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Bipolar $\mathbb{M}$-parametrized $\mathbb{N}$-soft sets: a gateway to informed decision-making

Sagvan Y. Musa[a], Baravan A. Asaad[b,c,*]

[a] Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Zakho, Iraq.

[b] Department of Computer Science, College of Science, Cihan University-Duhok, Duhok 42001, Duhok, Iraq.

[c] Department of Mathematics, College of Science, University of Zakho, Zakho 42002, Zakho, Iraq.

Abstract

$\mathbb{M}$-parametrized $\mathbb{N}$-soft set (MPNSS), an extension of $\mathbb{N}$-soft set ($\mathbb{N}$-SS) theory, is instrumental in addressing the challenges of assigning non-binary evaluations to both alternatives and attributes. Recognizing the inherent duality in human decision-making, where choices are influenced by both positive and negative aspects, we enhance the MPNSS framework by incorporating bipolarity. This addition, aimed at capturing the dual nature of decision processes, results in the development of bipolar $\mathbb{M}$-parametrized $\mathbb{N}$-soft set (BMPNSS) model. In the context of BMPNSS, we present some related definitions such as incomplete, negatively efficient, positively efficient, and totally efficient. Additionally, for the complement of MPNSS, we introduce four distinct definitions: complement, weak complement, top weak complement, and bottom weak complement. Set-theoretic operations, including extended and restricted union and intersection, are explored accompanied by a discussion of their properties, providing a comprehensive understanding of the behavior of these operations within the BMPNSS framework. To facilitate understanding, we include an illustrative example. The decision-making procedure introduces alternative ranking based on extended choice and extended weight choice values, demonstrated through a numerical example. In our comparative analysis, BMPNSS is positioned against existing models, emphasizing its distinctive features and advantages in diverse decision-making scenarios.

Keywords: Bipolar $\mathbb{M}$-parametrized $\mathbb{N}$-soft set, $\mathbb{M}$-parametrized $\mathbb{N}$-soft set, $\mathbb{N}$-soft set, decision-making, algorithm.

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1. Introduction

Several mathematical methodologies have been created to address issues related to uncertainty and vagueness. These range from fuzzy set theory [39], rough set theory [32], vague set theory [17] to soft set theory [26]. These theoretical frameworks find practical applications in a wide range of fields and disciplines. Soft sets, as introduced by Molodtsov [26] in 1999, present a mathematical framework designed for representing uncertainty and imprecision within sets and their elements. They offer a more adaptable and intuitive method for handling information that is incomplete, inconsistent, or uncertain.

*Corresponding author

Email addresses: sagvan.musa@uoz.edu.krd (Sagvan Y. Musa), baravan.asaad@uoz.edu.krd (Baravan A. Asaad)

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The hybridization of existing structures has consistently demonstrated its efficacy in enhancing decision-making and various applications within uncertainty modeling structures. Soft sets exemplify this trend, with several hybrid structures involving them having already been developed. Maji et al. [24] pioneered the concept of fuzzy soft sets by merging the principles of fuzzy sets and soft sets. The operations defined for soft sets and fuzzy soft sets by Maji et al. and Roy et al. [25, 36] have found applications in diverse fields. However, some researchers, such as Chen et al. [13], Pei and Miao [33], and others, have identified certain weaknesses in these approaches. Consequently, Çağman et al. [11] redefined fuzzy soft sets and their associated operations. Çağman et al. [10] introduced the concept of fuzzy parameterized soft sets, and in another contribution, Çağman et al. [9] proposed a hybrid model named fuzzy parameterized fuzzy soft sets. Shabir and Naz [38] introduced the concept of soft topology. Following these seminal contributions, several methods for extending soft topologies have been introduced, as documented in [1, 7, 8, 14, 15, 29, 30].

Recent studies [20, 21] indicate that the predominant focus among researchers has been on the development of hybrid models of soft sets, commonly employing binary evaluations or real numbers within the [0, 1] range as the degree of membership. However, in everyday scenarios, non-binary evaluations, such as rankings, are frequently encountered. Herawan and Deris [18] innovatively constructed a binary-valued information system using soft sets, assigning distinct rankings to each attribute, in contrast to the rating orders proposed by Chen et al. [12]. Conversely, Ali et al. [6] adopted a different approach, using a rating system to assess various elements of soft set attributes. Addressing limitations in the existing soft set model, Fatimah et al. [16] introduced the theory of N-SSs, aiming to offer a finer granularity in handling uncertainties in real-world situations, inspired by examples of this nature. Following the inception of N-SSs, researchers have dedicated considerable efforts to exploring and extending this influential concept. Shabir and Fatima [37] introduced N-bipolar soft sets (N-BSSs) and delved into their applications in decision-making. Akram et al. [2] presented a novel hybrid model called fuzzy N-SSs, integrating fuzzy set theory with N-SSs to address uncertainties in the attribute-based grading of alternatives. In this model, the assumption is made that membership degree and grade are directly proportional. However, Korkmaz et al. [19] proposed an alternative approach to fuzzy N-SSs, especially in cases where membership degree and grade are not inherently related. They demonstrated that the assumption used in [2] may not hold true in real-world situations. The literature contains numerous studies exploring fuzzy N-SSs from various perspectives [3–5, 22, 23]. Recently, Musa et al. [31] delved into N-hypersoft sets and developed decision-making algorithms for them. Musa [27] extended this exploration by introducing N-bipolar hypersoft sets, building upon the concept of bipolar hypersoft sets [28].

However, the existing models, including those mentioned, face limitations when dealing with alternatives and attributes, particularly in situations where decision-makers need to assign non-binary evaluations to both. In response to these challenges, Riaz et al. [35] introduced the concept of MPNSS, aiming to address these limitations by allowing independent non-binary evaluations for both attributes and alternatives. Subsequently, Ayesha and Riaz [34] further advanced the research by developing multi-attribute decision-making techniques based on MPNSSs, MPNS aggregation operators, and MPNS weighted aggregation operators. Building upon this foundation, the aim of our work is to extend the MPNSS framework by incorporating the notion of bipolarity. Recognizing the dual-sided nature of human decision-making, our proposed BMPNSS framework seeks to provide decision-makers with a more nuanced tool for expressing preferences. By integrating bipolarity into the evaluation process, BMPNSS aims to capture the complex interplay between positive and negative considerations, thereby enabling decision-makers to make more informed and balanced choices in scenarios where non-binary evaluations are required.

This simple organization is meant to help readers follow along easily and understand the BMPNSS framework and how it fits in with other decision-making models. Starting with Section 2, we review N-SS, N-BSS, and MPNSS, breaking down their definitions and results. Then, in Section 3, we dive into the extended MPNSS, known as BMPNSS, introducing the operators that come with it. Set-theoretic operations and their properties within the context of BMPNSS are discussed in Section 4. Moving forward to Section 5, we explain decision-making approaches tailored for BMPNSS. In Section 6, we compare

BMPNSS with other existing models to highlight its unique features. Finally, we bring the discussion to a close in Section 7.

2. Preliminaries

In this section, we provide an overview of definitions related to N-SS, N-BSS, and MPNSS. Throughout this work, let Ω denote the universal set of alternatives, E represent attributes, and $Q \subseteq E$. Let $R = \{0, 1, \ldots, N-1\}$ and $S = \{0, 1, \ldots, M-1\}$ be sets of ordered grades, where $M, N \in \{2, 3, \ldots\}$.

Definition 2.1 ([26]). A soft set over Ω, denoted by $(\mathfrak{f} : Q)$, is defined as a mapping $\mathfrak{f} : Q \to 2^{\Omega}$, where 2^{Ω} denotes the power set of Ω.

Definition 2.2 ([16]). We call $(\mathfrak{f}: Q, N)$ an N-SS over Ω, if, $\mathfrak{f} : Q \to 2^{\Omega\times R}$, with the property that for each $q \in Q$, there exists a unique $(\omega, r_q) \in \Omega \times R$ such that $(\omega, r_q) \in \mathfrak{f}(q)$, where $\omega \in \Omega$ and $r_q \in R$. Here, $2^{\Omega\times R}$ denotes the power set of $\Omega \times R$.

Definition 2.3 ([37]). We call $(\mathfrak{f}\mathfrak{g}: Q, N)$ an N-BSS over Ω, if, $\mathfrak{f} : Q \to 2^{\Omega\times R}$ and $\mathfrak{g} : \neg Q \to 2^{\Omega\times R}$, with the property that for each $q \in Q$, there exists a unique (ω, r_q), $(\omega, r_{\neg q}) \in \Omega \times R$ such that $(\omega, r_q) \in \mathfrak{f}(q)$ and $(\omega, r_{\neg q}) \in \mathfrak{g}(\neg q)$, subject to the condition $r_q + r_{\neg q} \leqslant N - 1$, where $\omega \in \Omega$ and r_q, $r_{\neg q} \in R$. Here, $\neg Q$ denotes the NOT set of Q: $\neg Q = \{\neg q : q \in Q\}$.

Now, the remaining definitions of this section are due to [34, 35] and suggest expressing MPNSS $(\mathfrak{f}\mathfrak{h}: \mathcal{P}_Q, M, N)$ in functional representation.

Definition 2.4. We call $(\mathfrak{f}\mathfrak{h}: \mathcal{P}_Q, M, N)$ an MPNSS over Ω, if, $\mathfrak{f} : \mathcal{P} \to 2^{\Omega\times R}$, where $\mathcal{P} = \{(q, \mathfrak{h}(q)) : q \in Q\}$ and $\mathfrak{h} : Q \to S$, with the property that for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$, there exists a unique $(\omega, r_{(q,\mathfrak{h}(q))}) \in \Omega \times R$ such that $(\omega, r_{(q,\mathfrak{h}(q))}) \in \mathfrak{f}((q, \mathfrak{h}(q)))$, where $\omega \in \Omega$ and $r_{(q,\mathfrak{h}(q))} \in R$.

Definition 2.5. The weak complement of $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is any MPNSS $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)^{\varpi} = ((\mathfrak{f}\mathfrak{h})^{\varpi} : \mathcal{P}_Q, M, N)$, where for each $q \in Q$, $\mathfrak{h}^{\varpi}(q) \cap \mathfrak{h}(q) = \emptyset$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}^{\varpi}((q, \mathfrak{h}(q)))(\omega) \cap \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = \emptyset$.

Definition 2.6. The top weak complement of $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)^t = ((\mathfrak{f}\mathfrak{h})^t : \mathcal{P}_Q, M, N)$, where for each $q \in Q$:

$$\mathfrak{h}^t(q) = \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$:

$$\mathfrak{f}^t((q, \mathfrak{h}(q)))(\omega) = \begin{cases} N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1. \end{cases}$$

Definition 2.7. The bottom weak complement of $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)^b = ((\mathfrak{f}\mathfrak{h})^b : \mathcal{P}_Q, M, N)$, for each $q \in Q$:

$$\mathfrak{h}^b(q) = \begin{cases} 0, & \text{if } \mathfrak{h}(q) > 0, \\ N-1, & \text{if } \mathfrak{h}(q) = 0, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$:

$$\mathfrak{f}^b((q, \mathfrak{h}(q)))(\omega) = \begin{cases} 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) > 0, \\ N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0. \end{cases}$$

Definition 2.8. An MPNSS $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is termed a relative null, denoted by Λ^0_0, if, for each $q \in Q$, $\mathfrak{h}(q) = 0$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0$.

Definition 2.9. An MPNSS $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is termed a relative whole, expressed as Λ_{N-1}^{M-1}, if, for each $q \in Q$, $\mathfrak{h}(q) = M-1$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1$.

Definition 2.10. An MPNSS $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is considered a subset of $(\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$, denoted by $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N) \sqsubseteq (\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$, if the following conditions are satisfied:

1. $\mathfrak{h}(q) \leqslant \mathfrak{h}_1(q)$ for each $q \in Q \subseteq Q_1$;
2. $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) \leqslant \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)$ for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$.

Definition 2.11. Two MPNSSs $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$ over Ω are considered equal if $\mathfrak{f} = \mathfrak{f}_1$, $\mathfrak{h} = \mathfrak{h}_1$, $\mathcal{P} = \mathcal{P}_1$, and $Q = Q_1$.

Definition 2.12. The extended union of MPNSSs $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is represented as $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N) \sqcup_\varepsilon (\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{h}_2 : (\mathcal{P} \cup \mathcal{P}_1)_{(Q \cup Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cup Q_1$:

$$\mathfrak{h}_2(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus Q_1, \\ \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q \cap Q_1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1. \end{cases}$$

Definition 2.13. The extended intersection of $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N) \sqcap_\varepsilon (\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{h}_2 : (\mathcal{P} \cup \mathcal{P}_1)_{(Q \cup Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cup Q_1$:

$$\mathfrak{h}_2(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus Q_1, \\ \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q, \\ \min\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q \cap Q_1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ \min\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1. \end{cases}$$

Definition 2.14. The restricted union of $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as $(\mathfrak{f}\mathfrak{h} : \mathcal{P}_Q, M, N) \sqcup_\mathfrak{R} (\mathfrak{f}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{h}_2 : (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cap Q_1$:

$$\mathfrak{h}_2(q) = \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}.$$

Definition 2.15. The restricted intersection of $(\mathfrak{f}\mathfrak{h}: \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{h}_1: \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as $(\mathfrak{f}\mathfrak{h}: \mathcal{P}_Q, M, N) \sqcap_\mathfrak{R} (\mathfrak{f}_1\mathfrak{h}_1: \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{h}_2: (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cap Q_1$:

$$\mathfrak{h}_2(q) = \min\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \min\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}.$$

3. Bipolar M-parameterized N-soft sets

In this section, we examine the groundbreaking notion of BMPNSSs and introduce a range of associated definitions.

Definition 3.1. We define $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ as a BMPNSS over Ω, if, $\mathfrak{f} : \mathcal{P} \to 2^{\Omega\times R}$ and $\mathfrak{g} : \neg\mathcal{P} \to 2^{\Omega\times R}$ where $\mathcal{P} = \{(q, \mathfrak{h}(q)) : q \in Q\}$ and $\mathfrak{h} : Q \to S$, with the property that for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$, there exists a unique $(\omega, r_{(q,\mathfrak{h}(q))})$, $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \Omega \times R$ such that $(\omega, r_{(q,\mathfrak{h}(q))}) \in \mathfrak{f}((q, \mathfrak{h}(q)))$ and $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))$, subject to the condition $r_{(q,\mathfrak{h}(q))} + r_{(\neg q,\mathfrak{h}(\neg q))} \leqslant N - 1$, where $\omega \in \Omega$ and $r_{(q,\mathfrak{h}(q))}$, $r_{(\neg q,\mathfrak{h}(\neg q))} \in R$.

For each graded attribute $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and alternative $\omega \in \Omega$, there is a unique evaluation in the assessment space R, denoted by $r_{(q,\mathfrak{h}(q))}$ and $r_{(\neg q,\mathfrak{h}(\neg q))}$, where $(\omega, r_{(q,\mathfrak{h}(q))}), (\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \Omega \times R$ such that $(\omega, r_{(q,\mathfrak{h}(q))}) \in \mathfrak{f}((q, \mathfrak{h}(q)))$ and $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))$. To simplify notation, we use $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = r_{(q,\mathfrak{h}(q))}$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = r_{(\neg q,\mathfrak{h}(\neg q))}$ as shorthand for $(\omega, r_{(q,\mathfrak{h}(q))}) \in \mathfrak{f}((q, \mathfrak{h}(q)))$ and $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))$, respectively. It is assumed that both $\Omega = \{\omega_i, i = 1, 2, \ldots, m\}$ and $Q = \{q_j, j = 1, 2, \ldots, n\}$ are finite, unless stated otherwise. In this scenario, the BMPNSS can be tabularly represented, where (r^+_{ij}, r^-_{ij}) denotes $(\omega_i, r^+_{ij}) \in \mathfrak{f}((q_j, \mathfrak{h}(q_j)))$ or $\mathfrak{f}((q_j, \mathfrak{h}(q_j)))(\omega_i) = r^+_{ij}$, and $(\omega_i, r^-_{ij}) \in \mathfrak{g}((\neg q_j, \mathfrak{h}(\neg q_j)))$ or $\mathfrak{g}((\neg q_j, \mathfrak{h}(\neg q_j)))(\omega_i) = r^-_{ij}$. This representation is illustrated in Table 1.

The BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ can be expressed in this simplified form: $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) = \big\{\big\langle (q, \mathfrak{h}(q)), \{\omega, \mathfrak{f}((q, \mathfrak{h}(q))(\omega), \mathfrak{g}((\neg q, \mathfrak{h}(\neg q))(\omega)\}\big\rangle : (q, \mathfrak{h}(q)) \in \mathcal{P}, \omega \in \Omega,$ and $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) \in R\big\}$.

Table 1: Tabular form of BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$.

$(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$	$(q_1, \mathfrak{h}(q_1))$	$(q_2, \mathfrak{h}(q_2))$	$\cdots$	$(q_n, \mathfrak{h}(q_n))$
ω_1	(r^+_{11}, r^-_{11})	(r^+_{12}, r^-_{12})	$\cdots$	(r^+_{1n}, r^-_{1n})
ω_2	(r^+_{21}, r^-_{21})	(r^+_{22}, r^-_{22})	$\cdots$	(r^+_{2n}, r^-_{2n})
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
ω_m	(r^+_{m1}, r^-_{m1})	(r^+_{m2}, r^-_{m2})	$\cdots$	(r^+_{mn}, r^-_{mn})

Remark 3.2. The grade $0 \in R$ and $0 \in S$, as defined in Definition 3.1, should not be interpreted as indicating incompleteness or lack of information. Instead, it signifies the lowest grade within the ordered set of grades.

Based on this observation, we propose the following definition.

Definition 3.3. We call $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ an incomplete BMPNSS over Ω if $\mathfrak{f} : \mathcal{P} \to 2^{\Omega\times R}$ and $\mathfrak{g} : \neg\mathcal{P} \to 2^{\Omega\times R}$, where $\mathcal{P} = \{(q, \mathfrak{h}(q)) : q \in Q\}$ and $\mathfrak{h} : Q \to S$. The property holds that for each $q \in Q$, there exists at most $\mathfrak{h}(q) \in S$, or for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$, there exists at most $(\omega, r_{(q,\mathfrak{h}(q))})$, $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \Omega \times R$ such that $(\omega, r_{(q,\mathfrak{h}(q))}) \in \mathfrak{f}((q, \mathfrak{h}(q)))$ and $(\omega, r_{(\neg q,\mathfrak{h}(\neg q))}) \in \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))$, subject to the condition $r_{(q,\mathfrak{h}(q))} + r_{(\neg q,\mathfrak{h}(\neg q))} \leqslant N - 1$, where $\omega \in \Omega$ and $r_{(q,\mathfrak{h}(q))}$, $r_{(\neg q,\mathfrak{h}(\neg q))} \in R$.

Remark 3.4. Any BMPNSS can be straightforwardly seen as a B$\acute{M}$P$\acute{N}$SS, where $\acute{M} > M$ and $\acute{N} > N$.

Following this observation, we introduce the subsequent definitions.

Definition 3.5. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is deemed positively efficient if there exists $q \in Q$ such that $\mathfrak{h}(q) = M - 1$ and there exists $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$ such that $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N - 1$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0$.

Definition 3.6. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is considered negatively efficient if there exists $q \in Q$ such that $\mathfrak{h}(q) = M - 1$ and there exists $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$ such that $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = N - 1$.

Definition 3.7. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is classified as totally efficient if there exists $q \in Q$ such that $\mathfrak{h}(q) = M-1$ and there exists $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$ such that $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0$, as well as $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = N-1$.

Definition 3.8. The complement of $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is denoted and defined as $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)^{\tilde{c}} = ((\mathfrak{f}\mathfrak{g}\mathfrak{h})^{\tilde{c}} : \mathcal{P}_Q, M, N)$, where for each $q \in Q$, $\mathfrak{h}^{\tilde{c}}(q) = (N-1) \setminus \mathfrak{h}(q)$ and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}^{\tilde{c}}((q, \mathfrak{h}(q)))(\omega) = \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega)$, and $\mathfrak{g}^{\tilde{c}}((\neg q, \mathfrak{h}(\neg q)))(\omega) = \mathfrak{f}((q, \mathfrak{h}(q)))(\omega)$.

Definition 3.9. The weak complement of $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is any BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)^{\tilde{\varpi}} = ((\mathfrak{f}\mathfrak{g}\mathfrak{h})^{\tilde{\varpi}} : \mathcal{P}_Q, M, N)$, where for each $q \in Q$, $\mathfrak{h}^{\tilde{\varpi}}(q) \cap \mathfrak{h}(q) = \emptyset$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}^{\tilde{\varpi}}((q, \mathfrak{h}(q)))(\omega) \cap \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = \emptyset$ and $\mathfrak{g}^{\tilde{\varpi}}((\neg q, \mathfrak{h}(\neg q)))(\omega) \cap \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = \emptyset$.

Definition 3.10. The top weak complement of $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)^{\tilde{t}} = ((\mathfrak{f}\mathfrak{g}\mathfrak{h})^{\tilde{t}} : \mathcal{P}_Q, M, N)$, where for each $q \in Q$:

$$\mathfrak{h}^{\tilde{t}}(q) = \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$:

$$\mathfrak{f}^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega) = \begin{cases} N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1, \end{cases}$$

$$\mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \\ N-1, & \text{otherwise.} \end{cases}$$

Definition 3.11. The bottom weak complement of $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)^{\tilde{b}} = ((\mathfrak{f}\mathfrak{g}\mathfrak{h})^{\tilde{b}} : \mathcal{P}_Q, M, N)$, for each $q \in Q$:

$$\mathfrak{h}^{\tilde{b}}(q) = \begin{cases} 0, & \text{if } \mathfrak{h}(q) > 0, \\ N-1, & \text{if } \mathfrak{h}(q) = 0, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$:

$$\mathfrak{f}^{\tilde{b}}((q, \mathfrak{h}(q)))(\omega) = \begin{cases} 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) > 0, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0 \text{ and } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) < N-1, \\ N-1, & \text{otherwise,} \end{cases}$$

$$\mathfrak{g}^{\tilde{b}}((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} N-1, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) < N-1, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = N-1. \end{cases}$$

4. Set-theoretic operations on bipolar M-parameterized N-soft sets and their properties

In this section, we shift our focus to the analysis of set-theoretic operations within the framework of BMPNSSs. We begin by introducing these operations and subsequently explore their essential properties.

Definition 4.1. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is termed a relative null, denoted by $\Lambda^0_{(0,N-1)}$, if, for each $q \in Q$, $\mathfrak{h}(q) = 0$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = 0$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = N-1$.

Definition 4.2. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is termed a relative whole, expressed as $\Lambda^{M-1}_{(N-1,0)}$, if, for each $q \in Q$, $\mathfrak{h}(q) = M-1$, and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$, $\mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1$ and $\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0$.

Definition 4.3. A BMPNSS $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N)$ is considered a subset of $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$, denoted by $(\mathfrak{f}\mathfrak{g}\mathfrak{h} : \mathcal{P}_Q, M, N) \,\tilde{\sqsubseteq}\, (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$, if the following conditions are satisfied:

1. $\mathfrak{h}(q) \leqslant \mathfrak{h}_1(q)$ for each $q \in Q \subseteq Q_1$;

2. $\mathfrak{f}((q,\mathfrak{h}(q)))(\omega) \leqslant \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)$ and $\mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) \leqslant \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega)$ for each $(q,\mathfrak{h}(q)) \in \mathcal{P}$ and $\omega \in \Omega$.

Definition 4.4. Two BMPNSSs $(\mathfrak{fgh} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M, N)$ over Ω are considered equal if $\mathfrak{f} = \mathfrak{f}_1$, $\mathfrak{g} = \mathfrak{g}_1$, $\mathfrak{h} = \mathfrak{h}_1$, $\mathcal{P} = \mathcal{P}_1$, and $Q = Q_1$.

Definition 4.5. The extended union of BMPNSSs $(\mathfrak{fgh} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is represented as

$$(\mathfrak{fgh} : \mathcal{P}_Q, M, N)\widetilde{\sqcup}_{\varepsilon}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2 : (\mathcal{P}\cup\mathcal{P}_1)_{(Q\,\cup\,Q_1)}, \max(M, M_1), \max(N, N_1)),$$

where for each $q \in Q \cup Q_1$:

$$\mathfrak{h}_2(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus Q_1, \\ \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q \cap Q_1, \end{cases}$$

and for each $(q,\mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}\setminus\mathcal{P}_1, \\ \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_1\setminus\mathcal{P}, \\ \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}\cap\mathcal{P}_1, \end{cases}$$

$$\mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1. \end{cases}$$

Definition 4.6. The extended intersection of $(\mathfrak{fgh} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as

$$(\mathfrak{fgh} : \mathcal{P}_Q, M, N)\widetilde{\sqcap}_{\varepsilon}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2 : (\mathcal{P}\cup\mathcal{P}_1)_{(Q\,\cup\,Q_1)}, \max(M, M_1), \max(N, N_1)),$$

where for each $q \in Q \cup Q_1$:

$$\mathfrak{h}_2(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus Q_1, \\ \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q, \\ \min\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q \cap Q_1, \end{cases}$$

and for each $(q,\mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}\setminus\mathcal{P}_1, \\ \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_1\setminus\mathcal{P}, \\ \min\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}\cap\mathcal{P}_1, \end{cases}$$

$$\mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ \max\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1. \end{cases}$$

Definition 4.7. The restricted union of $(\mathfrak{fgh} : \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as $(\mathfrak{fgh} : \mathcal{P}_Q, M, N)\ \widetilde{\sqcup}_{\mathfrak{R}}\ (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1 : \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2 : (\mathcal{P}\cap\mathcal{P}_1)_{(Q\cap Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cap Q_1$:

$$\mathfrak{h}_2(q) = \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q,\mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega) = \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\},$$
$$\mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega) = \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}.$$

Definition 4.8. The restricted intersection of $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)$ is denoted and defined as $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \widetilde{\sqcap}_{\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P}\cap\mathcal{P}_1)_{(Q\cap Q_1)}, \max(M, M_1), \max(N, N_1))$, where for each $q \in Q \cap Q_1$:

$$\mathfrak{h}_2(q) = \min\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \min\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\},$$
$$\mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega) = \max\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega)\}.$$

Example 4.9. Consider the set $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Let $\mathcal{P} = \{(q_1, 3), (q_2, 0), (q_3, 1)\}$ and $\mathcal{P}_1 = \{(q_1, 1), (q_2, 0), (q_4, 4)\}$. Take $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ as B4P5SS and B5P4SS, respectively, as presented in Tables 2 and 3. The extended union (intersection) and the restricted union (intersection) are detailed in Tables 4-7.

Table 2: The B4P5SS $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5)$ in Example 4.9.

$(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5)$	$(q_1, 3)$	$(q_2, 0)$	$(q_3, 1)$
ω_1	(4,0)	(2,1)	(2,1)
ω_2	(3,1)	(0,0)	(1,3)
ω_3	(1,1)	(4,0)	(0,4)

Table 3: The B5P4SS $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ in Example 4.9.

$(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$	$(q_1, 1)$	$(q_2, 0)$	$(q_4, 4)$
ω_1	(3,0)	(1,2)	(1,1)
ω_2	(1,2)	(1,2)	(3,0)
ω_3	(0,0)	(2,1)	(0,2)

Table 4: The extended union of $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ in Example 4.9.

$(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5) \widetilde{\sqcup}_{\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$	$(q_1, 3)$	$(q_2, 0)$	$(q_3, 1)$	$(q_4, 4)$
ω_1	(4,0)	(2,1)	(2,1)	(1,1)
ω_2	(3,1)	(1,0)	(1,3)	(3,0)
ω_3	(1,1)	(4,0)	(0,4)	(0,2)

Table 5: The extended intersection of $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ in Example 4.9.

$(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, 4, 5) \widetilde{\sqcap}_{\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$	$(q_1, 1)$	$(q_2, 0)$	$(q_3, 1)$	$(q_4, 4)$
ω_1	(3,0)	(1,2)	(2,1)	(1,1)
ω_2	(1,2)	(0,2)	(1,3)	(3,0)
ω_3	(0,0)	(2,1)	(0,4)	(0,2)

Table 6: The restricted union of $(\mathfrak{fgh}\colon \mathcal{P}_Q, 4, 5)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ in Example 4.9.

$(\mathfrak{fgh}\colon \mathcal{P}_Q, 4, 5) \tilde{\sqcup}_{\mathfrak{R}} (\mathfrak{f}_1, \mathfrak{g}_1, Q_1, 7)$	$(q_2, 0)$
ω_1	(2,1)
ω_2	(1,0)
ω_3	(4,0)

Table 7: The restricted intersection of $(\mathfrak{fgh}\colon \mathcal{P}_Q, 4, 5)$ and $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, 5, 4)$ in Example 4.9.

$(\mathfrak{f}, \mathfrak{g}, Q, 6) \tilde{\sqcap}_{\mathfrak{R}} (\mathfrak{f}_1, \mathfrak{g}_1, Q_1, 7)$	$(q_2, 0)$
ω_1	(1,2)
ω_2	(0,2)
ω_3	(2,1)

Proposition 4.10. *Let* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$, $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$, *and* $(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_Q, M, N)$ *be three BMPNSSs. Then,*

1. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} \Lambda^{M-1}_{(N-1,0)}$;
2. $\Lambda^{0}_{(0,N-1)} \tilde{\sqsubseteq} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$;
3. *if* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_Q, M, N)$, *then* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_Q, M, N)$.

Proof. Straightforward. □

Proposition 4.11. *Let* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$ *be two BMPNSSs. Then,*

1. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcup}_{\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$ *is the smallest BMPNSS that contains both* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$;
2. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcap}_{\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$ *is the largest BMPNSS that is contained in both* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$.

Proof. Straightforward. □

Proposition 4.12. *Let* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$ *be two BMPNSSs. Then,*

1. $(\Lambda^{0}_{(0,N-1)})^{\alpha} = \Lambda^{M-1}_{(N-1,0)}$, *where* $\alpha \in \{\tilde{\mathfrak{c}}, \tilde{\mathfrak{t}}, \tilde{\mathfrak{b}}\}$;
2. $(\Lambda^{M-1}_{(N-1,0)})^{\alpha} = \Lambda^{0}_{(0,N-1)}$, *where* $\alpha \in \{\tilde{\mathfrak{c}}, \tilde{\mathfrak{t}}, \tilde{\mathfrak{b}}\}$;
3. *if* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$, *then* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)^{\alpha} \tilde{\sqsubseteq} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\alpha}$, *where* $\alpha \in \{\tilde{\mathfrak{c}}, \tilde{\mathfrak{t}}, \tilde{\mathfrak{b}}\}$;
4. $((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{c}}})^{\tilde{\mathfrak{c}}} = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$, $((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{t}}})^{\tilde{\mathfrak{t}}} \tilde{\sqsubseteq} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$, *and* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} ((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{b}}})^{\tilde{\mathfrak{b}}}$;
5. $\Lambda^{0}_{(0,N-1)} \tilde{\sqsubseteq} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcap}_{\mathfrak{R}} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{c}}} \tilde{\sqsubseteq} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcup}_{\mathfrak{R}} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{c}}} \tilde{\sqsubseteq} \Lambda^{M-1}_{(N-1,0)}$;
6. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcap}_{\mathfrak{R}} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{b}}} = \Lambda^{0}_{(0,N-1)}$;
7. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcup}_{\mathfrak{R}} (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{t}}} = \Lambda^{M-1}_{(N-1,0)}$;
8. *if* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$, *then* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcap}_{\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N) = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$;
9. *if* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqsubseteq} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$, *then* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \tilde{\sqcup}_{\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N) = (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$.

Proof. Straightforward. □

Proposition 4.13. *Let* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)$ *be two BMPNSSs and* $\alpha \in \{\tilde{\mathfrak{c}}, \tilde{\mathfrak{t}}, \tilde{\mathfrak{b}}\}$*. Then,*

1. $\left((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)\right)^\alpha = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^\alpha \mathbin{\widetilde{\sqcap}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)^\alpha$;
2. $\left((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)\right)^\alpha = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^\alpha \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)^\alpha$;
3. $\left((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)\right)^\alpha = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^\alpha \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)^\alpha$;
4. $\left((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)\right)^\alpha = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^\alpha \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)^\alpha$.

Proof.

(1) Here, we prove the case where $\alpha = \tilde{\mathfrak{t}}$. Suppose that $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P}\cup\mathcal{P}_1)_{(Q\,\cup\,Q_1)}, M, N)$. Then, for each $q \in Q \cup Q_1$:

$$\mathfrak{h}_2(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus Q_1, \\ \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q \cap Q_1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \end{cases}$$

$$\mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1. \end{cases}$$

Now, $\left((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1_{Q_1}}, M, N)\right)^{\tilde{\mathfrak{t}}} = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P}\cup\mathcal{P}_1)_{(Q\,\cup\,Q_1)}, M, N)^{\tilde{\mathfrak{t}}}$. Then, for all $q \in Q \cup Q_1$:

$$\mathfrak{h}_2^{\tilde{\mathfrak{t}}}(q) = \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } q \in Q \setminus Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } q \in Q \setminus Q_1, \\ N-1, & \text{if } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q_1 \setminus Q, \\ 0, & \text{if } \mathfrak{h}_1(q) = N-1 \text{ and } q \in Q_1 \setminus Q, \\ N-1, & \text{if } \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\} < N-1 \text{ and } q \in Q \cap Q_1, \\ 0, & \text{if } \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\} = N-1 \text{ and } q \in Q \cap Q_1, \end{cases}$$

$$= \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } q \in Q \setminus Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } q \in Q \setminus Q_1, \\ N-1, & \text{if } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q_1 \setminus Q, \\ 0, & \text{if } \mathfrak{h}_1(q) = N-1 \text{ and } q \in Q_1 \setminus Q, \\ N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q \cap Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } \mathfrak{h}_1(q) = N-1, \text{ or } \mathfrak{h}(q) < N-1 \text{ and } \mathfrak{h}_1(q) = N-1, \\ & \quad \text{or } \mathfrak{h}(q) = N-1 \text{ and } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q \cap Q_1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2^{\tilde{\mathfrak{t}}}((q,\mathfrak{h}(q)))(\omega)$$
$$= \begin{cases} N-1, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ 0, & \text{if } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ N-1, & \text{if } \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\} < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \\ 0, & \text{if } \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\} = N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \end{cases}$$

$$= \begin{cases} N-1, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}\setminus\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}\setminus\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}_1\setminus\mathcal{P}, \\ 0, & \text{if } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}_1\setminus\mathcal{P}, \\ N-1, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}\cap\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) = N-1, \\ & \quad \text{or } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) = N-1, \\ & \quad \text{or } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) = N-1 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q,\mathfrak{h}(q)) \in \mathcal{P}\cap\mathcal{P}_1, \end{cases}$$

$$\mathfrak{g}_2^{\tilde{\mathfrak{t}}}((\neg q,\mathfrak{h}(\neg q)))(\omega)$$

$$= \begin{cases} 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}(q)(\omega) < N-1, \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1(q)(\omega) < N-1, \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ N-1, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ 0, & \text{if } \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\} > 0, \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1, \\ 0, & \text{if } \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\} = 0, \\ & \quad \text{and } \max\{\mathfrak{f}(q)(\omega), \mathfrak{f}_1(q)(\omega)\} < N-1 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1, \\ N-1, & \text{if } \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\} = 0, \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1, \end{cases}$$

$$= \begin{cases} 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\setminus\neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ N-1, & \text{if } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1\setminus\neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ when } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{when } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}\cap\neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{or } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ or } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{or } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q,\mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega) < N-1. \end{cases}$$

On the other hand, let $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)^{\tilde{\mathfrak{t}}} \mathbin{\widetilde{\sqcap}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)^{\tilde{\mathfrak{t}}} = (\mathfrak{f}_3\mathfrak{g}_3\mathfrak{h}_3\colon (\mathcal{P}\cup\mathcal{P}_1)_{(Q\,\cup\, Q_1)}, M, N)$, then for all $q \in Q\cup Q_1$:

$$\mathfrak{h}_3(q) = \begin{cases} \mathfrak{h}^{\tilde{\mathfrak{t}}}(q), & \text{if } q \in Q\setminus Q_1, \\ \mathfrak{h}_1^{\tilde{\mathfrak{t}}}(q), & \text{if } q \in Q_1\setminus Q, \\ \min\{\mathfrak{h}^{\tilde{\mathfrak{t}}}(q), \mathfrak{h}_1^{\tilde{\mathfrak{t}}}(q)\}, & \text{if } q \in Q\cap Q_1. \end{cases}$$

Hence,

$$\mathfrak{h}_3(q) = \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } q \in Q\setminus Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } q \in Q\setminus Q_1, \\ N-1, & \text{if } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q_1\setminus Q, \\ 0, & \text{if } \mathfrak{h}_1(q) = N-1 \text{ and } q \in Q_1\setminus Q, \\ N-1, & \text{if } \min\{\mathfrak{h}^{\tilde{\mathfrak{t}}}(q), \mathfrak{h}_1^{\tilde{\mathfrak{t}}}(q)\} < N-1 \text{ and } q \in Q\cap Q_1, \\ 0, & \text{if } \min\{\mathfrak{h}^{\tilde{\mathfrak{t}}}(q), \mathfrak{h}_1^{\tilde{\mathfrak{t}}}(q)\} = N-1 \text{ and } q \in Q\cap Q_1, \end{cases}$$

$$= \begin{cases} N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } q \in Q \setminus Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } q \in Q \setminus Q_1, \\ N-1, & \text{if } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q_1 \setminus Q, \\ 0, & \text{if } \mathfrak{h}_1(q) = N-1 \text{ and } q \in Q_1 \setminus Q, \\ N-1, & \text{if } \mathfrak{h}(q) < N-1 \text{ and } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q \cap Q_1, \\ 0, & \text{if } \mathfrak{h}(q) = N-1 \text{ and } \mathfrak{h}_1(q) = N-1, \text{ or } \mathfrak{h}(q) < N-1 \text{ and } \mathfrak{h}_1(q) = N-1, \\ & \quad \text{or } \mathfrak{h}(q) = N-1 \text{ and } \mathfrak{h}_1(q) < N-1 \text{ and } q \in Q \cap Q_1, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_3((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ \mathfrak{f}_1^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ \min\{\mathfrak{f}^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \end{cases}$$

$$\mathfrak{g}_3((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ \mathfrak{g}_1^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ \max\{\mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1. \end{cases}$$

Hence,

$$\begin{aligned} &\mathfrak{f}_3((q, \mathfrak{h}(q)))(\omega) \\ &= \begin{cases} N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ 0, & \text{if } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ N-1, & \text{if } \min\{\mathfrak{f}^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega)\} < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \\ 0, & \text{if } \min\{\mathfrak{f}^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1^{\tilde{t}}((q, \mathfrak{h}(q)))(\omega)\} = N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \end{cases} \\ &= \begin{cases} N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus \mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ 0, & \text{if } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}, \\ N-1, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \\ 0, & \text{if } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) = N-1, \\ & \quad \text{or } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) = N-1, \\ & \quad \text{or } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) = N-1 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1, \end{cases} \end{aligned}$$

$$\begin{aligned} &\mathfrak{g}_3((\neg q, \mathfrak{h}(\neg q)))(\omega) \\ &= \begin{cases} 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1, \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ N-1, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ 0, & \text{if } \max\{\mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega)\} > 0, \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1, \\ 0, & \text{if } \max\{\mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega)\} = 0, \\ & \text{and } \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\} < N-1, \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1, \\ N-1, & \text{if } \max\{\mathfrak{g}^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1^{\tilde{t}}((\neg q, \mathfrak{h}(\neg q)))(\omega)\} = 0, \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1, \end{cases} \end{aligned}$$

$$= \begin{cases} 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus \neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ N-1, & \text{if } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1, \\ 0, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) > 0 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ when } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{when } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1 \text{ and } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap \neg\mathcal{P}_1, \\ N-1, & \text{if } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{or } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ or } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0, \\ & \quad \text{or } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}((q, \mathfrak{h}(q)))(\omega) < N-1, \\ & \quad \text{or } \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega) = 0 \text{ and } \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega) < N-1. \end{cases}$$

Since $(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P} \cup \mathcal{P}_1)_{(Q \cup Q_1)}, M, N)^{\tilde{\mathfrak{t}}}$ and $(\mathfrak{f}_3\mathfrak{g}_3\mathfrak{h}_3\colon (\mathcal{P} \cup \mathcal{P}_1)_{(Q \cup Q_1)}, M, N)$ are equivalent for all $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup \mathcal{P}_1$ and $\omega \in \Omega$, we have completed the proof. The remaining parts can be proven in a similar manner. □

Proposition 4.14. *Let* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$ *be two BMPNSSs. Then,*

1. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$;
2. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_Q, M, N)$;
3. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$;
4. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} \Lambda^0_{(0,N-1)} = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} \Lambda^0_{(0,N-1)} = \Lambda^0_{(0,N-1)}$;
5. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} \Lambda^{M-1}_{(N-1,0)} = \Lambda^{M-1}_{(N-1,0)}$ *and* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} \Lambda^{M-1}_{(N-1,0)} = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$.

Proof. Straightforward. □

Proposition 4.15. *Let* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$, $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)$, *and* $(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)$ *represent a BMPNSS, a* BM_1PN_1SS*, and a* BM_2PN_2SS*, respectively. Let* $\oplus \in \{\widetilde{\sqcup}_\varepsilon, \widetilde{\sqcap}_\varepsilon, \widetilde{\sqcup}_{\mathfrak{R}}, \widetilde{\sqcap}_{\mathfrak{R}}\}$*. Then,*

1. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \oplus (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1) \oplus (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$;
2. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \oplus \big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1) \oplus (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \oplus, (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big) \oplus (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)$.

Proof. Straightforward. □

Proposition 4.16. *Let* $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$ *and* $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)$ *be two BMPNSSs. Then,*

1. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} \big((\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)\big) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$;
2. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_\varepsilon} \big((\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)\big) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$;
3. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_{\mathfrak{R}}} \big((\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)\big) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$;
4. $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} \big((\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)\big) = (\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)$.

Proof.

(1) Suppose that $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N) \mathbin{\widetilde{\sqcap}_{\mathfrak{R}}} (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N) = (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, M, N)$. Then, for all $q \in Q \cap Q_1$:

$$\mathfrak{h}_2(q) = \min\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega) = \min\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\},$$
$$\mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega) = \max\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega)\}.$$

Now, let $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \,\widetilde{\sqcup}_\varepsilon\, (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, M, N) = (\mathfrak{f}_3\mathfrak{g}_3\mathfrak{h}_3\colon \mathcal{P} \cup (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cup (Q \cap Q_1))}, M, N)$, then for each $q \in Q \cup (Q \cap Q_1)$:

$$\mathfrak{h}_3(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus (Q \cap Q_1), \\ \mathfrak{h}_2(q), & \text{if } q \in (Q \cap Q_1) \setminus Q = \emptyset, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_2(q)\}, & \text{if } q \in Q \cap (Q \cap Q_1), \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup (\mathcal{P} \cap \mathcal{P}_1)$ and $\omega \in \Omega$:

$$\mathfrak{f}_3((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus (\mathcal{P} \cap \mathcal{P}_1), \\ \mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in (\mathcal{P} \cap \mathcal{P}_1) \setminus \mathcal{P} = \emptyset, \\ \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap (\mathcal{P} \cap \mathcal{P}_1), \end{cases}$$
$$\mathfrak{g}_3((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus (\neg\mathcal{P} \cap \neg\mathcal{P}_1), \\ \mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in (\neg\mathcal{P} \cap \neg\mathcal{P}_1) \setminus \neg\mathcal{P} = \emptyset, \\ \min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap (\neg\mathcal{P} \cap \neg\mathcal{P}_1). \end{cases}$$

Hence,

$$\mathfrak{h}_3(q) = \begin{cases} \mathfrak{h}(q), & \text{if } q \in Q \setminus (Q \cap Q_1), \\ \mathfrak{h}(q), & \text{if } q \in Q \cap (Q \cap Q_1), \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cup (\mathcal{P} \cap \mathcal{P}_1)$ and $\omega \in \Omega$:

$$\mathfrak{f}_3((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \setminus (\mathcal{P} \cap \mathcal{P}_1), \\ \mathfrak{f}((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P} \cap (\mathcal{P} \cap \mathcal{P}_1), \end{cases}$$
$$\mathfrak{g}_3((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \setminus (\neg\mathcal{P} \cap \neg\mathcal{P}_1), \\ \mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P} \cap (\neg\mathcal{P} \cap \neg\mathcal{P}_1). \end{cases}$$

Therefore, $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \,\widetilde{\sqcup}_\varepsilon\, \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M, N)\big) = (\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$. The remaining parts can be demonstrated analogously. □

Proposition 4.17. *Let* $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$, $(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)$, *and* $(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)$ *represent a BMPNSS, a BM*$_1$*PN*$_1$*SS, and a BM*$_2$*PN*$_2$*SS, respectively. Then,*

1. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\varepsilon\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\varepsilon(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcap}_\mathfrak{R}\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\varepsilon(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$;
2. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcup}_\varepsilon(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcup}_\varepsilon\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$;
3. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\varepsilon\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\varepsilon(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcup}_\mathfrak{R}\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\varepsilon(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$;
4. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcap}_\varepsilon(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcap}_\varepsilon\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$;
5. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcap}_\mathfrak{R}\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$;
6. $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\widetilde{\sqcup}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = \big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\big)\widetilde{\sqcup}_\mathfrak{R}\big((\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcap}_\mathfrak{R}(\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big)$.

Proof.

(4) Suppose that $\big((\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1)\ \widetilde{\sqcap}_\varepsilon\ (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2)\big) = (\mathfrak{f}_3\mathfrak{g}_3\mathfrak{h}_3\colon (\mathcal{P}_1 \cup \mathcal{P}_2)_{(Q_1 \cup Q_2)}, \max(M_1, M_2), \max(N_1, N_2))$, then for all $q \in Q_1 \cup Q_2$:

$$\mathfrak{h}_3(q) = \begin{cases} \mathfrak{h}_1(q), & \text{if } q \in Q_1 \setminus Q_2, \\ \mathfrak{h}_2(q), & \text{if } q \in Q_2 \setminus Q_1, \\ \min\{\mathfrak{h}_1(q), \mathfrak{h}_2(q)\}, & \text{if } q \in Q_1 \cap Q_2, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}_1 \cup \mathcal{P}_2$ and $\omega \in \Omega$:

$$\mathfrak{f}_3((q,\mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_1 \setminus \mathcal{P}_2, \\ \mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega), & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_2 \setminus \mathcal{P}_1, \\ \min\{\mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in \mathcal{P}_1 \cap \mathcal{P}_2, \end{cases}$$

$$\mathfrak{g}_3((\neg q,\mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \setminus \neg\mathcal{P}_2, \\ \mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_2 \setminus \neg\mathcal{P}_1, \\ \max\{\mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_1 \cap \neg\mathcal{P}_2. \end{cases}$$

Now, let

$$\begin{aligned} &(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_{\mathfrak{R}}(\mathfrak{f}_3\mathfrak{g}_3\mathfrak{h}_3\colon (\mathcal{P}_1 \cup \mathcal{P}_2)_{(Q_1\cup Q_2)}, \max(M_1, M_2), \max(N_1, N_2)) \\ &\quad = (\mathfrak{f}_4\mathfrak{g}_4\mathfrak{h}_4\colon \mathcal{P} \cap (\mathcal{P}_1 \cup \mathcal{P}_2)_{Q\cap(Q_1 \cup Q_2)}, \max(M, \max(M_1, M_2)), \max(N, \max(N_1, N_2))) \\ &\quad = (\mathfrak{f}_4\mathfrak{g}_4\mathfrak{h}_4\colon (\mathcal{P}_3 \cup \mathcal{P}_4)_{(Q_3 \cup Q_4)}, \max(M, M_1, M_2)), \end{aligned}$$

where $\mathcal{P}_3 = \mathcal{P} \cap \mathcal{P}_1$, $\mathcal{P}_4 = \mathcal{P} \cap \mathcal{P}_2$, $Q_3 = Q \cap Q_1$ and $Q_4 = Q \cap Q_2$, then for each $q \in Q_3 \cup Q_4$:

$$\mathfrak{h}_4(q) = \max\{\mathfrak{h}(q), \mathfrak{h}_3(q)\}.$$

Hence,

$$\mathfrak{h}_4(q) = \begin{cases} \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q_3 \setminus Q_4, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_2(q)\}, & \text{if } q \in Q_4 \setminus Q_3, \\ \max\{\mathfrak{h}(q), \min\{\mathfrak{h}_1(q), \mathfrak{h}_2(q)\}\}, & \text{if } q \in Q_3 \cap Q_4, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}_3 \cup \mathcal{P}_4$ and $\omega \in \Omega$:

$$\mathfrak{f}_4((q,\mathfrak{h}(q)))(\omega) = \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_3((q,\mathfrak{h}(q)))(\omega)\}, \qquad \mathfrak{g}_4(\neg q)(\omega) = \min\{\mathfrak{g}(\neg q)(\omega), \mathfrak{g}_3(\neg q)(\omega)\}.$$

Hence,

$$\mathfrak{f}_4((q,\mathfrak{h}(q)))(\omega) = \begin{cases} \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in O \setminus P, \\ \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega)\}, & \text{if } (q,\mathfrak{h}(q)) \in P \setminus O, \\ \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \min\{\mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q,\mathfrak{h}(q)))(\omega)\}\}, & \text{if } (q,\mathfrak{h}(q)) \in O \cap P, \end{cases}$$

$$\mathfrak{g}_4((\neg q,\mathfrak{h}(\neg q)))(\omega) = \begin{cases} \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \setminus \neg\mathcal{P}_4, \\ \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_4 \setminus \neg\mathcal{P}_3, \\ \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), & \\ \max\{\mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q,\mathfrak{h}(\neg q)))(\omega)\}\}, & \text{if } (\neg q,\mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \cap \neg\mathcal{P}_4. \end{cases}$$

On the other hand, let $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)\widetilde{\sqcup}_{\mathfrak{R}}(\mathfrak{f}_1\mathfrak{g}_1\mathfrak{h}_1\colon \mathcal{P}_{1Q_1}, M_1, N_1) = (\mathfrak{f}_5\mathfrak{g}_5\mathfrak{h}_5\colon (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, \max(M, M_1), \max(N, N_1))$, then for all $q \in Q \cap Q_1$:

$$\mathfrak{h}_5(q) = \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_1$ and $\omega \in \Omega$:

$$\begin{aligned} \mathfrak{f}_5((q,\mathfrak{h}(q)))(\omega) &= \max\{\mathfrak{f}((q,\mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q,\mathfrak{h}(q)))(\omega)\}, \\ \mathfrak{g}_5((\neg q,\mathfrak{h}(\neg q)))(\omega) &= \min\{\mathfrak{g}((\neg q,\mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q,\mathfrak{h}(\neg q)))(\omega)\}. \end{aligned}$$

Let $(\mathfrak{f}\mathfrak{g}\mathfrak{h}\colon \mathcal{P}_Q, M, N)\ \widetilde{\sqcup}_{\mathfrak{R}}\ (\mathfrak{f}_2\mathfrak{g}_2\mathfrak{h}_2\colon \mathcal{P}_{2Q_2}, M_2, N_2) = (\mathfrak{f}_6\mathfrak{g}_6\mathfrak{h}_6\colon (\mathcal{P} \cap \mathcal{P}_2)_{(Q \cap Q_2)}, \max(M, M_2), \max(N, N_2))$, then for each $q \in Q \cap Q_2$:

$$\mathfrak{h}_6(q) = \max\{\mathfrak{h}(q)(\omega), \mathfrak{h}_2(q)(\omega)\},$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P} \cap \mathcal{P}_2$ and $\omega \in \Omega$:

$$\mathfrak{f}_6((q, \mathfrak{h}(q)))(\omega) = \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega)\},$$
$$\mathfrak{g}_6((\neg q, \mathfrak{h}(\neg q)))(\omega) = \min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2(\neg q)(\omega)\}.$$

Now, suppose that

$$(\mathfrak{f}_5\mathfrak{g}_5\mathfrak{h}_5\colon (\mathcal{P} \cap \mathcal{P}_1)_{(Q \cap Q_1)}, \max(M, M_1), \max(N, N_1)) \widetilde{\sqcap}_{\varepsilon} (\mathfrak{f}_6\mathfrak{g}_6\mathfrak{h}_6\colon (\mathcal{P} \cap \mathcal{P}_2)_{(Q \cap Q_2)}, \max(M, M_2), \max(N, N_2)) = (\mathfrak{f}_7\mathfrak{g}_7\mathfrak{h}_7\colon (\mathcal{P}_3 \cup \mathcal{P}_4)_{(Q_3 \cup Q_4)}, \max(M, M_1, M_2)),$$

where $\mathcal{P}_3 = \mathcal{P} \cap \mathcal{P}_1$, $\mathcal{P}_4 = \mathcal{P} \cap \mathcal{P}_2$, $Q_3 = Q \cap Q_1$ and $Q_4 = Q \cap Q_2$, then for each $q \in Q_3 \cup Q_4$:

$$\mathfrak{h}_7(q) = \begin{cases} \mathfrak{h}_5(q), & \text{if } q \in Q_3 \setminus Q_4, \\ \mathfrak{h}_6(q), & \text{if } q \in Q_4 \setminus Q_3, \\ \min\{\mathfrak{h}_5(q), \mathfrak{h}_6(q)\}, & \text{if } q \in Q_3 \cap Q_4, \end{cases}$$
$$= \begin{cases} \max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, & \text{if } q \in Q_3 \setminus Q_4, \\ \max\{\mathfrak{h}(q), \mathfrak{h}_2(q)\}, & \text{if } q \in Q_4 \setminus Q_3, \\ \min\{\max\{\mathfrak{h}(q), \mathfrak{h}_1(q)\}, \max\{\mathfrak{h}(q), \mathfrak{h}_2(q)\}\}, & \text{if } q \in Q_3 \cap Q_4, \end{cases}$$

and for each $(q, \mathfrak{h}(q)) \in \mathcal{P}_3 \cup \mathcal{P}_4$ and $\omega \in \Omega$:

$$\mathfrak{f}_7((q, \mathfrak{h}(q)))(\omega) = \begin{cases} \mathfrak{f}_5((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_3 \setminus \mathcal{P}_4, \\ \mathfrak{f}_6((q, \mathfrak{h}(q)))(\omega), & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_4 \setminus \mathcal{P}_3, \\ \min\{\mathfrak{f}_5((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_6((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_3 \cap \mathcal{P}_4, \end{cases}$$
$$= \begin{cases} \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_3 \setminus \mathcal{P}_4, \\ \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega)\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_4 \setminus \mathcal{P}_3, \\ \min\{\max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_1((q, \mathfrak{h}(q)))(\omega)\}, & \\ \max\{\mathfrak{f}((q, \mathfrak{h}(q)))(\omega), \mathfrak{f}_2((q, \mathfrak{h}(q)))(\omega)\}\}, & \text{if } (q, \mathfrak{h}(q)) \in \mathcal{P}_3 \cap \mathcal{P}_4, \end{cases}$$

$$\mathfrak{g}_7((\neg q, \mathfrak{h}(\neg q)))(\omega) = \begin{cases} \mathfrak{g}_5((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \setminus \neg\mathcal{P}_4, \\ \mathfrak{g}_6((\neg q, \mathfrak{h}(\neg q)))(\omega), & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_4 \setminus \neg\mathcal{P}_3, \\ \max\{\mathfrak{g}_5((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_6((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \cap \neg\mathcal{P}_4, \end{cases}$$
$$= \begin{cases} \min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \setminus \neg\mathcal{P}_4, \\ \min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_4 \setminus \neg\mathcal{P}_3, \\ \max\{\min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_1((\neg q, \mathfrak{h}(\neg q)))(\omega)\}, & \\ \min\{\mathfrak{g}((\neg q, \mathfrak{h}(\neg q)))(\omega), \mathfrak{g}_2((\neg q, \mathfrak{h}(\neg q)))(\omega)\}\}, & \text{if } (\neg q, \mathfrak{h}(\neg q)) \in \neg\mathcal{P}_3 \cap \neg\mathcal{P}_4. \end{cases}$$

Since $(\mathfrak{f}_4\mathfrak{g}_4\mathfrak{h}_4\colon (\mathcal{P}_3 \cup \mathcal{P}_4)_{(Q_3 \cup Q_4)}, \max(M, M_1, M_2))$ and $(\mathfrak{f}_7\mathfrak{g}_7\mathfrak{h}_7\colon (\mathcal{P}_3 \cup \mathcal{P}_4)_{(Q_3 \cup Q_4)}, \max(M, M_1, M_2))$ are equivalent for every $(q, \mathfrak{h}(q))$ in $\mathcal{P}_3 \cup \mathcal{P}_4$ and $\omega \in \Omega$, the proof is completed. Similar arguments can be applied to prove the other parts. □

5. Improving decision-making scenarios through bipolar M-parameterized N-soft sets

The decision-making methodology outlined in this section extends the approach used for soft sets, as discussed in [25], while keeping all relevant attributes intact. Algorithms 1 and 2 are utilized to prioritize alternatives based on their extended choice values and extended weight choice values, respectively. Subsequently, a thorough explanation of the step-by-step process involved in these algorithms is presented.

Algorithm 1 Extended choice values.

1. **Input**:
   (a) set of alternatives: $\Omega = \{\omega_i, i = 1, 2, \ldots, m\}$;
   (b) set of attributes: $Q = \{q_j, j = 1, 2, \ldots, n\}$;
   (c) BMPNSS $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ with $R = \{0, 1, \ldots, N-1\}$ and $S = \{0, 1, \ldots, M-1\}$, where $M, N \in \{2, 3, \ldots\}$; for each $(q_j, \mathfrak{h}(q_j)) \in \mathcal{P}$ and $\omega_i \in \Omega$, there are $r^+_{ij}, r^-_{ij} \in R$.
2. **Calculations**:
   (a) for each ω_i, compute $\sigma_i = \sigma^+_i - \sigma^-_i$, where $\sigma^+_i = \sum_{j=1}^{n} \mathfrak{h}(q_j) \cdot r^+_{ij}$ and $\sigma^-_i = \sum_{j=1}^{n} \mathfrak{h}(q_j) \cdot r^-_{ij}$;
   (b) let k be the index such that $\sigma_k = \max\{\sigma_i\}$.
3. **Output**: choose any ω_k corresponding to the index k that maximizes the calculated values σ_i.

Algorithm 2 Extended weight choice vlues.

1. **Input**:
   (a) set of alternatives: $\Omega = \{\omega_i, i = 1, 2, \ldots, m\}$;
   (b) set of attributes: $Q = \{q_j, j = 1, 2, \ldots, n\}$;
   (c) BMPNSS $(\mathfrak{fgh}\colon \mathcal{P}_Q, M, N)$ with $R = \{0, 1, \ldots, N-1\}$ and $S = \{0, 1, \ldots, M-1\}$, where $M, N \in \{2, 3, \ldots\}$; for each $(q_j, \mathfrak{h}(q_j)) \in \mathcal{P}$ and $\omega_i \in \Omega$, there are $r^+_{ij}, r^-_{ij} \in R$.
   (d) set of weights for graded attributes: $W = \{W_j, j = 1, 2, \ldots, n\}$.
2. **Calculations**:
   (a) for each ω_i, compute $\sigma^W_i = \sigma^{+W}_i - \sigma^{-W}_i$, where $\sigma^{+W}_i = \sum_{j=1}^{n} \big(\mathfrak{h}(q_j) W_j\big) \cdot r^+_{ij}$ and $\sigma^{-W}_i = \sum_{j=1}^{n} \big(\mathfrak{h}(q_j)(1 - W_j)\big) \cdot r^-_{ij}$;
   (b) let k be the index such that $\sigma^W_k = \max(\sigma^W_i)$.
3. **Output**: choose any ω_k corresponding to the index k that maximizes the calculated values σ^W_i.

Example 5.1. Consider a hiring committee in a company tasked with selecting the "Employee of the Year" from a pool of candidates. The committee wants to choose only one outstanding employee, and the guidance of managers is crucial to make the decision. Let Ω be the collection of qualified employees, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, who meet the demands for the "Employee of the Year" award. The committee defines a set of attributes $Q = \{q_1, q_2, q_3, q_4, q_5\}$ to evaluate the candidates, where

$$\begin{aligned} q_1 &= \text{Consistent performance},\\ q_2 &= \text{Team player},\\ q_3 &= \text{Innovative thinker},\\ q_4 &= \text{Effective communicator},\\ q_5 &= \text{Leadership qualities}. \end{aligned}$$

Then $\neg Q = \{\neg q_1, \neg q_2, \neg q_3, \neg q_4, \neg q_5\}$, where

$$\begin{aligned} \neg q_1 &= \text{Inconsistent performance},\\ \neg q_2 &= \text{Lacks team collaboration},\\ \neg q_3 &= \text{Lacks innovation},\\ \neg q_4 &= \text{Poor communicator},\\ \neg q_5 &= \text{Lacks leadership skills}. \end{aligned}$$

The committee establishes evaluation scales for attributes using $S = \{0, 1, 2, 3, 4, 5\}$, where 0 indicates "Critical"; 1 indicates "Essential"; 2 indicates "Significant"; 3 indicates "Crucial"; 4 indicates "Strikingly crucial"; 5 indicates "Exceptionally crucial". For alternatives, they use evaluation scales $R = \{0, 1, 2, 3, 4\}$ representing ordered grades, where 0 means "Not recommended"; 1 means "Entry-level"; 2 means "Competent"; 3 means "Experienced"; 4 means "Highly skilled". To assess each employee, the committee uses a B6P5SS $(\mathfrak{fgh}\colon \mathcal{P}_Q, 6, 5)$ to represent the assessment values for their respective skills. The combination of these assessments and evaluations from the managers will guide the committee in selecting the most deserving "Employee of the Year."

$$\begin{aligned}(\mathfrak{fgh}\colon \mathcal{P}_Q, 6, 5) = \Big\{ &\big\langle (q_1, 5), \{\omega_1, 3, 1\}, \{\omega_2, 3, 1\}, \{\omega_3, 2, 2\}, \{\omega_4, 2, 0\}, \{\omega_5, 4, 0\} \big\rangle, \\ &\big\langle (q_2, 3), \{\omega_1, 1, 1\}, \{\omega_2, 1, 1\}, \{\omega_3, 3, 1\}, \{\omega_4, 3, 1\}, \{\omega_5, 1, 0\} \big\rangle, \\ &\big\langle (q_3, 3), \{\omega_1, 3, 1\}, \{\omega_2, 2, 1\}, \{\omega_3, 2, 1\}, \{\omega_4, 2, 1\}, \{\omega_5, 1, 0\} \big\rangle, \\ &\big\langle (q_4, 4), \{\omega_1, 1, 3\}, \{\omega_2, 1, 2\}, \{\omega_3, 2, 2\}, \{\omega_4, 3, 1\}, \{\omega_5, 0, 4\} \big\rangle, \\ &\big\langle (q_5, 4), \{\omega_1, 3, 0\}, \{\omega_2, 2, 1\}, \{\omega_3, 2, 1\}, \{\omega_4, 0, 3\}, \{\omega_5, 1, 1\} \big\rangle \Big\}.\end{aligned}$$

Now, we can display this information in Table 8.

Table 8: $(\mathfrak{fgh}\colon \mathcal{P}_Q, 6, 5)$: the candidates skills assessment.

$(\mathfrak{fgh}\colon \mathcal{P}_Q, 6, 5)$	$(q_1, 5)$	$(q_2, 3)$	$(q_3, 3)$	$(q_4, 4)$	$(q_5, 4)$
ω_1	(3,1)	(1,1)	(3,1)	(1,3)	(3,0)
ω_2	(3,1)	(1,1)	(2,1)	(1,2)	(2,1)
ω_3	(2,2)	(3,1)	(2,1)	(2,2)	(2,1)
ω_4	(2,0)	(3,1)	(2,1)	(3,1)	(0,3)
ω_5	(4,0)	(1,0)	(1,0)	(0,4)	(1,1)

By utilizing Algorithm 1 on Example 5.1 presented in Table 9, we can deduce that the candidate ω_1 is chosen as the outcome.

Table 9: The extended choice values in Example 5.1.

$(\mathfrak{fgh}\colon \mathcal{P}_Q, 6, 5)$	$(q_1, 5)$	$(q_2, 3)$	$(q_3, 3)$	$(q_4, 4)$	$(q_5, 4)$	σ_i
ω_1	(3,1)	(1,1)	(3,1)	(1,3)	(3,0)	20
ω_2	(3,1)	(1,1)	(2,1)	(1,2)	(2,1)	13
ω_3	(2,2)	(3,1)	(2,1)	(2,2)	(2,1)	13
ω_4	(2,0)	(3,1)	(2,1)	(3,1)	(0,3)	15
ω_5	(4,0)	(1,0)	(1,0)	(0,4)	(1,1)	10

Example 5.2. Given the weights $W_1 = 0.9$, $W_2 = 0.6$, $W_3 = 0.7$, $W_4 = 0.8$, and $W_5 = 0.4$ assigned to the graded attributes $\mathfrak{h}(q_j)$, $j = 1, 2, \dots, n$, in Example 5.1, the analysis presented in Table 10 indicates that the candidate ω_1 is chosen again.

6. Comparative analysis

In this section, we conduct a comparative assessment of the proposed BMPNSS model in relation to existing models, considering essential evaluation factors such as non-binary evaluation on alternatives (NbEA), non-binary evaluation on both alternatives and attributes (NbEAA), and bipolarity setting (BS). The aim is to underscore the BMPNSS model's versatility and efficacy in addressing these critical features when compared to other models within the field.

Table 10: The extended weight choice values in Example 5.2.

$(\mathfrak{fgh}: \mathcal{P}_Q, 6, 5)$	$(q_1, 5)$ $W_1 = 0.9$	$(q_2, 3)$ $W_2 = 0.6$	$(q_3, 3)$ $W_3 = 0.7$	$(q_4, 4)$ $W_4 = 0.8$	$(q_5, 4)$ $W_5 = 0.4$	σ_i^W
ω_1	(3,1)	(1,1)	(3,1)	(1,3)	(3,0)	24.6
ω_2	(3,1)	(1,1)	(2,1)	(1,2)	(2,1)	19.3
ω_3	(2,2)	(3,1)	(2,1)	(2,2)	(2,1)	21.1
ω_4	(2,0)	(3,1)	(2,1)	(3,1)	(0,3)	18.1
ω_5	(4,0)	(1,0)	(1,0)	(0,4)	(1,1)	17.9

Table 11 presents a thorough comparison, illustrating the applicability or limitations of each model concerning the identified evaluation factors. The presence of a checkmark (✓) signifies successful incorporation of the respective feature, while the symbol (×) indicates its absence or limited implementation.

The analysis unequivocally reveals that the BMPNSS model adeptly handles all evaluated features, establishing itself as a comprehensive and versatile approach. In contrast, comparative models exhibit limitations in one or more of these aspects, highlighting the unique strengths and advancements offered by the BMPNSS model.

Through this comparative analysis, we assert the superior capability of the proposed BMPNSS model in accommodating NbEA, NbEAA, and BS, setting it apart from existing models. This analysis reinforces the importance and applicability of our research, providing valuable insights for the further advancement and adoption of the BMPNSS model in practical scenarios.

Table 11: Comparing the BMPNSS model against other relevant existing models.

Authors	Models	NbEA	NbEAA	BS
Fatimah et al. [16]	N-SS	✓	×	×
Shabir et al. [37]	N-BSS	✓	×	✓
Riaz et al. [35]	MPNSS	✓	✓	×
Proposed model	BMPNSS	✓	✓	✓

7. Concluding remarks

MPNSS framework was explored extensively, presenting an effective solution to the challenges associated with assigning non-binary evaluations to both alternatives and attributes. Recognizing the intrinsic duality in human decision-making, characterized by the interplay of positive and negative influences, led to a significant enhancement of the MPNSS model through the incorporation of bipolarity. This augmentation culminated in the development of BMPNSS model, which introduced key definitions such as incomplete, negatively efficient, positively efficient, and totally efficient within the BMPNSS context. The complement of MPNSS was also addressed, introducing four distinct definitions: complement, weak complement, top weak complement, and bottom weak complement. A comprehensive exploration of set-theoretic operations, specifically extended and restricted union and intersection, was undertaken, with a detailed discussion of their properties contributing to a thorough understanding of their behavior within the BMPNSS framework. To enhance comprehension, an illustrative example was provided. The decision-making procedure introduced an alternative ranking system based on extended choice and extended weight choice values, demonstrated through a numerical example. In the comparative analysis, BMPNSS was positioned against existing models, emphasizing its unique features and advantages in diverse decision-making scenarios.

A promising avenue for future research involves extending the proposed BMPNSS model to incorporate fuzzy BMPNSS, integrating fuzzy logic principles to enhance its ability to handle uncertainty. This extension allows for a more nuanced representation of evaluations with varying degrees of membership. Additionally, considering approaches such as those proposed by Korkmaz et al. [19], where membership degree and grade are not necessarily correlated, could further refine the model. Furthermore, investigating topological and algebraic structures using the BMPNSS model opens avenues for a deeper understanding of its mathematical foundations.

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