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Theoretical and computational results for implicit non-singular hybrid fractional differential equation subject to multi-terms non-local initial conditions

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Abstract

This research work is devoted to investigate a class of hybrid fractional differential equations with $n+1$ terms initial conditions. The aforesaid problem is considered under the Atangana-Baleanue-Caputo fractional order derivative. Here it is remarkable that hybrid differential equations with linear perturbations have significant applications in modeling various dynamical problems. Sufficient conditions are established for the existence and uniqueness of solution to the problem under investigation by using the Banach and Krasnoselsikii's fixed point theorems. Since stability theory plays important role in establishing various numerical and optimizations results, therefore, Hyers-Ulam type stability results are deduced for the considered problems using the tools of nonlinear functional analysis. Additionally, a numerical method based on Euler procedure is established to study some approbation results for the proposed problem. By a pertinent example, we demonstrate our results. Also some graphical illustrations for different fractional orders are given.

Keywords: Atangana-Baleanu Caputo derivative, Hyers-Ulam stability, fixed point theorem, Euler numerical method.

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1. Introduction

Calculus with fractional order derivatives and integral has been given attention in the last few decades. Because, the said area has been found highly applicable in modeling various real world problems. In fact the differential and integral operators with real or complex orders are the natural extension of classical integer order operators of integration and differentiation. In additions, in many situation fractional order derivatives have been found more efficient to use as compared to ordinary derivatives. For instance certain real-world phenomena may be more accurately modelled via fractional differential operators, particularly when the dynamics are impacted by constrains of systems. Therefore, recently, researchers have focused to use the mentioned area in study of various problems devoted to engineering, physical, image processing, and biological science. For instance some novel work, we refer to [8, 18, 36–38]. Further, for more applications and new results using new concepts of fractional calculus, we refer to [2, 3, 22, 28].

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It is important to mention that fractional differential and integrals operators have been defined variously. There are numerous definitions given by researchers in literature. Based on the definitions, the said operators may be divided in to two classes including singular and non-singular operators. But remember that both kinds of operators are global operators in nature. Those operators involving power law type kernels like the Reimann-Liouville or Caputo derivatives are referred to be singular operators. For relevant results on the said area we refer to [26]. On the other hand those operators introduced by Caputo and Fabrizio and Atangana, Baleanue in Caputo sense abbreviated as ABC are known as non-singular derivatives. We remark that both kinds of derivatives have been used very well in investigating various real world problems. Here we refer to see details of said operators in [16]. The Caputo-Fabrizio derivative has used in various real world applications (refer to [7]). In the same way, the ABC derivative was proposed to address some limitations of the traditional Caputo fractional derivative and to provide a more comprehensive mathematical tool for describing real world phenomena, particularly in the field of fractional calculus [9]. The ABC-fractional derivative is based on the concept of non-locality, mean it takes into account the entire history of the function not just at a single point. The non-local nature of the new kernel allows for a comprehensive account of memory within structures and media of various scales, a task beyond the capabilities of classical fractional derivatives or CF-type derivatives. Moreover, we have the opinion that ABC-fractional derivatives will be able to play a significant impact on investigation of the micro-structural tendencies of certain materials, especially those characterized by non-local interactions [10]. As a result, ABC-fractional derivatives proves highly valuable in elucidating a diverse array of scientific, engineering, and technological challenges.

Initial value problems (IVPs) play a very important role in different fields of natural and physical sciences (we refer to [35]). Numerous applications of IVPs appear in applied fields such as chemical engineering, blood flow problems, population dynamics, water flow, and general relativity (see [19]). Therefore researchers have significantly worked on the mentioned area from different aspects including qualitative theory [30], numerical analysis [13], stability theory [35]. Also the IVPs were increasingly investigated under the concepts of fractional calculus. In this regards plenty of research work has been published. We refer some novel results devoted to the existence, stability, and numerical analysis of various problems as [1, 29, 32]. Both kinds of operators, singular and non-singular, were used to study various IVPs and their applications, here we refer to [17, 24, 33]. An important class of differential equations is devoted to hybrid differential equations (HDEs). Researchers have studied HDEs under the concepts of fractional order derivatives of Riemann-Liouville and Caputo sense very well. For instance HDEs with linear and quadratic perturbations were studied using the Caputo derivative of fractional order in [11, 21]. Also some researchers studied HFDEs with ABC and Caputo-Fabrizio derivative [23]. Various biological models have studied using the concept of non-singular type fractional derivatives recently. For instance we refer to [4, 5].

Since HDEs have significant applications in various real world problems, for instance, hybrid model for learning space technology was studied in [34], further, authors [27] comprehensively reviewed building hybrid models of physical systems. Most of the biological models of diseases are formulated in the form of systems of HDEs. But to the best of our information, HDEs with ABC fractional derivatives were very rarely studied for theoretical and computational purposes. Also the importance of ABC derivative mentioned in [9], it is needed to investigate a class of HFDEs under multi point initial conditions from theoretical and numerical perspectives. In particularly, implicit HDEs under ABC fractional order derivative with $n+1$ terms initial conditions have not studied for the existence, stability, and numerical solutions.

Due to the importance applicability of ABC derivative and significant applications of HDEs, we consider the following class of $n+1$ terms IVPs involving ABC derivative of fractional order with $t \in [0, T] = \mathbb{J}$ as

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha}[\mu(t) - f(t, \mu(\delta t))] = g(t, \mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)), \ \alpha \in (0, 1], \\ \mu(0) = \sum_{j=0}^{n} k_j h(\mu), \end{cases} \tag{1.1}$$

where $f : \mathbb{J}\times \mathbb{R}\to\mathbb{R}$, $g : \mathbb{J}\times \mathbb{R}^2\to\mathbb{R}$, $h \in C(\mathbb{J})$ are continuous functions and $f(0, \mu(0)) = 0$. Here, we state that the term denoted by δ is called proportional type delay which lies in $(0, 1)$. If $\delta > 1$, then the problem becomes ill posed. The mentioned delay type problems have many applications in electro-locomotive and other physical systems (see [20]). By using Banach and Krasnoselsikii's fixed point theorems [12] enough prerequisites are established for a solution to be unique and exist. Also, we derive some results related to H-U stability following [31]. The concerned stability has very well studied for numerous problems recently, we refer to [6, 15, 25]. The mentioned stability can be found about the best approximate solution of the suggested problem. The mentioned stability has been studied in various articles for different kind of problems of fractional calculus. Since for the considered nonlinear problem, numerical results are crucial requirement, therefore, following the generalized Taylor formula given by [14], we deduce a numerical method of Euler type for the given problem. Several graphical illustrations are provided for the test example, to demonstrate our theoretical results.

Our article is organized as follows. We give introduction in Section 1. Preliminaries are given in Section 2. Theoretical results are given in Section 3. Computational results are written in Section 4. Section 5 is devoted to conclusion.

2. Preliminaries

Definition 2.1 ([9]). Here C[0T] is the space of continuous function, then the ABC derivative for fractional order $\alpha \in (0, 1)$ of $\mu \in C[0, T]$ is defined by

$$^{ABC}\mathbb{D}^{\alpha}\mu(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \mu'(s)\mathscr{E}_{\alpha}\left[\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right] ds, \tag{2.1}$$

where the normalization function is defined by $M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 2.2 ([27]). Let $\mu \in L[0, T]$, then for $\alpha \in (0, 1]$, the integral is described as

$$^{AB}_{0}\mathbb{I}^{\alpha}_{t}\mu(t) = \frac{1-\alpha}{M(\alpha)}\mu(t) + \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds,$$

provided that integral on right side exists.

Lemma 2.3 ([9]). *Let $h \in L[0, T]$, such that $h \to 0$ at $t \to 0$, then*

$$^{ABC}_{0}\mathbb{D}^{\alpha}_{t}\mu(t) = h(t),\ 0 < \alpha \leqslant 1,$$

has a unique solution given by

$$\mu(t) = \mu(0) + \frac{1-\alpha}{M(\alpha)}h(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds.$$

We use the space $\mathscr{X} = C[0, 1]$ with norm $\|\mu\| = \sup_{t\in\mathbb{J}} |\mu(t)|$, which is a Banach space.

Theorem 2.4 ([12]). *Let $\mathscr{S} \subseteq \mathscr{X}$ and T_1, T_2 be two operators satisfying*

i) T_1 *is a contraction map;*
ii) T_2 *is completely continuous mapping,*

then, the $T_1(\mu) + T_2(\mu) = \mu$ has at least one solution.

3. Existence theory

Here, we move to present our main results.

Lemma 3.1. *The solution of linear fractional order hybrid problem under ABC derivative where* $f_1, f_2 \in C[0,T]$, *such that* $f_1(0) = 0$, *given by*

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha}[\mu(t) - f_1(t)] = f_2(t), \ \alpha \in (0,1], \\ \mu(0) = \sum_{j=0}^{n} k_j h(\mu), \end{cases} \tag{3.1}$$

is deduced as

$$\mu(t) = \sum_{j=0}^{n} k_j h(\mu) + f_1(t) + \frac{(1-\alpha)}{M(\alpha)} f_2(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s) ds.$$

Proof. Let a_0 be the constant and applying ${}^{AB}\mathbb{I}^{\alpha}$ on (3.1) and using Lemma 2.3, one has

$$\mu(t) - f_1(t) = a_0 + \frac{(1-\alpha)}{M(\alpha)} f_2(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s) ds. \tag{3.2}$$

By using the initial condition, one has $a_0 = k_j h(\mu)$. Thus, equation (3.2) becomes

$$\mu(t) = \sum_{j=0}^{n} k_j h(\mu) + f_1(t) + \frac{(1-\alpha)}{M(\alpha)} f_2(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s) ds.$$

□

In view of Lemma 3.1, we obtain the desired integral representation of (1.1) as follows.

Corollary 3.2. *The equivalent integral equation of* (1.1) *is given by*

$$\begin{aligned} \mu(t) = {} & \sum_{j=0}^{n} k_j h(\mu) + f(t, \mu(\delta t)) + \frac{(1-\alpha)}{M(\alpha)} g(t, \mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t) \\ & + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s)) ds. \end{aligned}$$

Following assumptions hold.

(H_1) There is a constant $C_h > 0$, such that for each $\mu, \bar{\mu} \in \mathscr{X}$, one has $|h(\mu) - h(\bar{\mu})| \leqslant C_h|\mu - \bar{u}|$.
(H_2) There exists constant $C_f > 0$, such that for every $\mu, \bar{\mu}, \in \mathscr{X}$, one has $|f(t,\mu) - f(t,\bar{u})| \leqslant C_f|\mu - \bar{u}|$.
(H_3) There is a constant $C_g > 0$, such that for every $\mu, v, \bar{\mu}, \bar{v} \in \mathscr{X}$, we have

$$|g(t,\mu,v) - g(t,\bar{\mu},\bar{v})| \leqslant C_g\left[|\mu - \bar{\mu}| + |v - \bar{v}|\right].$$

(H_4) Let there exist constants $K_f, K_g, K_h, M_f, M_g, M_h > 0$, then following assumptions hold:

$$|f(t,\mu(t))| \leqslant K_f|\mu| + M_f, \ |g(t,\mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t))| \leqslant K_g\left[|\mu| + |{}^{ABC}\mathbb{D}^{\alpha}\mu|\right] + M_g, \ |h(\mu)| \leqslant K_h|\mu| + M_h.$$

Let us define two operators $T_1, T_2 : \mathscr{X} \to \mathscr{X}$ by

$$\begin{aligned} T_1(\mu) &= \sum_{j=0}^{n} k_j h(\mu) + f(t, \mu(\delta t)) + \frac{(1-\alpha)}{M(\alpha)} g(t, \mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)), \\ T_2(\mu) &= \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s)) ds. \end{aligned}$$

Theorem 3.3. *Under the hypothesis* (H_1)-(H_4) *and if*

$$\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)}\left(\frac{C_g}{1-C_g}\right) < 1,$$

then the problem (1.1) *has at least one solution.*

Proof. We prove that T_1 is a contraction. Let $\mu, \bar{\mu} \in \mathscr{X}$, consider

$$\begin{aligned}
\|T_1(\mu) - T_1(\bar{\mu})\| = \max_{t\in[0,T]} \Bigg\| &\sum_{j=0}^{n} k_j h(\mu) + f(t, \mu(\delta t)) + \frac{(1-\alpha)}{M(\alpha)} g(t, \mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)) \\
&- \sum_{j=0}^{n} k_j h(\bar{\mu}) - f(t, \bar{\mu}(\delta t)) - \frac{(1-\alpha)}{M(\alpha)} g(t, \bar{\mu}(t), {}^{ABC}\mathbb{D}^{\alpha}\bar{\mu}(t)) \Bigg\| \\
&\leqslant \sum_{j=0}^{n} k_j C_h \|mu - \bar{\mu}\| + \frac{(1-\alpha)}{M(\alpha)}\left(\frac{C_g}{1-C_g}\|\mu - \bar{\mu}\|\right. \\
&\leqslant \left[\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)}\left(\frac{C_g}{1-C_g}\right)\right]\|\mu - \bar{\mu}\|.
\end{aligned} \tag{3.3}$$

Thus (3.3) shows that T_1 is a contraction mapping. Similarly, we now show that the operator T_2 is completely continuous. Let $D = \{\mu \in \mathscr{X} : \|\mu\| \leqslant r\}$ be closed and bounded subset of $\mathscr{X}$, then

$$\|T_2(\mu)\| = \max_{t\in[0,T]}\left|\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))ds\right| \leqslant \frac{1}{M(\alpha)\Gamma(\alpha)}\left(\frac{K_g}{1-K_g} + M_g\right) r := K.$$

Hence, T_2 is bounded. Also for $t_1 < t_2 \in [0, T]$, we have

$$\begin{aligned}
|T_2(\mu(t_2)) - T_2(\mu(t_1))| = \| &\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1} g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))ds \\
&- \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1} g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))ds\| \\
\leqslant \alpha &\int_0^{t_1} \frac{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{M(\alpha)\Gamma(\alpha)}\|g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))\| ds \\
&+ \alpha\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{M(\alpha)\Gamma(\alpha)}\|g(s, \mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))\| ds \leqslant \frac{t_2^{\alpha} - t_1^{\alpha}}{M(\alpha)\Gamma(\alpha)}\left[M_f + \frac{K_g r}{1-K_g}\right],
\end{aligned}$$

as right side tends to zero at $t_1 \to t_2$, so

$$\|T_2\mu(t_2) - T_2\mu(t_1)\| \to 0 \ \text{ as } \ t_1 \to t_2$$

implies that T_2 is equi-continuous. As T_2 is bounded and equi-continuous, therefore by Arzelá-Ascoli theorem, T_2 is completely continuous. Hence by using Theorem 2.4, the problem (1.1) has at least one solution. □

Theorem 3.4. *Using hypothesis* (H_1)-(H_3) *and if*

$$\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)} C_g + \frac{T^{\alpha}}{M(\alpha)\Gamma(\alpha)}\frac{C_g}{1-C_g} < 1,$$

then the problem (1.1) *has a unique solution.*

Proof. First, we define the operator $T_3 : \mathscr{X} \to \mathscr{X}$ by

$$\begin{aligned} T_3\mu(t) &= \sum_{j=0}^{n} k_j h(\mu) + f(t,\mu(\delta t)) + \frac{(1-\alpha)}{M(\alpha)} g(t,\mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)) \\ &\quad + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))ds. \end{aligned} \tag{3.4}$$

Now, consider $\mu, \bar{\mu} \in \mathscr{X}$, then, using ($H_1$)-($H_3$) we can easily show that

$$\|T_3\mu - T_3\bar{\mu}\| \leqslant \left[\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)} C_g + \frac{T^{\alpha}}{M(\alpha)\Gamma(\alpha)} \frac{C_g}{1-C_g} \right] \|\mu - \bar{\mu}\|.$$

Thus, by Banach contraction theorem, problem (1.1) has a unique solution. □

4. Stability analysis

In this section, we study the H-U and generalized H-U stability of the problem (1.1). We recall the definition from [31].

Definition 4.1. Let

$$\mu(t) = P\mu(t) \tag{4.1}$$

be the operator equation, then for any $\epsilon > 0$ such that the inequality given by

$$|\mu(t) - P\mu(t)| \leqslant \epsilon, \; t \in [0,T],$$

holds, then (4.1) is said to be H-U stable if there exists a constant $C_P > 0$ and a unique fixed point $\bar{\mu} \in \mathscr{X}$, such that $|\mu(t) - \bar{\mu}(t)| \leqslant C_P\epsilon$. In addition, for any $\epsilon > 0$, such that the inequality given by

$$|\mu(t) - P\mu(t)| \leqslant \vartheta(\epsilon), \; t \in [0,T],$$

holds, then (4.1) is generalized H-U stable if there exists a constant $C_P > 0$ and a unique fixed point $\bar{\mu} \in \mathscr{X}$, such that $|\mu(t) - \bar{\mu}(t)| \leqslant C_P\vartheta(\epsilon)$.

Remark 4.2. We define a function $\psi : [0,T] \to \mathbb{R}$ independent of μ that statistics

(i) $|\psi(t)| \leqslant \epsilon$, $t \in [0,T]$;
(ii) $\psi(0) = 0$.

Consider the problem

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha}[\mu(t) - f(t,\mu(\delta t))] = g(t,\mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)) + \psi(t), \\ \mu(0) = \sum_{j=0}^{n} k_j h(\mu), \end{cases}$$

which is equivalent to integral equation described by

$$\begin{aligned} \mu(t) &= \sum_{j=0}^{n} k_j h(\mu) + f(t,\mu(\delta t)) + \frac{(1-\alpha)}{M(\alpha)} g(t,\mu(t), {}^{ABC}\mathbb{D}^{\alpha}\mu(t)) \\ &\quad + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,\mu(s), {}^{ABC}\mathbb{D}^{\alpha}\mu(s))ds + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\psi(s)ds, \end{aligned}$$

which can be written in view of (3.4) as

$$\mu(t) = T_3\mu(t) + \frac{(1-\alpha)}{M(\alpha)}\psi(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(s)ds. \tag{4.2}$$

From (4.2), one has

$$|\mu(t) - T_3\mu(t)| \leqslant \frac{(1-\alpha)}{M(\alpha)}|\psi(t)| + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\psi(s)|ds \leqslant \Lambda_{\alpha,T}\epsilon, \tag{4.3}$$

where $\Lambda_{\alpha,T} = \frac{(1-\alpha)}{M(\alpha)} + \frac{T^\alpha}{M(\alpha)\Gamma(\alpha)}$. Thus, from (4.3), one has

$$|\mu(t) - T_3\mu(t)| \leqslant \Lambda_{\alpha,T}\epsilon.$$

Theorem 4.3. *The solution of problem* (1.1) *is H-U stable and consequently generalized H-U stable if*

$$\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)}C_g + \frac{T^\alpha}{M(\alpha)\Gamma(\alpha)}\frac{C_g}{1-C_g} = \Delta < 1.$$

Proof. We deduce the proof by using hypothesis (H_1)-(H_3) and Remark 4.2. Let $\mu \in \mathscr{X}$ be any solution of (1.1) and $\bar{\mu} \in \mathscr{X}$ is the unique solution of (1.1), take

$$\begin{aligned}
\|\mu - \bar{\mu}\| &= \max_{t\in[0,T]} |\mu(t) - T_3\bar{\mu}(t)| \\
&\leqslant \max_{t\in[0,T]} \left[|\mu(t) - T_3\mu(t)| + |T_3\mu(t) - T_3\bar{\mu}(t)|\right] \\
&\leqslant \Lambda_{\alpha,T}\epsilon + \max_{t\in[0,T]} |T_3\mu(t) - T_3\bar{\mu}(t)| \\
&\leqslant \Lambda_{\alpha,T}\epsilon + \sum_{j=0}^{n} k_j C_h + C_f + \left[\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)}C_g + \frac{T^\alpha}{M(\alpha)\Gamma(\alpha)}\frac{C_g}{1-C_g}\right]\|\mu - \bar{\mu}\|,
\end{aligned}$$

then we have

$$\|\mu - \bar{\mu}\| \leqslant \Lambda_{\alpha,T}\epsilon + \Delta\|\mu - \bar{\mu}\|,$$

which implies

$$\|\mu - \bar{\mu}\| \leqslant \frac{\Lambda_{\alpha,T}}{1-\Delta}\epsilon.$$

Hence, solution of problem (1.1) is H-U Stable. □

Moreover, if we have a non-decreasing function $\vartheta : (0, T) \to \mathbb{R}$, such that $\vartheta(\epsilon) = \epsilon$, then inview of Definition 4.1, we have

$$\|\mu - \bar{\mu}\| \leqslant \frac{\Lambda_{\alpha,T}}{1-\Delta}\vartheta(\epsilon).$$

Clearly, $\vartheta(0) = 0$. So solution of problem (1.1) is generalized H-U stable.

5. Numerical method

Here we extend the numerical method presented in [14] called fractional Euler's method for ABC derivative. In this regards recall the general Taylor formula for a function ϕ about a as

$$\phi(t) = \sum_{m=0}^{n}\left[\left({}^{ABC}\mathbb{D}^\alpha\right)^{n+1}\sum_{l=0}^{n+1}\frac{t^{\alpha l}\Gamma(n+2)\alpha^l(1-\alpha)^{n-l+1}}{\Gamma(l+1)\Gamma(n-l+2)\Gamma(l\alpha+1)(M(\alpha))^{n+1}}\right.$$

$$+\left({}^{ABC}\mathbb{D}^{\alpha}\right)^{m}\sum_{l=0}^{m}\frac{t^{\alpha l}\Gamma(m+1)\alpha^{l}(1-\alpha)^{m-l}}{\Gamma(l+1)\Gamma(m-l+1)\Gamma(l\alpha+1)(M(\alpha))^{m}}(t-a)\phi(a).$$

Now consider a general problem given by

$$^{ABC}\mathbb{D}^{\alpha}\phi(t)=\psi(t,\phi(t)),\ \phi(0)=\phi_0. \tag{5.1}$$

Let $h=\frac{a}{l}$, and $t_j=t_j+jh$ be the nodes with $j=0,1,2,\dots,l$ by dividing the interval $[0,a)\subset[0,T]$ in to l-subintervals, then expansion of $\phi(t)$ about $t=t_0$, and obtain the following numerical Euler formula for (5.1) by ignoring terms containing highest power of h as

$$\phi(t_1)=\phi(t_0)+\psi(t_0,\phi(t_0))\frac{\alpha(h-a)^{\alpha}}{M(\alpha)\Gamma(\alpha+1)}. \tag{5.2}$$

In general, we can write (5.2) as

$$\phi(t_{n+1})=\phi(t_n)+\psi(t_n,\phi(t_n))\frac{\alpha(h-a)^{\alpha}}{M(\alpha)\Gamma(\alpha+1)},\quad n=0,1,2,\dots. \tag{5.3}$$

Using (5.3), we can write the numerical scheme for the proposed problem as

$$\mu(t_{n+1})=\sum_{j=0}^{n}k_jh(\mu)+f(t_n,\mu(\delta t_n))+g(t_n,\mu(t_n),\kappa(t_n))\frac{\alpha(h-a)^{\alpha}}{M(\alpha)\Gamma(\alpha+1)},\quad n=0,1,2,\dots, \tag{5.4}$$

where ${}_0^{ABC}\mathbb{D}^{\alpha}\mu(t)=\kappa(t)$, which we can further write on replacing n by $n-1$ in (5.4) as

$$\mu(t_n)=\mu(t_{n-1})+\kappa(t_{n-1})\frac{\alpha(h-a)^{\alpha}}{M(\alpha)\Gamma(\alpha+1)},\quad n=1,2,\dots,$$

which yields that

$$\kappa(t_n)=[\mu(t_n)-\mu(t_{n-1})]M(\alpha)\Gamma(\alpha)(h-a)^{-\alpha}. \tag{5.5}$$

Using (5.5) in (5.4), we get the final formula as

$$\mu(t_{n+1})=\sum_{j=0}^{n}k_jh(\mu)+f(t_n,\mu(\delta t_n))+g\Big(t_n,\mu(t_n),[\mu(t_n)-\mu(t_{n-1})]M(\alpha)\Gamma(\alpha)(h-a)^{-\alpha}\Big)\frac{\alpha(h-a)^{\alpha}}{M(\alpha)\Gamma(\alpha+1)},$$

where $n=0,1,2,\dots$.

6. Test problem

Here, we provide a test problem as illustration of our existence, uniqueness, and stability results.

Example 6.1. Consider

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha}[\mu(t)-\exp(-t)-\frac{1}{98}|\mu(0.5t)|]=\frac{|\mu(t)|+\sin|\mu(t)|+{}^{ABC}\mathbb{D}^{\alpha}\mu(t)}{t^2+30},\\ u(0)=\sum_{j=0}^{2}k_j\frac{\sqrt{|\mu|}}{25}, \end{cases}$$

where $k_0=\sqrt{2}$, $k_1=\exp(-11)$, $k_2=\sqrt{\pi}$, $\delta=0.5$. Then $f(t,\mu(\delta t))=\exp(-t)+\frac{1}{98}|\mu(0.5t)|$,

$$g(t,\mu(t),{}^{ABC}\mathbb{D}^{\alpha}\mu(t))=\frac{|\mu(t)|+\sin|\mu(t)|+{}^{ABC}\mathbb{D}^{\alpha}\mu(t)}{t^2+30}\quad\text{and}\quad h(\mu)=\frac{\sqrt{|\mu|}}{25}.$$

On calculation, $C_h = \frac{1}{25}$, $C_f = \frac{1}{98}$, $C_g = \frac{1}{31}$, $K_f = 2$, $K_g = 1$, $K_h = \frac{1}{25}$, $M_f = 2$, $M_g = 1$, $M_h = 1$. Using $M(\frac{1}{2}) = 1$, $T = 1$, we compute that

$$\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)}\left(\frac{C_g}{1-C_g}\right) = 0.304338114 < 1.$$

Also, for instance at $\alpha = \frac{1}{2}$, one has

$$\sum_{j=0}^{n} k_j C_h + C_f + \frac{(1-\alpha)}{M(\alpha)} C_g + \frac{T^{\alpha}}{M(\alpha)\Gamma(\alpha)} \frac{C_g}{1-C_g} = 0.162402162 < 1.$$

Hence, all the Hypothesis holds. Thus, Theorem 3.3 follows that Example 6.1 has at least one solution and Theorem 3.4 follows that Example 6.1 has a unique solution also. In addition, Theorem 4.3 follows that the solution of Example 6.1 is H-U stable. In Figures 1, 2, and 3, we present the numerical solutions against different fractional orders graphically of Example 6.1.

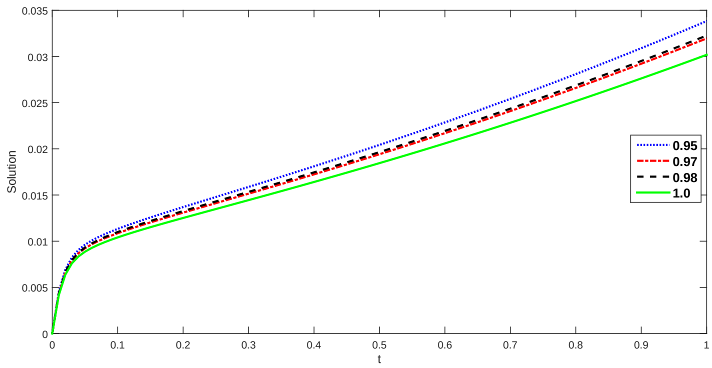

Figure 1: Numerical results for Example 6.1 with various fractional orders.

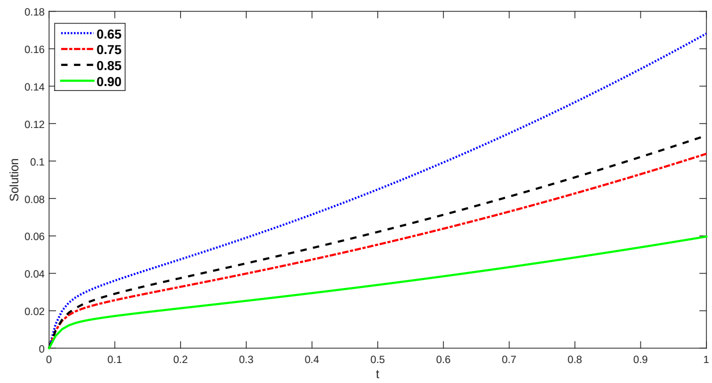

Figure 2: Numerical results for Example 6.1 with various fractional orders.

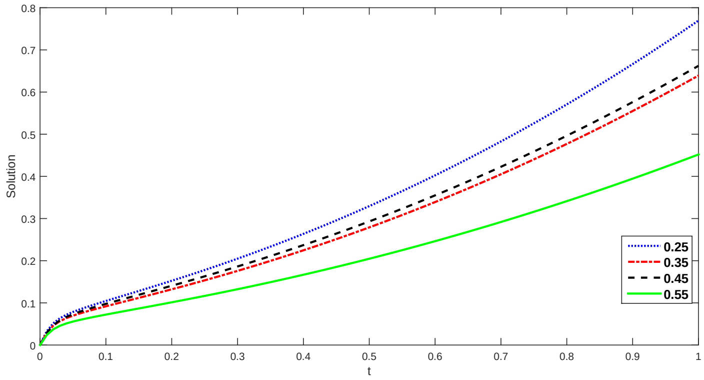

Figure 3: Numerical results for Example 6.1 with various fractional orders.

7. Conclusion

This work was devoted to the existence and uniqueness theory of a fractional differential equation with initial conditions on $n+1$ terms. Moreover, we used the ABC fractional differential operator to study the afore mentioned problem. The conditions for the existence and uniqueness of the solution were established by applying the Banach and the Krasnoselskii's fixed point theorems. Additionally, some results related to H-U and generalized H-U stabilities were derived using the functional analysis tools. Using the Euler concept, a numerical scheme has been developed for the computation of the numerical solution to the given problem. All the derived results have been demonstrated by a suitable example with graphical presentations using different fractional orders. The analysis we established here for a class of HDEs under the ABC fractional order derivative with multi conditions can be extended to boundary value problems of HDEs in the future. Further, more various classes of coupled HDEs by using fractals fractional concepts can be investigated in this future in the same way.

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