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Design an efficient neural network for solving steady state problems

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Abstract

In this article, the mathematical model of steady state problems based on horizontal radial flow in homogenous confined aquifers has been presented. Then we design efficient neural network (ANN) to solve the equation in polar coordinates. A reliable unconstrained optimization method has been used as training algorithm to get high accuracy results. The results illustrated by contour maps. The new effective Levenberg-Marquardt method (NLM) has been implemented to solve the problem. A comparison between the training, testing and validation results has been presented. The weight of the ANN will be chosen such that satisfied local minimizer. Furthermore, the quadratic convergence of NLM has been proved. The results reveal that the suggested design is effective, time saver, and applicable for solving steady state problems.

Keywords: Steady state problems, neural networks, unconstrained optimization, Levenberg-Marquardt algorithm, anisotropic confined aquifers model.

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1. Introduction

Differential equations are essential to many fields of science and engineering. Differential equations are incredibly useful tools for comprehending and resolving challenging issues, from simulating the development of diseases to forecasting the behavior of electrical circuits. Population dynamics: modeling population dynamics is one of the most well-known uses of differential equations [3, 14, 16, 26, 29, 32, 46, 48]. In economics, differential equations are also utilized to model a variety of economic events. For instance, a system of differential equations called the supply and demand equations can be used to simulate the behavior of a market. In biology, differential equations are also frequently employed to model a variety of biological phenomena. For instance, the logistic equation system of differential equations can be used to simulate population expansion. For more information in physics, engineering, optimization, epidemiology see [17, 21–23, 28, 47], and ecology see [2, 9, 27, 33, 45].

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Partial differential equation-based mathematical models can be used to describe a wide variety of physical issues. The partial differential equations (PDEs) govern a wide field of physical, chemical, and biological events. A mathematical model is a condensed, mathematically stated depiction of physical reality [31]. Nonlinear PDEs are also crucial for study in a wide field of domains, including hydrodynamics, engineering, quantum field theory, optics, plasma physics, etc. [38]. Non-linear steady state problems SSPs have an important role in representing different applied science such physical or chemical phenomena arising in engineering [42, 49]. Therefore, researchers focus their attention on capturing the behaviors of these problems. Since having an exact solution for such problems is not easy, researchers have tried to developing analytic and numerical methods [4, 30, 43] to investigate the behaviors of these problems. Herein, artificial neural networks techniques have been proposed for the following problem. During the last several decades, there has been a lot of interest research of various machine intelligence approaches, particularly artificial neural networks (ANNs) is used to solve differential equations [13, 42, 50]. Because ANNs are known to have universal approximation capabilities [19], parallel processing technique, when compared to other traditional numerical approaches. So, many authors used ANN for solving ODEs, PDEs, integral equations and integro equations [35]. The authors proposed various design of ANNs depending on architectural of network: number of layers, number of nodes in each layers, partial or fully connected between layers and/or between nodes in layers, way of feeding the data forward or backward, or depending on training supervise or unsupervised learning [11]. Lee and kang [15] proposed Hopfield neural network for solving differential equations that is unsupervised learning. Lagaris et al. [24] suggested type of neural networks a multi-layer perceptron and used optimization approach to solve each of ODEs and PDEs. Also they solved some types of PDEs specially two and three dimensional space with uneven boundaries using multilayer architecture of ANN [25]. Aarts and Van der veer proposed evolutionary ANN for solving IVP; for more details see [40, 44]. Shirvany et al. in [34] suggested a multilayer perceptron as type of ANN. Depending on the nature of the problem a variety of transfer function have been used, such as ridge basis function, radial basis function (RBF) and others. In [18] the authors used RBF for solving the non-linear Schrodinger problem. Hoda and Nagla [41] used a multilayer ANN technique to address mixed BVPs. Mai-Duy and Tran-Cong in [39] introduced ANN with a radial basis function of type multi quadric for solving ODEs and elliptic PDEs. Jianye et al. in [5] employed ANN with RBF to solve elliptical PDEs. Parisi et al. in [36] used a different strategy to tackle a steady-state heat transport problem.

This article is organized as follows. In Section 2, definition and preliminaries of the ANNs have been introduced. Mathematical model of steady state problem based on confined aquifers is introduced in Section 3. Design efficient ANN for solving SSPDE and illustrated through example and implementation with discussions of the results are presented in Section 4. Global, local minimizer, strong local minimizer, and convergence of the results are presented in Section 5. Finally, the conclusions are presents in Section 6.

2. Artificial neural networks

Artificial neural networks (ANNs) is a structure of parallel processing for distributing information in the form of connected layers consisting of a set of nodes called neurons (also called processing elements) is the basic processor in ANNs, along with directed line segments between them called links (also are called connections) [10]. All nodes can take any number of arrival connections and can have any number of coming-out connections, but the signs must be the same [37]. In effect, all nodes have a one coming-out connection that can branch out to form multiple output connections, each of which carries the same sign. Each node possesses a transfer (activation) function which can use input signs, and which produces the node's output sign. Generally, ANNs have been generalizations of mathematical models of the human brain, based on the processing of information that occurs at many connection nodes; signs are passed between nodes over connection links which have an associated weight; each node applies a transfer function to its weighted input net to determine its sign of output.

The treatment of given data held by income these data as weighted input vector x, as form $W_j^T x$ to enter in hidden layers. In suggested design every hidden neurons has the same activation function σ, but that bias b_j. So the output of j^{th} hidden neuron in hidden layer is $\sigma(W_j^T x + b_j)$ and again weighted by-product with v_j then entered to output layer as the form:

$$g(x) = \sum_{j=1}^{k} v_j \sigma(W_j^T x + b_j),$$

where $g(x)$ represents the output of the NN. Note that, sigma must be choosing sigmoidal transfer functions, so herein we choose suitable effective sigmoidal σ defined as [1]:

$$\sigma(n_i) = \frac{e^{n_i} - 1}{e^{n_i} + 1}.$$

Therefore, the equation of input-output ANN process is: $\hat{Y} = \phi(x^T W^T + b^T) v^T$, where $W \in R^{n \times r}$ is adjustable input weights; $v \in R^{1 \times n}$ is adjustable output weights and $b \in R^{n \times 1}$ is biased.

The architecture of interconnections ANN can be classified as different types of ANNs sometimes depending on feeding the data such as feed forward neural network (FFNN): organized of nodes are in the form of layers and arrival input from the previous layer then feed their output to the next layer, in a strictly the data goes from the input node to the output node as feed-forward way, i.e., forward loops. Feedback neural network (FBNN): All possible connections are allowed between layers and neurons. The data transfer in the network as back loops [6]. Herein we choose FFNN.

3. Steady state confined aquifers

A flow is considered to be steady when the conditions at any point in the fluid do not change with time, i.e., $\partial h/\partial t = 0$ and also the properties do not change with time.

In this section we discussed flow in a completely confined aquifer where no recharge occurs. Most confined aquifers, however, are not totally isolated from sources of vertical recharge. For example, aquitards (confining layers) above or below the aquifer may leak water into the confined aquifer under favorable hydraulic gradient conditions. So, the general three-dimensional groundwater flow equation to calculating hydraulic head for confined flow with vertical leakage in radial coordinates is:

$$\partial/\partial x(T_x \partial h/\partial x) + \partial/\partial y(T_y \partial h/\partial y) + \partial/\partial z(T_z \partial h/\partial z) + Q + l(h_1 - h) = S\partial h/\partial t,$$

where, l is leakage factor (L_2/T) and $h1$ initial hydraulic head. There are several special cases for the above equation depending on status of wells, region, and soil. In the confined aquifer with source, we didn't have leakage flow, therefore, $l = 0$ and the equation becomes:

$$\partial/\partial x(T_x \partial h/\partial x) + \partial/\partial y(T_y \partial h/\partial y) + \partial/\partial z(T_z \partial h/\partial z) + Q(x, y, z, t) = S\partial h/\partial t$$

When a fully penetrating well pumps a confined aquifer the influence of the pumping extends radially outwards from the well with time, and the pumped water is withdrawn entirely from the storage within the aquifer. In theory, because the pumped water must come from a reduction of storage within the aquifer, only unsteady-state flow can exist. In practice, however, the flow to the well is considered to be in a steady state if the change in drawdown has become negligibly small with time. In this article, the equation of flow will be study and solve with the assumptions and conditions underlying such as:

- the aquifer is confined;
- the aquifer has a seemingly infinite areal extent;

- the aquifer is homogeneous, anisotropic, and of uniform thickness over the area influenced by the test;
- prior to pumping, the piezometric surface is horizontal (or nearly so) over the area that will be influenced by the test;
- the aquifer is pumped at a constant discharge rate;
- the well penetrates the entire thickness of the aquifer and thus receives water by horizontal flow.

The standard equation that governs steady state confined aquifer is [7, 20]:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} + T_z \frac{\partial^2 h}{\partial z^2} = S \frac{\partial h}{\partial t}, \tag{3.1}$$

where S is storage coefficient, T_i is transitivity in the i-direction $\frac{L^2}{T}$, (i may be x, y, or z). Since the flow is horizontal in aquifers, assumption (3.1) becomes:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = S \frac{\partial h}{\partial t}. \tag{3.2}$$

For polar coordinate suppose that:

$$r = \sqrt{x^2 + y^2}, \quad r' = \sqrt{\frac{T_y x^2 + T_x y^2}{T}}, \quad T = \sqrt{T_x T_y}. \tag{3.3}$$

Then

$$\frac{\partial r'}{\partial x} = \frac{T_y x}{T r'}, \quad \frac{\partial r'}{\partial y} = \frac{T_x y}{T r'}, \quad \frac{\partial^2 r'}{\partial x^2} = \frac{T T_y r'^2 - T_y^2 x^2}{T^2 r'^3}, \quad \frac{\partial^2 r'}{\partial y^2} = \frac{T T_x r'^2 - T_x^2 y^2}{T^2 r'^3}.$$

To get

$$T_x \frac{\partial^2 h}{\partial x^2} = T_x \left[\frac{\partial h}{\partial r'} \frac{\partial^2 r'}{\partial x^2} + \frac{\partial^2 h}{\partial^2 r'^2} \left(\frac{\partial r'}{\partial x} \right)^2 \right] = T_x \left[\frac{\partial h}{\partial r'} \frac{T T_y r'^2 - T_y^2 x^2}{T^2 r'^3} + \frac{\partial^2 h}{\partial r'^2} \left(\frac{T_y x}{T r'} \right)^2 \right].$$

So,

$$T_x \frac{\partial^2}{\partial x^2} = \frac{T}{r'} \frac{\partial h}{\partial r'} - \frac{T_y x^2}{r'^3} \frac{\partial h}{\partial r'} + \frac{T_y x^2}{r'^2} \frac{\partial^2 h}{\partial r'^2}, \qquad T_y \frac{\partial^2}{\partial y^2} = \frac{T}{r'} \frac{\partial h}{\partial r'} - \frac{T_x y^2}{r'^3} \frac{\partial h}{\partial r'} + \frac{T_x y^2}{r'^2} \frac{\partial^2 h}{\partial r'^2}.$$

Here

$$T_x \frac{\partial^2}{\partial x^2} + T_y \frac{\partial^2}{\partial y^2} = \frac{T}{r'} \frac{\partial h}{\partial r'} - \frac{T_y x^2}{r'^3} \frac{\partial h}{\partial r'} + \frac{T_y x^2}{r'^2} \frac{\partial^2 h}{\partial r'^2} + \frac{T}{r'} \frac{\partial h}{\partial r'} - \frac{T_x y^2}{r'^3} \frac{\partial h}{\partial r'} + \frac{T_x y^2}{r'^2} \frac{\partial^2 h}{\partial r'^2},$$

i.e.,

$$T_x \frac{\partial^2}{\partial x^2} + T_y \frac{\partial^2}{\partial y^2} = T \left[\frac{1}{r'} \frac{\partial h}{\partial r'} + \frac{\partial^2 h}{\partial r'^2} \right]. \tag{3.4}$$

Now, substituting (3.4) in (3.2):

$$\left[\frac{1}{r'} \frac{\partial h}{\partial r'} + \frac{\partial^2 h}{\partial r'^2} \right] = \frac{S}{T} \frac{\partial h}{\partial t}. \tag{3.5}$$

Then (3.5) governs the radial flow with homogeneous confined aquifer. Now by the polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, where θ is the angle between the positive x-axis and the vector, then (3.3) can be written in polar coordinates as

$$r' = r\sqrt{\frac{T_y\cos^2(\theta) + T_x\sin^2(\theta)}{T}}.$$

Let $T_\theta = \sqrt{\frac{T_y\cos^2(\theta)+T_x\sin^2(\theta)}{T}}$, then

$$r' = rT_\theta, \tag{3.6}$$

in the steady state case, $\partial h/\partial t = 0$. So, Eq. (3.5) becomes $\frac{1}{r'}\frac{\partial h}{\partial r'} + \frac{\partial^2 h}{\partial r'^2} = 0$. This can be rewritten as $\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial h}{\partial r'}\right) = 0$. Integrate both sides of this equation to get:

$$r'\frac{\partial h}{\partial r'} = C_1.$$

From Darcy's law:

$$\frac{Q}{2\pi T_r} = \frac{\partial h}{\partial r}',$$

where T_r is the transmissivities in the direction r and defined in [12] as

$$T_r = \frac{T_y T_x}{T_y\cos^2(\theta) + T_x\sin^2(\theta)}.$$

From Eq. (3.6) $\frac{\partial r'}{\partial r} = T_\theta$ and $r = \frac{r'}{T_\theta}$, so

$$\frac{Q}{2\pi T_r} = r'\frac{\partial h}{\partial r'} = C_1$$

and

$$\partial h = \frac{Q}{2\pi T_r}\frac{\partial r'}{r'}. \tag{3.7}$$

Integrating both sides of Eq. (3.7) gives

$$h = \frac{Q}{2\pi T_r}\ln r' + C_2 \tag{3.8}$$

Consequently, from Eqs. (3.6) and (3.8), one can get

$$h = \frac{Q}{2\pi T_r}\ln(rT_\theta) + C_2, r > 0. \tag{3.9}$$

If r_v is the radius of the well and its hydraulic head at a distance from the center of the well and in all directions is equal to h_v, then $h_v = Q/(2\pi T_{r_v})\ln(r_v T_{\theta_v}) + C_2$ and so on $h - h_v = Q/(2\pi T_r)[\ln(rT_\theta) - \ln(r_v T_\theta)] = Q/(2\pi T_r)\ln(r/r_v)$ that is the result

$$h = h_v + \frac{Q}{2\pi T_r}\ln(\frac{r}{r_v}r).$$

Depending on boundary conditions, we can get another estimate to the constant C_2 from, which specifies $h_0 = h$ at $R = r$, where R is the largest radius of effect, then

$$h_0 = \frac{Q}{2\pi T_r} \ln\left(R\sqrt{\frac{T_y \cos^2(\theta_0) + T_x \sin^2(\theta_0)}{T}} \right) + C_2.$$

After choosing the same direction, the above equation takes the following form

$$C_2 = h_0 - \frac{Q}{2\pi T_r} \ln\left(R\sqrt{\frac{T_y \cos^2(\theta_0) + T_x \sin^2(\theta_0)}{T}} \right).$$

Then Eq. (3.9) becomes

$$\begin{aligned} h_0 - h &= \frac{Q}{2\pi T_r}\left[\ln\left(R\sqrt{\frac{T_y \cos^2(\theta_0) + T_x \sin^2(\theta_0)}{T}} \right) - \ln\left(r\sqrt{\frac{T_y \cos^2(\theta_0) + T_x \sin^2(\theta_0)}{T}} \right) \right] \\ &= \frac{Q}{2\pi T_r}\left[\ln\left(\frac{R}{r}\right) \right], \end{aligned}$$

where h_0–h, is the drawdown s_r.

4. Design efficient ANN

The authors suggest suitable design consist three fully interconnection layers, input layer consist three neurons, but hidden layer consist seven neurons of type (tanhsig.) ridge transfer function but the output layer consist two neurons of type (linsig.) transfer function (see 1). The suggested ANN is trained by the back propagation rule based on unconstrained optimization herein we chose the Levenberg-Marquardt (LM) training algorithm which has the following update rule of the weights $w_{k+1} = w_k - (J^T J + \mu I)^{-1} J_k E(w_k)$, where I is the identity matrix; μ is coefficient always chosen positive; J is the Jacobean matrix, which consists 1^{st} derivatives of the ANN errors (energy functions or objective functions) with respect to the weights and biases.

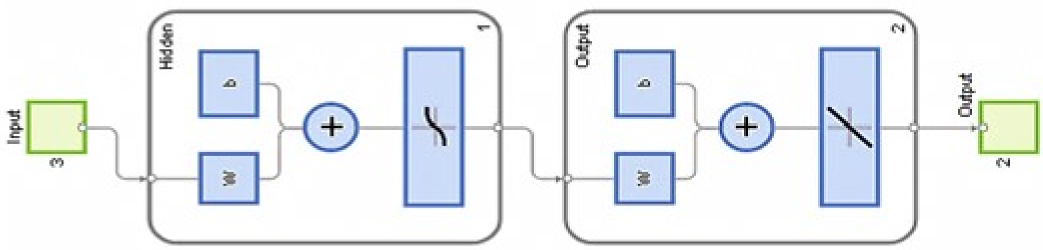
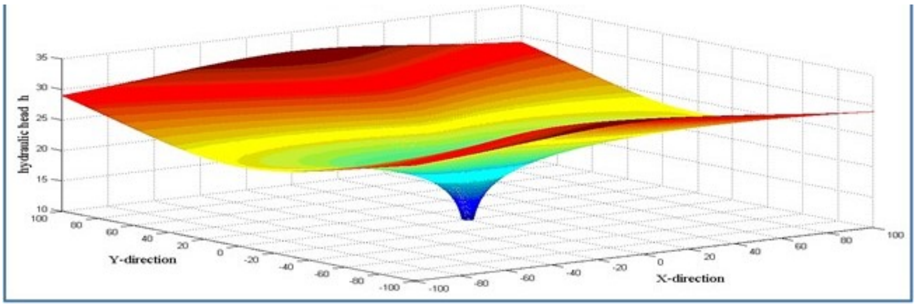


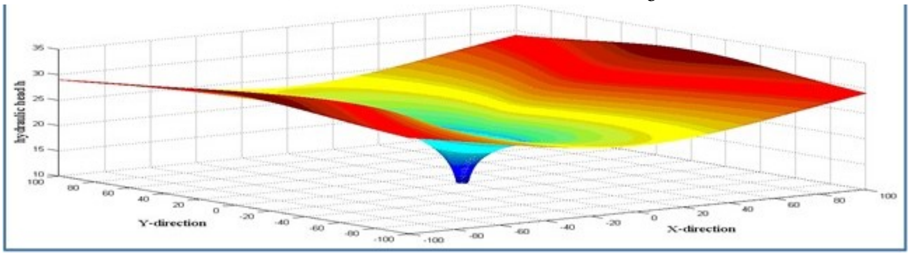
Figure 1: Suggested design of ANN.

Let the system of the pumping test and pumping rate $Q = 10000(\pi m^3)/s, h_v = 10m$, $r_v = 1m$, the distance $1000m$ on all direction and the transmissivity studying in three case as follows

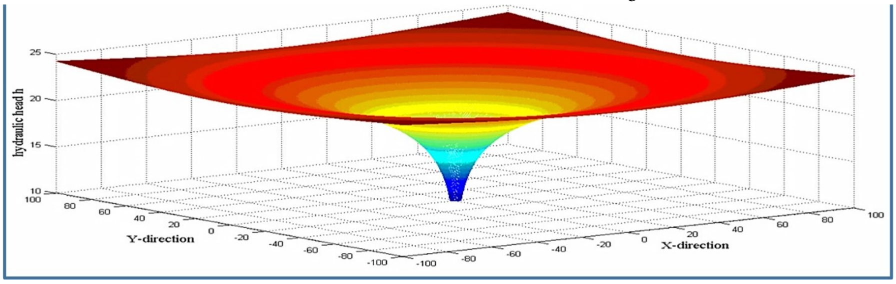
a) $T_x = 5\pi m^2/6s, T_y = \pi m^2/2s$;
b) $T_x = \pi m^2/2s, T_y = 5\pi m^2/6s$;
c) $T_x = 5\pi m^2/6s, T_y = 5\pi m^2/6s$.



(a) The hydraulic head with $T_x = 150, T_y = 90$.



(b) The hydraulic head with $T_x = 90, T_y = 150$.



(c) The hydraulic head with $T_x = 150, T_y = 150$.

Figure 2: The effect of transmissivity on the hydraulic head by ANN.

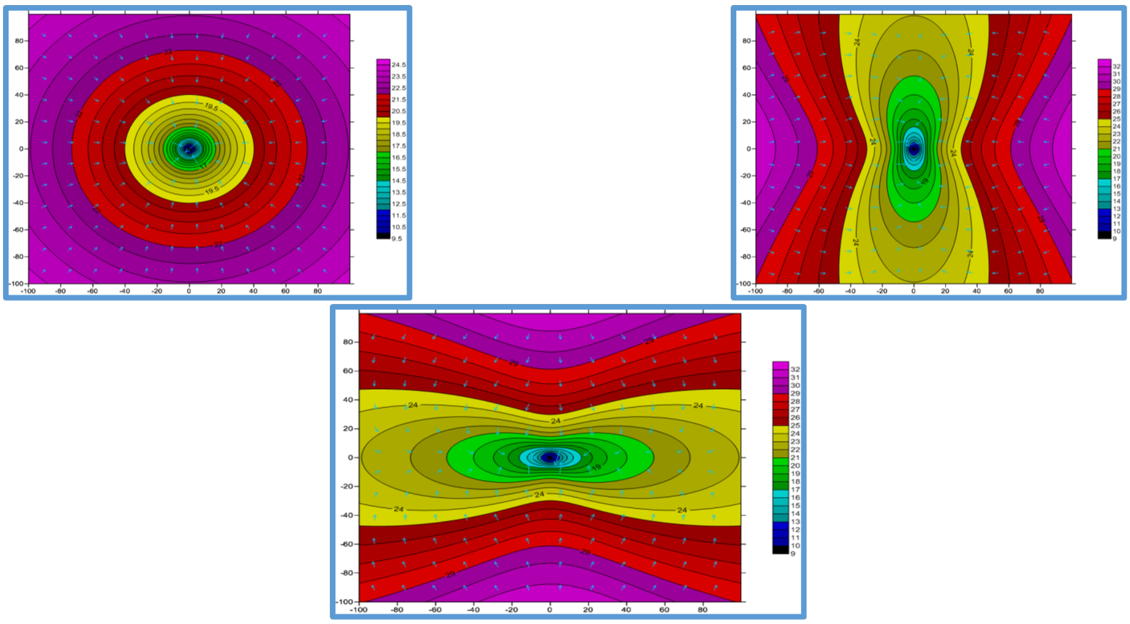


Figure 3: Contour maps.

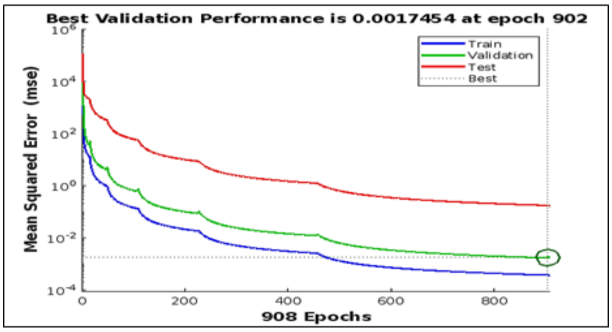


Figure 4: Comparison between training, testing and validation results for ANN.

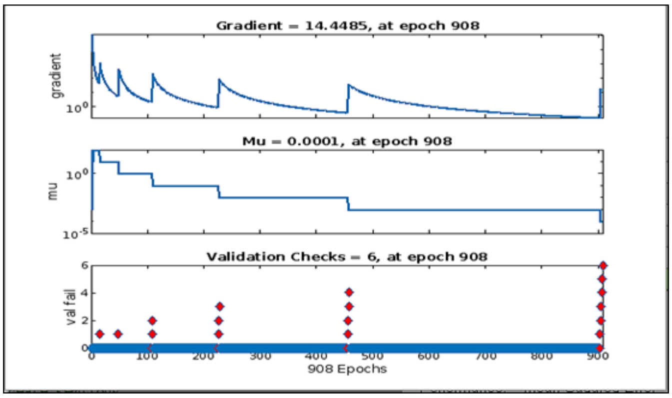


Figure 5: Performance of suggested design.

By training suggested ANN, we get the results in Fig. 2 and observe the effect of transmissivity on the hydraulic head by the effect on the cone. In Fig. 2a, since $T_x > T_y$, main axes of the ellipses are parallel to the x-axis, while in Fig. 2b, $T_x < T_y$. So, the main axes of the ellipses are parallel to the y-axis, but in Fig. 2c the conics are circles, because of $T_x = T_y$, the effect of the anisotropic aquifer on the flow by the contour maps as in Fig. 3. Fig. 4 illustrates a comparison between the results of training, testing and validation for suggested design. Fig. 5 illustrates the performance of suggested design.

5. Global minimizer

A weight of the network w^* is said to be a global minimizer if $f(w^*) \leqslant f(w), \forall w$, where $w \in R^n$. Note, the essential assumption is that the global minimizer can be difficult to prove, because our knowledge of f is only local, so we will show a local minimizer at the least.

5.1. *Local minimizer*

A weight of the network w^* is said to be a local minimizer if the following condition is satisfied: there exist a neighborhood $N, f(w^*) \leqslant f(w), \forall w \in N$, such that $\left\| w - w^* \right\| < \epsilon, \forall \epsilon > 0$.

5.2. *Strong local minimizer*

A weight of the network w^* is called strong local minimizer if there is a neighborhood N of w^* and $f(w^*) < f(w), \forall w \in N$ with $w \neq w^*$, such that $0 < \left\| w - w^* \right\| < \epsilon$. Note that, for the neural network choose the weights w^* such that satisfies strong local minimizer condition.

5.3. Local convergence of the NLM method

The convergence of NLM is present in this section; firstly we suppose the following hypothesis.

Hypothesis 1:

(a) $\Big\| E(w) \Big\|$ is smooth (continuously differentiable), and the Jacobian of weights $J(w)$ is satisfies Lipschitz condition about some neighborhood of $w^* \in W^*$, i.e., there exist $+ve$ constants L_1 and $b_1 < 1$ such that

$$\Big\|J(w_1) - J(w_2)\Big\| \leqslant L_1\Big\|w_1 - w_2\Big\|, \quad \forall w_1, w_2 \in N(w^*, b_1) = w|\Big\|w^* - w\Big\| \leqslant b_1.$$

(b) A local error bound $\Big\| E(w) \Big\|$ on $N(w^*, b_1)$, i.e., there is a constant $c_1 > 0$ such that

$$\Big\|E(w)\Big\| \geqslant c_1 \mathrm{dist}(w, w^*), \quad \forall w \in N(w^*, b_1). \tag{5.1}$$

Note that, by Hypothesis 1 (a), we get:

$$\Big\|E(w_1) - E(w_2) - J(w_2)(w_1 - w_2)\Big\| \leqslant L_1\Big\|w_1 - w_2\Big\|^2, \forall w_1, w_2 \in N(w^*, b_1)$$

and, there is a constant $L_2 > 0$ such that:

$$\Big\|E(w_1) - E(w_2)\Big\| \leqslant L_2\Big\|w_1 - w_2\Big\|, \quad \forall w_1, w_2 \in N(w^*, b_1).$$

For simplification, we use $E_k = E(w_k)$ and $J_k = J(w_k)$.

Hypothesis 2: we choose $\mu_k = \Big\|E_k\Big\|^\delta \forall k$, where $\delta \in [1, 2]$.

In [13], Yamashita and Fukushima proved the convergence is quadratic of the LM method when choosing $\mu_k = \Big\|E_k\Big\|^2$, based on the properties of an unconstrained optimization. Herein, first we prove the NLM method has super linear convergence if we choose $\mu_k = \Big\|E_k\Big\|^\delta$, then, depending on the properties of singular value decomposition (SVD) of the Jacobian matrix, we get the quadratic convergence. In the next, we denote $\bar{w}_k$ the vector in W^* that satisfies

$$\Big\|w_k - \bar{w}_k\Big\| = \mathrm{dist}(w_k, W^*).$$

Theorem 5.1. *Under the conditions of Hypothesis 1 and 2, if the initial weight w_0 is chosen close to W^* sufficiently, then $w_{k+1} = w_k + \rho_k$ converges to the solution $\bar{w}$ super linearly.*

Proof. Suppose $r = \min\{\frac{b_1}{2(1+11c_2)}, \frac{1}{2c_3^{\frac{2}{\delta}}}\}$, first we show that if $w_0 \in N(w^*, r)$, then $w_k \in N(w^*, \frac{b_1}{2})$, $\forall k$. By using induction and since $[w_{k+1} = w_k + \rho_k]$, we have

$$\Big\|w_1 - w^*\Big\| = \Big\|w_0 + \rho_0 - w^*\Big\| \leqslant \Big\|\rho_0\Big\| + \Big\|w_0 - w^*\Big\|.$$

The search direction ρ_k can be chosen such that

$$\Big\|\rho_k\Big\| = c_2 \mathrm{dist}(w_k, W^*), \tag{5.2}$$

where a constant $c_2 > 0$. From (5.2) we have

$$\Big\|w_1 - w^*\Big\| \leqslant \Big\|w_0 - w^*\Big\| + c_2\Big\|w_0 - \bar{w}_0\Big\|.$$

Since $\left[w_0 \in N(w^*, r) \to \left\|w_0 - w^*\right\| \leqslant r\right]$, then we get

$$\left\|w_1 - w^*\right\| \leqslant (c_2+1)r \leqslant \frac{b_1}{2},$$

which means $w_1 \in N(w^*, (b_1)/2)$. Suppose $w_i \in N(w^*, b_1, 2)$, $\forall i = 2, \dots, k$, Then we have

$$\begin{aligned}
\left\|w_i - \bar{w}_i\right\| &\leqslant c_3 \left\|w_{i-1} - \bar{w}_{i-1}\right\|^{\frac{2+\delta}{2}} \\
&\vdots \\
&\leqslant c_3^{(\frac{2+\delta}{2})^i - 1} \left\|w_0 - w^*\right\|^{(\frac{2+\delta}{2})^i} \\
&\leqslant c_3^{\frac{2}{\delta}\left((\frac{2+\delta}{2})^i - 1\right)} \left\|w_0 - w^*\right\|^{(\frac{2+\delta}{2})^i} \\
&\leqslant r\left(\frac{1}{2}\right)^{(\frac{2+\delta}{2})^i - 1} = 2r\left(\frac{1}{2}\right)^{(\frac{2+\delta}{2})^i} \leqslant 2r\left(\frac{1}{2}\right)^{(\frac{3}{2})^i}, \quad \text{since} \quad \delta \in [1,2].
\end{aligned}$$

Hence, from the definition of r we have

$$\begin{aligned}
\left\|w_{k+1} - w^*\right\| &= \left\|w_{k+1} - w_k + w_k - w^*\right\| \\
&\leqslant \left\|w_{k+1} - w_k\right\| + \left\|w_k - w^*\right\| \\
&= \left\|\rho_k\right\| + \left\|w_k - w_{k-1} + w_{k-1} - w^*\right\| \\
&\leqslant \left\|\rho_k\right\| + \left\|w_k - w_{k-1}\right\| + \left\|w_{k-1} - w^*\right\| \\
&= \left\|\rho_k\right\| + \left\|\rho_{k-1}\right\| + \left\|w_{k-1} - w_{k-2} + w_{k-2} - w^*\right\| \\
&\leqslant \left\|\rho_k\right\| + \left\|\rho_{k-1}\right\| + \left\|w_{k-1} - w_{k-2}\right\| + \left\|w_{k-2} - w^*\right\| \\
&= \left\|\rho_k\right\| + \left\|\rho_{k-1}\right\| + \left\|\rho_{k-2}\right\| + \left\|w_{k-2} - w^*\right\| \\
&\leqslant \left\|\rho_k\right\| + \left\|\rho_{k-1}\right\| + \left\|\rho_{k-2}\right\| + \cdots + \left\|\rho_{k-(k-1)}\right\| + \left\|w_{k-(k-1)} - w^*\right\| \\
&\leqslant \left\|\rho_k\right\| + \left\|\rho_{k-1}\right\| + \left\|\rho_{k-2}\right\| + \cdots + \left\|\rho_1\right\| + \left\|w_1 - w^*\right\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|w_{k+1} - w^*\right\| &\leqslant \left\|w_1 - w^*\right\| + \sum_{i=1}^{k} \left\|\rho_i\right\| \\
&\leqslant (1+c_2)r + c_2 \sum_{i=1}^{k} \left\|w_i - \bar{w}_i\right\| \\
&\leqslant (1+c_2)r + 2rc_2 \sum_{i=1}^{k} (\frac{1}{2})^{(\frac{3}{2})^i} \\
&\leqslant (1+c_2)r + 2rc_2 \left(4 + \sum_{i=1}^{k} (\frac{1}{2})^{(\frac{3}{2})^i}\right) \\
&\leqslant (1+9c_2)r + 2rc_2 \sum_{i=1}^{\infty} (\frac{1}{2})^i \leqslant (1+11c_2)r \leqslant \frac{b_1}{2},
\end{aligned}$$

so, $w_{k+1} \in N(w^*, \frac{b_1}{2}$. Therefore, if w_0 is chosen close to W^* sufficiently, then all w_k are in. □

Theorem 5.2. *Under the conditions of Hypothesis 1 and 2, if the sequence w_k is generated by the NLM method with w_0 close to w^* sufficiently, then w_k converges to the solution quadratically.*

Proof. By the SVD of J_k, the step at the current iterates is

$$\rho_k = -(J_k^T J_k + \mu_k I)^{-1} J_k^T E_k,$$

when $J_k = U_k \Sigma_k V_k^T$, then we have

$$\begin{aligned}
\rho_k &= -\left[(U_k\Sigma_k V_k^T)^T (U_k\Sigma_k V_k^T) + \mu_k I\right]^{-1} (U_k\Sigma_k V_k^T) E_k \\
&= -\left[V_k\Sigma_k^T \quad U_k^T U_k \quad \Sigma_k V_k^T + \mu_k I\right]^{-1} (V_k\Sigma_k^T U_k^T) E_k, \{U \text{ orthogonal, i.e., } U_k^{-1} = U_k^T\} \\
&= -(V_k\Sigma_k^2 V_k^T + \mu_k I)^{-1} (V_k \Sigma_k U_k^T) E_k, \{\text{since } \Sigma_k^T = \Sigma_k \text{ is diagonal matrix}\} \\
&= -\left[V_k\Sigma_k^2 V_k^T + \mu_k I(V_k V_k^T)\right]^{-1} (V_k\Sigma_k U_(k)^T) E_k, \{V \text{ orthogonal, i.e., } V_k^{-1} = V_k^T\} \\
&= -V_k \left(\Sigma_k^2 + \mu_k I\right)^{-1} \left(V_k^T V_k\right) \Sigma_k U_k^T E_k.
\end{aligned}$$

Hence

$$\rho_k = -V_k(\Sigma_k^2 + \mu_k I)^{-1} \Sigma_k U_k^T E_k.$$

Now when $J_k = U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T$, so, we have

$$\rho_k = -V_1(\Sigma_1^2 + \mu_k I)^{-1}\Sigma_1 U_1^T E_k - V_2(\Sigma_2^2 + \mu_k I)^{-1}\Sigma_2 U_2^T E_k.$$

Now we need to show

$$\left\|(w_{k+1} - w^*)\right\| = o\left(\left\|w_k - w^*\right\|^2\right).$$

According to the Taylor expansion, $E(w_k)$ can be written as

$$E(w_{k+1}) = E(w_k) + E'(w_k)(w_{k+1} - w_k) + o(w_k^2) = E_k + J_k\rho_k,$$

since $w_{k+1} = w_k + \rho_k$. Now we have

$$\begin{aligned}
E_k + J_k\rho_k &= E_k + (U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T) - \left[V_1(\Sigma_1^2 + \mu_k I)^{-1}\Sigma_1 U_1^T E_k - V_k(\Sigma_2^2 + \mu_k I)^{-1}\Sigma_2 U_2^T E_2\right] \\
&= E_k - \left[U_1\Sigma_1(V_1^T V_1)(\Sigma_1^2 + \mu_k I)^{-1}\Sigma_1 U_1^T E_k\right] - \left[U_1\Sigma_1(V_1^T V_2)(\Sigma_2^2 + \mu_k I)^{-1}\Sigma_2 U_2^T E_k\right] \\
&\quad - \left[U_2\Sigma_2(V_2^T V_1)(\Sigma_1^2 + \mu_k I)^{-1}\Sigma_1 U_1^T E_k\right] - \left[U_2\Sigma_2(V_2^T V_2)(\Sigma_2^2 + \mu_k I)^{-1}\Sigma_2 U_2^T E_k\right] \\
&= E_k - U_1\Sigma_1(\Sigma_1^2 + \mu_k I)^{-1}\Sigma_1 U_1^T E_k - U_2\Sigma_2(\Sigma_2^2 + \mu_k I)^{-1}\Sigma_2 U_2^T E_k.
\end{aligned}$$

Since V is orthogonal, i.e., $V_k^{-1} = V_k^T$ and $V_k^T V_i = 0, \forall k \neq i$, so, by the SVD we get

$$E_k + J_k\rho_k = \mu_k U_1(\Sigma_1^2 + \mu_k I)^{-1} U_1^T E_k + \mu_k U_2(\Sigma_2^2 + \mu_k I)^{-1} U_2^T E_k + U_3 U_3^T E_k.$$

Since w_k converges to w^* super-linearly, without loss of generality, we suppose that

$$L_1\left\|w_k - w^*\right\| < \frac{\sigma_r^*}{2}, \quad \forall k \text{ sufficiently large.} \tag{5.3}$$

Then we get

$$\left\|(\Sigma_1^2 + \mu_k I)^{-1}\right\| \leqslant \left\|\Sigma_1^{-2}\right\|, \qquad \left\|\Sigma - \Sigma_1^*\right\| \leqslant L_1\left\|w_k - w^*\right\|,$$

$$\left\|\Sigma_1\right\| + \left\|\Sigma_1^*\right\| \leqslant L_1\left\|w_k - w^*\right\|, \qquad \left\|\Sigma_1\right\| \leqslant L_1\left\|w_k - w^*\right\| - \left\|\Sigma_1^*\right\|,$$

$$\left\|\Sigma_1\right\|^2 \leqslant (\left\|\Sigma_1^*\right\| - L\Sigma_1\left\|w_k - w^*\right\|)^2.$$

Therefore

$$\left\|\Sigma_1\right\|^{-2} \leqslant \frac{1}{(\sigma_r^* - L_1\left\|w_k - w^*\right\|)^2}. \tag{5.4}$$

From (5.3) we obtain

$$\left\|\Sigma_1\right\|^{-2} < \frac{1}{\left(\sigma_r^* - \frac{\sigma_r^*}{2}\right)^2} = \frac{4}{\sigma_r^{*2}}.$$

And

$$\Sigma_2^2 + \mu_k I \geqslant \mu_k \; (\Sigma_2^2 + \mu_k I)^{-1} \leqslant \mu_k^{-1}.$$

Then

$$\left\|(\Sigma_2^2 + \mu_k I)^{-1}\right\| \leqslant \mu_{k^{-1}}. \tag{5.5}$$

By above two inequalities (5.4) and (5.5), we have

$$\left\|E_k + J_k\rho_k\right\| \leqslant \mu_k U_1 \frac{4}{\sigma_r^{*2}} U_1^T E_k + \mu_k U_2 \mu_k^{-1} U_2^T E_k + U_3 U_3^T E_k \leqslant \left(\frac{4}{\sigma_r^{*2}} L_2^{1+\delta} + 2L_1\right)\left\|w_k - w^*\right\|^2, \quad \delta \in [1,2].$$

This result with help that U_k is orthogonal, i.e., $U_k^{-1} = U_k^T$. Let $c_4 = \frac{4}{\sigma_r^{*2}} L_2^{1+\delta} + 2L_1$, then we get

$$\left\|E_k + J_k\rho_k\right\| \leqslant c_4\left\|w_k - w^*\right\|^2. \tag{5.6}$$

From (5.1) we get

$$c_1 \mathrm{dist}(w_{k+1}, W^*) \leqslant \left\|E(w_{k+1})\right\| = \left\|E(w_k + \rho_k)\right\|.$$

From Taylor series we have

$$c_1 \mathrm{dist}(w_{k+1}, W^*) \leqslant \left\|E_k + J_k\rho_k\right\| + L_1\left\|\rho_k\right\|^2.$$

From (5.6) we get

$$c_1 \mathrm{dist}(w_{k+1}, W^*) \leqslant c_4\left\|w_k - w^*\right\|^2 + c_2^2 L_1\left\|w_k - w^*\right\|^2 \leqslant (c_4 + c_2^2 L_1)\left\|w_k - w^*\right\|^2.$$

That is $\left\|\rho_{k+1}\right\| = o(\left\|\rho_k\right\|^2)$, which implies that $\{w_k\}$ converges quadratically to w^*, namely, $\left\|(w_{k+1} - w^*)\right\| = o(\left\|w_k - w^*\right\|^2)$. □

Remark 5.3. From the above theorem, if the parameter of Levenberg-Marquardt is chosen as $\mu_k = \left\|E_k\right\|^\delta$ with $\delta \in [1,2]$, then under the condition of local error bound we have

$$\frac{\mu_{k+1}}{\mu_k^2} = \frac{\left\|E_{k+1}\right\|^\delta}{\left\|E_k\right\|^{2\delta}} \leqslant o\left(\frac{\left\|w_{k+1} - \bar{w}_{k+1}\right\|^\delta}{\left\|w_k - \bar{w}_k\right\|^{2\delta}}\right) = o\left(\frac{\left\|\rho_{k+1}\right\|^\delta}{\left\|\rho_k\right\|^{2\delta}}\right) = o\left(\frac{\left\|\rho_k\right\|^{2\delta}}{\left\|\rho_k\right\|^{2\delta}}\right) = o(1).$$

Hence, $\mu_{k+1} \leqslant \mu_k^2$, which implies the parameter $\{\mu_k\}$ and $\{\left\|E_k\right\|\}$ converges quadratically to zero as the sequence $\{w_k\}$ converges quadratically to the solution of the nonlinear equations.

6. Conclusions

In this article, the effective parallel processing technique based on ANN and the novel effective training algorithm based on modified NLM have been implemented to solve nonlinear SSPDE involving initial value problems. The nonlinear problems are reduced to nonlinear algebraic system of equations solved with Mathematica @12. The novel approximate solutions were obtained and proved accurate and reliable, even within a few polynomial orders. Moreover, the Mse for the proposed ANN were calculated. The results show that the proposed design has higher accuracy and less error. It is also observed that the Mse results of the proposed ANN decrease vastly compared to the ADM. Therefore, the proposed novel MLMTA have better accuracy than the usual LM. The main conclusion from the results is that the ANN-based one hidden layer with 5 nodes has slightly better accuracy than the other methods for solving the WLE. Moreover, the ANN based on the one hidden layer is more accurate than the ANN with two hidden layer in solving the SSPDE. In addition, the ANN based on the modified BFGS training algorithm is slightly more accurate than the usual LM in solving the SSPDE. Furthermore, the suggested design has sampling time is 0.5 seconds with/without noise and disturbances. The average computation time for the parallel and serial implementations are approximately 31.5259 and 4.4117 seconds, respectively at each identification. Note that none of the serial and parallel implementation meets the sampling time of 0.5 seconds for the suggested design. Thus, the parallel implementation of the combined identification and control strategies on real-time embedded multiprocessor systems, such as field programmable gate arrays, is recommended. So that the proposed algorithm can be used much with the sampling time.

In future the authors can be design other types of ANNs or use other architectural for ANNs. Also can be used other training algorithms to solve SSPDEs or any problems in the polar coordinates.

Data availability statement

All data generated or analyzed during this study are included in this article.

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