+\mathfrak{k}}(\kappa_2, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| \\ &+ \left| \sum_{\vartheta=\alpha+n-\rho}^{\kappa_1-\rho} h_{\rho-1}(\kappa_1, \sigma(\vartheta)) - \sum_{\vartheta=\alpha+n-\rho}^{\kappa_2-\rho} h_{\rho-1}(\kappa_2, \sigma(\vartheta)) \right| (\mathfrak{w}_1 + \mathfrak{m}).\end{aligned}$$

This implies $\|\mathcal{P}_2\varpi(\kappa_1) - \mathcal{P}_2\varpi(\kappa_2)\| \leqslant \mathfrak{C}\|\kappa_1 - \kappa_2\| < \epsilon_2$, whenever $\|\kappa_1 - \kappa_2\| < \frac{\epsilon_2}{\mathfrak{C}}$, where

$$\mathfrak{C} = \sum_{\mathfrak{k}=0}^{n-1} \frac{(\mathfrak{d}-\alpha-n+\gamma)^{(\gamma-n+\mathfrak{k}-1)}}{\Gamma(\gamma-n+\mathfrak{k}+1)} |\mathfrak{z}_{\mathfrak{k}}| + \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \frac{(\mathfrak{d}-\vartheta-1)^{(\rho-2)}}{\Gamma(\rho)} |\mathfrak{w}_1 + \mathfrak{m}|.$$

This implies that $\mathcal{P}_2(\mathcal{W})$ is equicontinuous. Next, let $\varpi \in \mathfrak{Q}$, $\mathfrak{v} \in \mathcal{W}$ such that $\varpi = \mathcal{P}_1\varpi\mathcal{P}_2\mathfrak{v}$. Then,

$$\begin{aligned}|\varpi(\kappa)| &= |\mathcal{P}_1\varpi(\kappa)||\mathcal{P}_2 v(\kappa)| \\ &\leqslant |\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)| \Bigg| \sum_{\mathfrak{k}=0}^{n-1} h_{\gamma-n+\mathfrak{k}}(\kappa, \alpha+n-\gamma)\mathfrak{z}_{\mathfrak{k}} + \sum_{\vartheta=\alpha+n-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta)) \big[\mathfrak{w}(\vartheta) \\ &\quad + \Upsilon_2\big(\vartheta, \mathfrak{v}(\vartheta+\rho-n), \mathfrak{v}(\mathfrak{p}(\vartheta+\rho-n))\big) - \Upsilon_1\big(\vartheta, \mathfrak{v}(\vartheta+\rho-n), \mathfrak{v}(\mathfrak{p}(\vartheta+\rho-n))\big)\big] \Bigg| \\ &\leqslant |\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big) - \mathfrak{F}(0,0,0)|\mathfrak{A} + |\mathfrak{F}(0,0,0)|\mathfrak{A}.\end{aligned}$$

Thus, $\|\varpi\| \leqslant \frac{\mathfrak{M}_0\mathfrak{A}}{1-\mathfrak{M}_1\mathfrak{A}} \leqslant \Omega$, and hence $\varpi \in \mathcal{W}$. Finally, for $\mathfrak{B} = \|\mathcal{P}_2(\mathcal{W})\|$,

$$\mathfrak{M}_1\mathfrak{B} = \mathfrak{M}_1\|\mathcal{P}_2(\mathcal{W})\| = \mathfrak{M}_1 \sup_{\varpi\in\mathcal{W}} \{|\mathcal{P}_2(\varpi)|\} \leqslant \mathfrak{M}_1\mathfrak{A} < 1.$$

Therefore, by Theorem 2.5, the HGPHFDP (1.1) has a solution $\varpi(\kappa)$ for $\kappa \in \mathfrak{D}$. Hence, the proof is complete. $\square$

**Theorem 3.2** (Uniqueness). *Assume $(\mathscr{B}.1)$-$(\mathscr{B}.4)$ hold and that there exists $\mathfrak{M}_2 \in \mathbb{R}^+$ such that $|\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)| \leqslant \mathfrak{M}_2$, for every $\varpi$ in $\mathfrak{Q}$ and $\kappa$ in $\mathfrak{D}$. Then $\varpi(\kappa)$ is a unique solution of the HGPHFDP (1.1) provided*

$$1 > \Omega_1 = \mathfrak{M}_1 \sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\mathfrak{d}, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \big|h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))\big| \big(\mathfrak{M}_1(\mathfrak{w}_1+\mathfrak{m}) + \mathfrak{M}_2(\mathfrak{m}_1+\mathfrak{m}_2)\big).$$

*Proof.* First, let us prove that the mapping $\mathcal{T}$ defined in (3.1) is a contraction. For all $\kappa \in \mathfrak{D}$ and $\varpi, \varpi^* \in \mathfrak{Q}$, using $(\mathscr{B}.1)$-$(\mathscr{B}.4)$, we have

$$\begin{aligned}\|\mathcal{T}\varpi - \mathcal{T}\varpi^*\| \leqslant \Bigg\{ &\mathfrak{M}_1 \sum_{\mathfrak{k}=0}^{n-1} |h_{\gamma-n+\mathfrak{k}}(\mathfrak{d}, \alpha+n-\gamma)||\mathfrak{z}_{\mathfrak{k}}| + \sum_{\vartheta=\alpha+n-\rho}^{\mathfrak{d}-\rho} \big|h_{\rho-1}(\mathfrak{d}, \sigma(\vartheta))\big| \big(\mathfrak{M}_1(\mathfrak{w}_1+\mathfrak{m}) \\ &+ \mathfrak{M}_2(\mathfrak{m}_1+\mathfrak{m}_2)\big) \Bigg\} \|\varpi - \varpi^*\| = \Omega_1\|\varpi - \varpi^*\|.\end{aligned}$$

Thus, the mapping $\mathcal{T}$ is a contraction mapping with $\Omega_1 < 1$. Therefore, by Theorem 2.7, the uniqueness follows. $\square$

## 4. Oscillation results

In this section, we figure out the oscillation results for HGPHFDP (1.1). To get the results, we take into account the following considerations:

($\mathscr{C}$.1) $\frac{\Upsilon_{\mathfrak{k}}(\kappa,\varpi,\varpi(\mathfrak{p}))}{\varpi} > 0$, $(\mathfrak{k}=1,2)$, $\varpi \neq 0$, $\kappa \in \mathbb{N}_{\alpha+\mathfrak{n}-\rho}$;

($\mathscr{C}$.2) $|\Upsilon_1(\kappa,\varpi,\varpi(\mathfrak{p}))| \geqslant \mathfrak{q}_1(\kappa)|\varpi|^{\iota_1}$, $|\Upsilon_2(\kappa,\varpi,\varpi(\mathfrak{p}))| \leqslant \mathfrak{q}_2(\kappa)|\varpi|^{\iota_2}$, $\varpi \neq 0$, $\kappa \in \mathbb{N}_{\alpha+\mathfrak{n}-\rho}$, for some continuous functions $\mathfrak{q}_{\mathfrak{k}} : \mathbb{N}_{\alpha+\mathfrak{n}-\rho} \to \mathbb{R}^+$, $\mathfrak{k}=1,2$ and for some positive real numbers $\iota_1 > \iota_2$;

($\mathscr{C}$.3) $|\Upsilon_1(\kappa,\varpi,\varpi(\mathfrak{p}))| \leqslant \mathfrak{q}_1(\kappa)|\varpi|^{\iota_1}$, $|\Upsilon_2(\kappa,\varpi,\varpi(\mathfrak{p}))| \geqslant \mathfrak{q}_2(\kappa)|\varpi|^{\iota_2}$, $\varpi \neq 0$, $\kappa \in \mathbb{N}_{\alpha+\mathfrak{n}-\rho}$, for some continuous functions $\mathfrak{q}_{\mathfrak{k}} : \mathbb{N}_{\alpha+\mathfrak{n}-\rho} \to \mathbb{R}^+$, $\mathfrak{k}=1,2$ and for some positive real numbers $\iota_1 < \iota_2$.

**Theorem 4.1.** *Assume that ($\mathscr{C}$.1), ($\mathscr{C}$.2), and $\mathfrak{F}(\kappa,\varpi(\kappa),\varpi(\mathfrak{q}(\kappa))) > 0$ hold. If*

$$\liminf_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\left[\mathfrak{w}(\vartheta)+\mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\right] = -\infty \tag{4.1}$$

*and*

$$\limsup_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\left[\mathfrak{w}(\vartheta)-\mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\right] = \infty, \tag{4.2}$$

*where $\mathfrak{K}_{\iota_1,\iota_2}(\vartheta) = (\iota_1/\iota_2-1)[\iota_2\mathfrak{q}_2(\vartheta)/\iota_1]^{\iota_1/(\iota_1-\iota_2)}\mathfrak{q}_1^{\iota_2/(\iota_2-\iota_1)}(\vartheta)$, then every solution of HGPHFDP (1.1) is oscillatory for sufficiently large $\vartheta_1$.*

*Proof.* Let us show the result through contradiction. Assume that $\varpi(\kappa)$ is a non-oscillatory solution of HGPHFDP (1.1). Then, it will eventually be either positive or negative. Suppose that $\varpi(\kappa)$ turns out to be positive. In this case, there exists a sufficiently large $\vartheta_1 > \alpha+\mathfrak{n}$ such that $\varpi(\kappa) > 0$ for $\kappa \in \mathbb{N}_{\vartheta_1}$. Take $\mathfrak{G}(\kappa) = \mathfrak{w}(\kappa)+\Upsilon_2(\kappa,\varpi(\kappa+\rho-\mathfrak{n}),\varpi(\mathfrak{q}(\kappa+\rho-\mathfrak{n})))-\Upsilon_1(\kappa,\varpi(\kappa+\rho-\mathfrak{n}),\varpi(\mathfrak{q}(\kappa+\rho-\mathfrak{n})))$. Then, from Lemma 2.10, we have

$$\begin{aligned}\varpi(\kappa) \leqslant \mathfrak{F}(\kappa,\varpi(\kappa),\varpi(\mathfrak{q}(\kappa)))&\Bigg[\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1} h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa,\alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}} + \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\vartheta_1-1-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\mathfrak{G}(\vartheta)\\ &+\sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\left[\mathfrak{w}(\vartheta)+\mathfrak{q}_2(\vartheta)\varpi_2^{\iota}(\vartheta+\rho-\mathfrak{n})-\mathfrak{q}_1(\vartheta)\varpi_1^{\iota}(\vartheta+\rho-\mathfrak{n})\right]\Bigg].\end{aligned}$$

Define

$$\Phi(\kappa) = \sum_{\mathfrak{k}=0}^{\mathfrak{n}-1} h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa,\alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}} \quad \text{and} \quad \Psi(\kappa,\vartheta_1) = \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\vartheta_1-1-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\mathfrak{G}(\vartheta).$$

Hence,

$$\begin{aligned}\varpi(\kappa) \leqslant \varpi(\kappa,\varpi(\kappa),\varpi(\mathfrak{q}(\kappa)))&\Bigg[\Phi(\kappa)+\Psi(\kappa,\vartheta_1)\\ &+\sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa,\sigma(\vartheta))\left[\mathfrak{w}(\vartheta)+\mathfrak{q}_2(\vartheta)\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n})-\mathfrak{q}_1(\vartheta)\varpi^{\iota_1}(\vartheta+\rho-\mathfrak{n})\right]\Bigg],\end{aligned} \tag{4.3}$$

for $\kappa \in \mathbb{N}_{\vartheta_1}$. By choosing $\chi = \varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n})$, $\zeta = \frac{\mathfrak{v}\mathfrak{q}_2(\vartheta)}{\iota_1\mathfrak{q}_1(\vartheta)}$, $\mathfrak{u} = \iota_1/\iota_2$, and $\mathfrak{v} = \iota_1/(\iota_1-\iota_2)$ in Lemma 2.8 (i), we have

$$\mathfrak{q}_2(\vartheta)\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n})-\mathfrak{q}_1(\vartheta)\varpi^{\iota_1}(\vartheta+\rho-\mathfrak{n}) = \frac{\iota_1\mathfrak{q}_1(\vartheta)}{\iota_2}\left[\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n})\frac{\iota_2\mathfrak{q}_2(\vartheta)}{\iota_1\mathfrak{q}_1(\vartheta)} - \frac{(\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n}))^{\iota_1/\iota_2}}{\iota_1/\iota_2}\right]$$

$$= \frac{\iota_1 q_1(\vartheta)}{\iota_2}\left[\chi\zeta - \frac{1}{\mathfrak{u}}\chi^{\mathfrak{u}}\right] \leqslant \frac{\iota_1 q_1(\vartheta)}{\iota_2}\frac{\zeta^{\mathfrak{v}}}{\mathfrak{v}} = \mathfrak{K}_{\iota_1,\iota_2}(\vartheta).$$

Then, for $\kappa \in \mathbb{N}_{\vartheta_1}$, (4.3) becomes

$$\varpi(\kappa) \leqslant \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)\left[\Phi(\kappa) + \Psi(\kappa, \vartheta_1) + \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\right].$$

Taking the aforementioned inequality and multiplying it by by $\kappa^{1-\gamma}$, we get

$$0 \leqslant \kappa^{1-\gamma}\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)\left[\Phi(\kappa) + \Psi(\kappa, \vartheta_1) + \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\right], \tag{4.4}$$

for $\kappa \in \mathbb{N}_{\vartheta_1}$. Consider

$$\begin{aligned}
\kappa^{1-\gamma}\Phi(\kappa) &= \kappa^{1-\gamma}\sum_{\mathfrak{k}=0}^{n-1} h_{\gamma-n+\mathfrak{k}}(\kappa, \alpha+n-\gamma)\mathfrak{z}_{\mathfrak{k}} \\
&= \sum_{\mathfrak{k}=0}^{n-1} \frac{\Gamma(\kappa-\alpha-n+\gamma+1)}{\kappa^{\gamma-1}\Gamma(\kappa-\alpha-\mathfrak{k}+1)\Gamma(\gamma-n+\mathfrak{k}+1)}\mathfrak{z}_{\mathfrak{k}} \\
&= \sum_{\mathfrak{k}=0}^{n-1} \frac{1}{(\kappa-\alpha+\gamma-\mathfrak{k}-1)\cdots(\kappa-\alpha+\gamma-(n-1))}\frac{\Gamma(\kappa-\alpha-\mathfrak{k}+\gamma)}{\Gamma(\kappa-\alpha-\mathfrak{k}+1)}\frac{1}{\Gamma(\gamma-n+\mathfrak{k}+1)}\mathfrak{z}_{\mathfrak{k}}
\end{aligned}$$

and

$$\kappa^{1-\gamma}\Psi(\kappa, \vartheta_1) = \kappa^{1-\gamma}\sum_{\vartheta=\alpha+n-\rho}^{\vartheta_1-1-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\mathfrak{G}(\vartheta) = \frac{1}{\Gamma(\rho)}\sum_{\vartheta=\alpha+n-\rho}^{\vartheta_1-1-\rho}\frac{1}{\kappa^{\omega(n-\rho)}}\frac{\Gamma(\kappa-\vartheta)}{\kappa^{\rho-1}\Gamma(\kappa-\vartheta+1-\rho)}\mathfrak{G}(\vartheta).$$

Then, using the asymptotic expansion formula, $\kappa^{1-\gamma}\Phi(\kappa) + \kappa^{1-\gamma}\Psi(\kappa, \vartheta_1)$ is convergent, and so bounded. That is,

$$|\kappa^{1-\gamma}\Phi(\kappa) + \kappa^{1-\gamma}\Psi(\kappa, \vartheta_1)| \leqslant \mathcal{M}_1,$$

for some $\mathcal{M}_1 \in \mathbb{R}^+$. Thus, (4.4) becomes

$$0 \leqslant \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)\left[\mathcal{M}_1 + \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\right],$$

and hence, we have

$$-\mathcal{M}_1 < \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big].$$

This contradicts (4.1). Similarly, for an eventually negative solution $\varpi(\kappa)$ of HGPHFDP (1.1), we get a contradiction to (4.2). Hence, the proof is complete. □

**Theorem 4.2.** *Assume that* ($\mathscr{C}$.**1**), ($\mathscr{C}$.**2**), *and* $\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big) < 0$ *hold. If*

$$\liminf_{\kappa\to\infty} \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_2-\rho}^{\mathfrak{r}-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) - \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big] = -\infty \tag{4.5}$$

*and*

$$\limsup_{\kappa\to\infty} \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big] = \infty, \tag{4.6}$$

*where* $\mathfrak{K}_{\iota_1,\iota_2}(\vartheta)$ *is defined as in Theorem 4.1, then every solution of HGPHFDP (1.1) is oscillatory for sufficiently large* $\vartheta_2$.

*Proof.* Let us show the result through contradiction. Assume that $\varpi(\kappa)$ is a non-oscillatory solution of HGPHFDP (1.1). Then, it will eventually be either positive or negative. Suppose that $\varpi(\kappa)$ turns out to be positive. In this case, there exists a sufficiently large $\vartheta_2 > \alpha + \mathfrak{n}$ such that $\varpi(\kappa) > 0$ for $x \in \mathbb{N}_{\vartheta_2}$. Take $\mathfrak{G}(\kappa) = \mathfrak{w}(\kappa) + \Upsilon_2(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{q}(\kappa+\rho-\mathfrak{n}))) - \Upsilon_1(\kappa, \varpi(\kappa+\rho-\mathfrak{n}), \varpi(\mathfrak{q}(\kappa+\rho-\mathfrak{n})))$. Then, from Lemma 2.10, we have

$$\varpi(\kappa) \leqslant \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[\sum_{\mathfrak{k}=0}^{\mathfrak{n}-1} h_{\gamma-\mathfrak{n}+\mathfrak{k}}(\kappa, \alpha+\mathfrak{n}-\gamma)\mathfrak{z}_{\mathfrak{k}} + \sum_{\vartheta=\alpha+\mathfrak{n}-\rho}^{\vartheta_2-1-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\mathfrak{G}(\vartheta)$$
$$+ \sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{q}_2(\vartheta)\varpi_2^{\iota}(\vartheta+\rho-\mathfrak{n}) - \mathfrak{q}_1(\vartheta)\varpi_1^{\iota}(\vartheta+\rho-\mathfrak{n})\big]\Bigg].$$

Defining $\Phi(\kappa)$ and $\Psi(\kappa, \vartheta_2)$ as in the proof of Theorem 4.1, we get

$$\begin{aligned}\varpi(\kappa) \leqslant \varpi\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[&\Phi(\kappa) + \Psi(\kappa, \vartheta_2)\\ &+ \sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{q}_2(\vartheta)\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n}) - \mathfrak{q}_1(\vartheta)\varpi^{\iota_1}(\vartheta+\rho-\mathfrak{n})\big]\Bigg]\end{aligned} \tag{4.7}$$

for $\kappa \in \mathbb{N}_{\vartheta_2}$. By choosing $\chi = \varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n})$, $\zeta = \frac{\iota_2\mathfrak{q}_2(\vartheta)}{\iota_1\mathfrak{q}_1(\vartheta)}$, $\mathfrak{u} = \iota_1/\iota_2$, and $\mathfrak{v} = \iota_1/(\iota_1-\iota_2)$ in Lemma 2.8 (i), we attain

$$\mathfrak{q}_2(\vartheta)\varpi^{\iota_2}(\vartheta+\rho-\mathfrak{n}) - \mathfrak{q}_1(\vartheta)\varpi^{\iota_1}(\vartheta+\rho-\mathfrak{n}) \leqslant \mathfrak{K}_{\iota_1,\iota_2}(\vartheta).$$

Then, for $\kappa \in \mathbb{N}_{\vartheta_2}$, (4.7) becomes

$$\varpi(\kappa) \leqslant \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[\Phi(\kappa) + \Psi(\kappa, \vartheta_2) + \sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\Bigg].$$

Taking the aforementioned inequality and multiplying it by $\kappa^{1-\gamma}$, we get

$$0 \leqslant \kappa^{1-\gamma}\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[\Phi(\kappa) + \Psi(\kappa, \vartheta_2) + \sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\Bigg], \tag{4.8}$$

for $\kappa \in \mathbb{N}_{\vartheta_2}$. While pursuing as in the proof of Theorem 4.1, we get

$$|\kappa^{1-\gamma}\Phi(\kappa) + \kappa^{1-\gamma}\Psi(\kappa, \vartheta_2)| \leqslant \mathcal{M}_2,$$

for some $\mathcal{M}_2 \in \mathbb{R}^+$. Thus, (4.8) becomes

$$0 \leqslant \mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(\mathfrak{q}(\kappa))\big)\Bigg[\mathcal{M}_2 + \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big]\Bigg],$$

and hence, we have

$$-\mathcal{M}_2 > \kappa^{1-\gamma}\sum_{\vartheta=\vartheta_2-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big].$$

This contradicts (4.5). Similarly, for an eventually negative solution $\varpi(\kappa)$ of (1.1), we get a contradiction to (4.6). Hence, the proof is complete. □

**Theorem 4.3.** *Assume $\gamma \geqslant 1$ and that ($\mathscr{C}$.1) and ($\mathscr{C}$.3) hold. If there exists $\mathfrak{V} \in \mathbb{R}^+$ such that*

$$\left| \frac{\varpi(\kappa)}{\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)} \right| \leqslant \mathfrak{V},$$

$$\liminf_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) - \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big] = -\infty, \tag{4.9}$$

*and*

$$\limsup_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big] = \infty, \tag{4.10}$$

*where $\mathfrak{K}_{\iota_1,\iota_2}$ is defined as in Theorem 4.1, then every solution of (1.1) is oscillatory for sufficiently large $\vartheta_3$.*

*Proof.* Let us show the result through contradiction. Assume that $\varpi(\kappa)$ is a non-oscillatory solution of HGPHFDP (1.1). Then, it will eventually be either positive or negative. Suppose that $\varpi(\kappa)$ turns out to be positive. Then, there exists a sufficiently large $\vartheta_3 > \alpha + \mathfrak{n}$ such that $\varpi(\kappa) > 0$ for $\kappa \in \mathbb{N}_{\vartheta_3}$. As in the proof of Theorem 4.1, define $\mathfrak{G}(\kappa)$, $\Phi(\kappa)$, and $\Psi(\kappa, \vartheta_3)$. Then, by Lemma 2.8 (ii), we have, for $\kappa \in \mathbb{N}_{\vartheta_3}$,

$$\mathfrak{V} \geqslant \frac{\varpi(\kappa)}{\mathfrak{F}\big(\kappa, \varpi(\kappa), \varpi(q(\kappa))\big)} \geqslant \Phi(\kappa) + \Psi(\kappa, \vartheta_3) + \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big].$$

Taking the aforementioned inequality and multiplying it by $\kappa^{1-\gamma}$, we get

$$\kappa^{1-\gamma}\mathfrak{V} \geqslant \kappa^{1-\gamma}\Phi(\kappa) + \kappa^{1-\gamma}\Psi(\kappa, \vartheta_3) + \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big].$$

While pursuing as in the proof of Theorem 4.1, we get

$$|\kappa^{1-\gamma}\Phi(\kappa) + \kappa^{1-\gamma}\Psi(\kappa, \vartheta_3)| \leqslant \mathcal{M}_3,$$

for some $\mathcal{M}_3 \in \mathbb{R}^+$. Thus,

$$\kappa^{1-\gamma} \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\big[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\big] \leqslant \frac{\mathfrak{V}}{\kappa^{\gamma-1}} + \mathcal{M}_3.$$

This contradicts (4.10). Similarly, for an eventually negative solution $\varpi(\kappa)$ of (1.1), we get a contradiction to (4.9). Hence, the proof is complete. □

## 5. Examples

We provide some numerical examples in this part to highlight our key findings.

**Example 5.1.** Examine the HGPHFDP

$$\begin{cases} \Delta_1^{\frac{1}{2},\frac{1}{3}}\left(\frac{\varpi(\kappa)}{1+\varpi^2(\kappa)}\right) + \varpi^5(\kappa - \frac{1}{2}) = 5\sin\kappa - 1 + \varpi^3(\kappa - \frac{1}{2}), \\ \Delta_1^{-\frac{1}{3}}\mathfrak{z}((\kappa)| - \kappa = \frac{4}{3} = z_0, \quad x \in \mathbb{N}_{\frac{3}{2}}. \end{cases} \tag{5.1}$$

Here $\rho = \frac{1}{2}$, $\omega = \frac{1}{3}$, $\alpha = 1$, $\gamma = \frac{2}{3}$, $\Upsilon_1(\kappa, \varpi, \varpi(\mathfrak{p})) = \varpi^5$, $\Upsilon_2(\kappa, \varpi, \varpi(\mathfrak{p})) = \varpi^3$, and $\mathfrak{w}(\kappa) = 5\sin\kappa - 1$. By choosing $\mathfrak{q}_1(\kappa) = \mathfrak{q}_2(\kappa) = 1$, $\iota_1 = 5$, and $\iota_2 = 3$, we get

$$\begin{aligned}\liminf_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\left[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\right] &= \liminf_{\kappa\to\infty} \kappa^{\frac{1}{3}} \sum_{\vartheta=\vartheta_1-\frac{1}{2}}^{\kappa-\frac{1}{2}} h_{\frac{-1}{2}}(\kappa, \sigma(\vartheta))\left[5\sin\vartheta - 1 + \mathfrak{K}_{5,3}(\vartheta)\right] \\ &\leqslant \liminf_{\kappa\to\infty} \frac{5\kappa^{\frac{11}{6}}}{\sqrt{\pi}} \sum_{\vartheta=\vartheta_1-\frac{1}{2}}^{\kappa-\frac{1}{2}} \frac{\sin\vartheta}{\kappa-\vartheta-1/2} = -\infty\end{aligned}$$

and

$$\begin{aligned}\limsup_{\kappa\to\infty} \kappa^{1-\gamma} \sum_{\vartheta=\vartheta_1-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\left[\mathfrak{w}(\vartheta) - \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\right] &= \limsup_{\kappa\to\infty} \kappa^{\frac{1}{3}} \sum_{\vartheta=\vartheta_1-\frac{1}{2}}^{\kappa-\frac{1}{2}} h_{\frac{-1}{2}}(\kappa, \sigma(\vartheta))\left[5\sin\vartheta - 1 - \mathfrak{K}_{5,3}(\vartheta)\right] \\ &\geqslant \limsup_{\kappa\to\infty} \frac{\kappa^{\frac{1}{3}}}{\sqrt{\pi}} \sum_{\vartheta=\vartheta_1-\frac{1}{2}}^{\kappa-\frac{1}{2}} \frac{(\kappa-t-1)^{3/2}}{\kappa-\vartheta-1/2}[5\sin\vartheta - 2] = \infty\end{aligned}$$

with $\mathfrak{K}_{5,3}(\vartheta) = \frac{2.3^{3/2}}{5^{5/2}}$. Hence, according to the Theorem 4.1, every solution of (5.1) is oscillatory.

**Example 5.2.** Examine the HGPHFDP

$$\begin{cases} \Delta_1^{2,\frac{1}{2}}\left(\frac{x}{x+3\varpi(\kappa)}\right) + x\varpi(\kappa) = \kappa^2 - \kappa^3 + \kappa\varpi^2 \operatorname{sgn}\varpi(\kappa), \\ \Delta_1^0\left(\frac{x}{\kappa+3\varpi(\kappa)}\right)\Big|_{-\kappa=2} = \frac{1}{3}, \\ \Delta_1^1\left(\frac{x}{\kappa+3\varpi(\kappa)}\right)\Big|_{-\kappa=2} = 0, \quad \kappa \in \mathbb{N}_1. \end{cases} \tag{5.2}$$

Here $\rho = 2, \omega = \frac{1}{2}$, $\alpha = 1$, $\gamma = 2$, $\Upsilon_1(\kappa, \varpi, \varpi(\mathfrak{p})) = \kappa\varpi$, $\Upsilon_2(\kappa, \varpi, \varpi(\mathfrak{p})) = \kappa\varpi^2 \operatorname{sgn}\varpi$, and $\mathfrak{w}(\kappa) = \kappa^2 - \kappa^3$. By choosing $\mathfrak{q}_1(\kappa) = \kappa^2$, $\mathfrak{q}_2(\kappa) = \frac{x}{4}$, $\iota_1 = 1$, and $\iota_2 = 2$, we get $\mathfrak{K}_{\iota_1,\iota_2}(\vartheta) = -\vartheta^3$. For $\vartheta \in \mathbb{N}_1$, we have

$$\begin{aligned}\kappa^{1-\gamma} \sum_{\vartheta=\vartheta_3-\rho}^{\kappa-\rho} h_{\rho-1}(\kappa, \sigma(\vartheta))\left[\mathfrak{w}(\vartheta) + \mathfrak{K}_{\iota_1,\iota_2}(\vartheta)\right] &= \frac{1}{\kappa}\sum_{\vartheta=1}^{\kappa-2}(\kappa-\vartheta+1)[\vartheta^2 - \vartheta^3 - \vartheta^3] \\ &= \frac{1}{\kappa}\sum_{\vartheta=1}^{\kappa-2}(\kappa-\vartheta+1)\vartheta^2(1-2\vartheta) < 0.\end{aligned}$$

Thus the condition (4.10) of Theorem 4.3 fails. In fact, we observe that (5.2) has a non oscillatory solution $\varpi(\kappa) = \kappa$.

## 6. Conclusion

In this paper, we discussed the existence and uniqueness of the HGPHFDP (1.1). Also, we examined the oscillatory behavior of HGPHFDP (1.1). The main findings are obtained through some discrete fractional calculus features and specific mathematical inequalities. Numerical examples are given to ensure compatibility with the theoretical findings. We assert that our findings are new and have not previously been taken into account. Further, in a similar way, one can study the oscillatory behavior of hybrid generalized pantograph problem using Hilfer type nabla or Hilfer generalized proportional nabla difference operator in the future.

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## Authors' contributions

The problem investigation and manuscript drafting were done equally by all authors, who also read and approved the outcome.

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