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Hyers-Ulam stability of a-type additive functional equations in Banach space using direct method

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Abstract

In this paper, we introduce the new kind of a-type additive functional equation and investigate the Hyers-Ulam stability of a-type additive functional equations within the Banach spaces using the direct method. This approach provides a straightforward and efficient way to establish the stability of functional equations without relying on more complex or indirect techniques. We begin by defining the specific a-type additive functional equation under consideration and outlining the conditions required for its stability. Through rigorous mathematical analysis, we demonstrate that under certain constraints, the functional equation exhibits Hyers-Ulam stability.

Keywords: Additive functional equation, Banach space, direct method, Hyers-Ulam stability.

2020 MSC: 39B52, 32B72, 32B82.

1. Introduction

The stability analysis of functional equations has achieved widespread recognition in the recent eight decades. During in 1940, the Mathematical Colloquium of the University of Wisconsin, Ulam [34] covered several difficult and mainly unsolved subjects. Stability analysis is one of these and serves as the basis for a novel kind of study. For the first time, functional equation stability was demonstrated by Hyers in 1941 [16].

The most simple approach is a direct method offered by Hyers that yields the stability of additive function. This strategy's most significant and effective tool is for analyzing the stability of various types of functional equations. Since the Hyers theorem's publication, numerous research have been written in

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relation to different generalizations of Ulam's problem and the Hyers theorem (see [7]). Aoki [4] improved the Hyers theorem for roughly linear transformations in Banach spaces in 1950 by lowering the requirement for the Cauchy difference for the addition of powers of norms. For the "sum" the Hyers-Ulam-Aoki-Rassias theorem was proved by Rassias [28] and Maligranda [26]. Rassias [29] replaced the sum with the norm powers product. The Ulam-Gavruta-Rassias stability is the stability discussed by [6, 14, 19, 27, 33] in their theorem. The theorem was established in 1994 by Gavruta [13], which used the control function and generalized all of the previous findings. This property of the functional equation is called the generalized Hyers-Ulam-Rassias stability. Over the past 35 years, a number of authors have conducted extensive studies on the Hyers-Ulam-Rassias stability of different functional equations (see [3, 11, 15, 31, 32]). Hyers-Ulam stability is a central theme in the study of functional equations. Abdollahpour et al. [1, 2] have made significant contributions to this area and they examined the Hyers-Ulam stability of hypergeometric differential equations. Earlier, they studied the Hyers-Ulam stability of associated Laguerre differential equations within a subclass of analytic functions, further enriching the literature. Aczel [3] provides a foundational understanding of functional equations, offering a broad and deep exploration of the subject. Similarly, Bourgin's [5] work on classes of transformations is seminal, discussing various transformations and their properties. Baias et al. [7] explored set-valued solutions of Popoviciu's functional equation. Their study addresses the existence and properties of these solutions, providing a comprehensive analysis of this specific functional equation type. Czerwik [8, 9] has made notable contributions, particularly in the stability of quadratic mappings and general functional equations. His books and papers discuss functional equations and inequalities in multiple variables, providing insights into their stability and applications. Elqorachi et al. [10] studied the generalized Hyers-Ulam stability of trigonometric functional equations, adding to the understanding of stability in specific functional forms. Gajda's [12] work on the stability of additive mappings also contributes to this area by examining the conditions under which additive functions remain stable. Hyers et al. [17, 18] have authored extensive works on the stability of functional equations, including "Stability of Functional Equations in Several Variables" and papers on approximate homomorphisms. Jung [20–23] studies on Hyers-Ulam-Rassias stability offers a detailed analysis of stability in nonlinear functional equations, while his other works address specific functional equations and their stability properties. Kannappan's [24] functional equations and inequalities discusses various applications, providing a practical perspective on the theoretical concepts. Rassias [25] and Sahoo [30] have also contributed significantly with their works on functional inequalities and introductory texts on functional equations, respectively .

A rapidly growing field of mathematics with a wide range of applications, functional equations are being used more and more to address problems in the fields of mathematical analysis, biology, combinatorics, statistics, information theory, and physical sciences. The symmetry types of functions used to solve an equation or inequality can be evaluated to find lowest quality solutions. Conceptually speaking, knowledge of the symmetry characteristics of important mappings, including special polynomials and hypergeometric mappings, may lead to intriguing results. Additionally, symmetry types for different types of operators related to the notion of quantum calculus can be examined.

The Cauchy functional equation

$$\mu(x_a + y_a) = \mu(x_a) + \mu(y_a) \tag{1.1}$$

for all $x_a, y_a \in R$ was determined by Cauchy. Linear functions are the solutions to this additive Cauchy functional equation. In this paper, we introduce the solution and stability of the new kind of functional equation of the form

$$\begin{aligned}
&\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(-ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\
&\quad + \mu(ax_{a_1} - a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} - a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\
&\quad + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} - a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} - a^5x_{a_5} + a^6x_{a_6}) \\
&\quad + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} - a^6x_{a_6}) \\
&= 5\left(\mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6})\right)
\end{aligned}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$, where a is positive integer with $a \neq 0$ in Banach space using direct method and its application.

2. General solution of the functional equation (1.1)

This section examines the general solution of the functional equation (1.1) in a real vector space.

Lemma 2.1. *If $\mu : \chi \to \psi$ is an odd mapping satisfies the functional equation*

$$\mu(x_a + y_a) = \mu(x_a) + \mu(y_a), \tag{2.1}$$

for all $x_a, y_a \in \chi$, if and only if $\mu : \chi \to \psi$ satisfies the functional equation

$$\begin{aligned} &\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(-ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} \\ &\quad + a^6x_{a_6}) + \mu(ax_{a_1} - a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} - a^3x_{a_3} + a^4x_{a_4} \\ &\quad + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} - a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} \\ &\quad + a^4x_{a_4} - a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} - a^6x_{a_6}) \\ &= 5\left(\mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6})\right), \end{aligned} \tag{2.2}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$.

Proof. Let $x = y = 0$ in (2.1), we obtain $\mu(0) = 0$. Replacing (x_a, y_a) by (x_a, x_a) and $(x_a, 2x_a)$ in (2.1), we get $\mu(2x_a) = 2\mu(x_a)$ and $\mu(3x_a) = 3\mu(x_a)$, for all $x_a \in \chi$. In general

$$\mu(bx_a) = b\mu(x_a), \text{ for any positive integer } a. \tag{2.3}$$

It is easy to verify from (2.3) that,

$$\mu(b^2x_a) = b^2\mu(x_a) \quad \text{and} \quad \mu(b^3x_a) = b^3\mu(x_a),$$

for all $x_a \in \chi$. Replacing (x_a, y_a) by $(ax_{a_1}, a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6})$ in (2.1), we obtain

$$\begin{aligned} &\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\ &\quad = \mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6}), \end{aligned} \tag{2.4}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Applying (x_a, y_a) by $(-a^2x_{a_2}, ax_{a_1} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6})$ in (2.1), we obtain

$$\begin{aligned} &\mu(ax_{a_1} - a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\ &\quad = \mu(ax_{a_1}) + \mu(-a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6}), \end{aligned}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Iterating (x_a, y_a) by $(-a^3x_{a_3}, ax_{a_1} + a^2x_{a_2} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6})$ in (2.1), we get

$$\begin{aligned} &\mu(ax_{a_1} + a^2x_{a_2} - a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\ &\quad = \mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(-a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6}), \end{aligned}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Switching (x_a, y_a) by $(-a^4x_{a_4}, ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^5x_{a_5} + a^6x_{a_6})$ in (2.1), we obtain

$$\begin{aligned} &\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} - a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) \\ &\quad = \mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(-a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6}), \end{aligned}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Replacing (x_a, y_a) by $(-a^5x_{a_5}, ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^6x_{a_6})$ in (2.1), we have

$$\begin{aligned}&\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} - a^5x_{a_5} + a^6x_{a_6})\\&\quad = \mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(-a^5x_{a_5}) + \mu(a^6x_{a_6}),\end{aligned}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Interchanging (x_a, y_a) by $(-a^6x_{a_6}, ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5})$ in (2.1), we arrive

$$\begin{aligned}&\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} - a^6x_{a_6})\\&\quad = \mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(-a^6x_{a_6}),\end{aligned}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Applying (x_a, y_a) by $(-ax_{a_1}, a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6})$ in (2.1), we obtain

$$\begin{aligned}&\mu(-ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6})\\&\quad = \mu(-ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5}) + \mu(a^6x_{a_6}),\end{aligned} \tag{2.5}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Adding (2.4) to (2.5), we obtain

$$\begin{aligned}&\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(-ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5}\\&\quad + a^6x_{a_6}) + \mu(ax_{a_1} - a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} - a^3x_{a_3} + a^4x_{a_4}\\&\quad + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} - a^4x_{a_4} + a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3}\\&\quad + a^4x_{a_4} - a^5x_{a_5} + a^6x_{a_6}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5} - a^6x_{a_6})\\&= 7\mu(ax_{a_1}) + 7\mu(a^2x_{a_2}) + 7\mu(a^3x_{a_3}) + 7\mu(a^4x_{a_4}) + 7\mu(a^5x_{a_5}) + 7\mu(a^6x_{a_6}) + \mu(-ax_{a_1}) + \mu(-a^2x_{a_2})\\&\quad + \mu(-a^3x_{a_3}) + \mu(-a^4x_{a_4}) + \mu(-a^5x_{a_5}) + \mu(-a^6x_{a_6}),\end{aligned} \tag{2.6}$$

$\forall x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Using the oddness of μ, and re-modifying (2.6) we arrive at our desired result (2.2). Conversely, substituting $(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6})$ by $(x_a, 0, 0, 0, 0, 0, 0)$, $(0, x_a, 0, 0, 0, 0)$, $\dots$, $(0, 0, 0, 0, 0, x_a)$, respectively in (2.6), we obtain

$$\mu(2x_a) = 2\mu(x_a) \quad \text{and} \quad \mu(3x_a) = 3\mu(x_a), \tag{2.7}$$

for all $x_a \in \chi$. One can easily verify from (2.7) that

$$\mu\left(\frac{x_a}{n}\right) = \frac{1}{n}\mu(x_a), \quad n = 1, 2, 3, 4, 5, 6, \tag{2.8}$$

for all $x_a \in \chi$. Substituting $(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6})$ by $\left(\frac{x_a}{1}, \frac{y_a}{2}, 0, 0, 0, 0\right)$ in (2.6) and using oddness of μ and (2.8), we arrive at our result. □

Remark 2.2. If $\mu : \chi \to \psi$ is an odd mapping that satisfies the functional equation

$$\mu(x_a + y_a) = \mu(x_a) + \mu(y_a)$$

for all $x_a, y_a \in \chi$, if and only if $\mu : \chi \to \psi$ satisfies the functional equation

$$\begin{aligned}&\mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5}) + \mu(-ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5})\\&\quad + \mu(ax_{a_1} - a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5}) + \mu(ax_{a_1} + a^2x_{a_2} - a^3x_{a_3} + a^4x_{a_4} + a^5x_{a_5})\\&\quad + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} - a^4x_{a_4} + a^5x_{a_5}) + \mu(ax_{a_1} + a^2x_{a_2} + a^3x_{a_3} + a^4x_{a_4} - a^5x_{a_5})\\&\quad = 5\left(\mu(ax_{a_1}) + \mu(a^2x_{a_2}) + \mu(a^3x_{a_3}) + \mu(a^4x_{a_4}) + \mu(a^5x_{a_5})\right),\end{aligned}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5} \in \chi$.

Proof. The proof is similar to that of the Lemma 2.1. □

3. Stability results of (1.1): direct method

This section, we use the direct technique to discuss the Hyers-Ulam stability of the functional equation (1.1).

Theorem 3.1. *Let $\alpha : \chi^6 \to [0, \infty)$ be a function and $j \in \{-1, 1\}$ such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{kj}x_{a_1}, 2^{kj}x_{a_2}, 2^{kj}x_{a_3}, 2^{kj}x_{a_4}, 2^{kj}x_{a_5}, 2^{kj}x_{a_6})}{2^{kj}}$$

converges in R *and*

$$\lim_{k\to\infty} \frac{\alpha(2^{kj}x_{a_1}, 2^{kj}x_{a_2}, 2^{kj}x_{a_3}, 2^{kj}x_{a_4}, 2^{kj}x_{a_5}, 2^{kj}x_{a_6})}{2^{kj}} = 0,$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. *Let* $\mu_a : \chi \to \psi$ *be an odd function satisfying the inequality*

$$\|D\mu_a(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6})\| \leqslant \alpha(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6}), \tag{3.1}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$.*Then there exists a unique additive mapping* $A : \chi \to \psi$, *which satisfies* (1.1) *and*

$$\|\mu_a(x_a) - A(x_a)\| \leqslant \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x_a, 0, 0, 0, 0, 0)}{2^{kj}}, \tag{3.2}$$

for all $x_a \in \chi$. *The mapping* $A(x_a)$ *is defined by* $A(x_a) = \lim_{k\to\infty} \frac{\mu_a(2^{kj}x_a)}{2^{kj}}$, *for all* $x_a \in \chi$.

Proof. Assume that $j = 1$. Replacing$(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6})$ by $(x, 0, 0, 0, 0, 0)$ in (3.1) and using oddness of μ_a, we get

$$\|2\mu_a(2x_a) - 4\mu_a(x_a)\| \leqslant \alpha(x_a, 0, 0, 0, 0, 0), \tag{3.3}$$

for all $x_a \in \chi$. It follows from (3.3) that

$$\left\|\frac{\mu_a(2x_a)}{2} - \mu_a(x_a)\right\| \leqslant \frac{\alpha}{4}(x_a, 0, 0, 0, 0, 0), \tag{3.4}$$

for all $x_a \in \chi$. Applying x_a by $2x_a$ in (3.4) and dividing by 2, we obtain

$$\left\|\frac{\mu_a(4x_a)}{4} - \frac{\mu_a(2x_a)}{2}\right\| \leqslant \frac{\alpha}{8}(2x_a, 0, 0, 0, 0, 0), \tag{3.5}$$

for all $x_a \in \chi$. It follows from (3.4) and (3.5) that

$$\left\|\frac{\mu_a(4x_a)}{4} - \mu_a(x_a)\right\| \leqslant \frac{1}{4}\{\alpha(x_a, 0, 0, 0, 0, 0) + \frac{\alpha}{2}(2x_a, 0, 0, 0, 0, 0)\},$$

for all $x_a \in \chi$. Generalizing, we have

$$\left\|\mu_a(x_a) - \frac{\mu_a(2^k x_a)}{2^k}\right\| \leqslant \frac{1}{4}\sum_{k=0}^{n-1} \frac{\alpha(2^k x_a, 0, 0, 0, 0, 0)}{2^k} \leqslant \frac{1}{4}\sum_{k=0}^{\infty} \frac{\alpha(2^k x_a, 0, 0, 0, 0, 0)}{2^k}, \tag{3.6}$$

for all $x_a \in \chi$. In order to prove convergence of the sequence $\left\{\frac{\mu_a(2^k x_a)}{2^k}\right\}$, replacing x_a by $2^l x_a$ and dividing by 2^l, we obtain

$$\left\|\frac{\mu_a(2^l x_a)}{2^l} - \frac{\mu_a(2^{k+l} x_a)}{2^{k+l}}\right\| = \frac{1}{2^l}\left\|\mu_a(2^l x_a) - \frac{\mu_a(2^k 2^l x_a)}{2^k}\right\|$$

$$\leqslant \frac{1}{4}\sum_{k=0}^{1}\frac{\alpha(2^{k+l}x_a,0,0,0,0,0)}{2^{k+l}} \leqslant \frac{1}{4}\sum_{k=0}^{\infty}\frac{\alpha(2^{k+l}x_a,0,0,0,0,0)}{2^{k+l}} \to 0 \text{ as } l\to\infty,$$

for all $x_a \in x_a$. Hence the sequence $\{\frac{\mu_a(2^k x_a)}{2^k}\}$ is a Cauchy sequence. Since ψ is complete, there exists a mapping $A:\chi\to\psi$ such that $A(x_a)=\lim_{k\to\infty}\frac{\mu_a(2^k x_a)}{2^k}$, $\forall x_a\in\chi$. Letting $k\to\infty$ in (3.6), we see that (3.2) holds for all $x_a\in\chi$. To prove that A satisfies (1.1), replacing $(x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5},x_{a_6})$ by $(2^{kj}x_{a_1},2^{kj}x_{a_2},2^{kj}x_{a_3},2^{kj}x_{a_4},2^{kj}x_{a_5},2^{kj}x_{a_6})$ and dividing 2^k in (3.1), we obtain

$$\begin{aligned}&\frac{1}{2^k}\|D\mu_a(2^{kj}x_{a_1},2^{kj}x_{a_2},2^{kj}x_{a_3},2^{kj}x_{a_4},2^{kj}x_{a_5},2^{kj}x_{a_6})\|\\&\leqslant \frac{1}{2^k}\alpha(2^{kj}x_{a_1},2^{kj}x_{a_2},2^{kj}x_{a_3},2^{kj}x_{a_4},2^{kj}x_{a_5},2^{kj}x_{a_6}),\end{aligned}$$

for all $x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5},x_{a_6}\in\chi$. Letting $k\to\infty$ in the above inequality and using the definition of $A(x_a)$, we see that

$$DA(x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5},x_{a_6})=0.$$

Hence A satisfies (1.1) for all $x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5},x_{a_6}\in\chi$. To show that A is unique, let $B(x_a)$ be another additive mapping satisfying (1.1) and (3.2), then

$$\begin{aligned}\|A(x_a)-B(x_a)\| &\leqslant \frac{1}{2^l}\|A(2^l x_a)-\mu_a(2^l x_a)\|+\|\mu_a(2^l x_a)-B(2^l x_a)\|\\&\leqslant \frac{1}{4}\sum_{k=0}^{\infty}\frac{\alpha(2^{k+l}x_a,0,0,0,0,0)}{2^{k+l}} \to 0 \text{ as } l\to\infty,\end{aligned}$$

for all $x_a\in\chi$. Hence A is unique. Now, switching x_a by $\left(\frac{x_a}{2}\right)$ in (3.3), we get

$$\left\|2\mu_a(x_a)-4\mu_a\left(\frac{x_a}{2}\right)\right\| \leqslant \alpha\left(\frac{x_a}{2},0,0,0,0,0\right), \tag{3.7}$$

for all $x_a\in\chi$. It follows from (3.7) that

$$\left\|\mu_a(x_a)-2\mu_a\left(\frac{x_a}{2}\right)\right\| \leqslant \frac{1}{2}\alpha\left(\frac{x_a}{2},0,0,0,0,0\right),$$

for all $x_a\in\chi$. The rest of the proof is similar to that of $j=1$. Hence for $j=-1$ also the theorem is true. □

Remark 3.2. Let $j\in\{-1,1\}$ and $\alpha:\chi^5\to[0,\infty)$ be a function such that $\sum_{k=0}^{\infty}\frac{\alpha(2^{kj}x_{a_1},2^{kj}x_{a_2},2^{kj}x_{a_3},2^{kj}x_{a_4},2^{kj}x_{a_5})}{2^{kj}}$ converges in $\mathbb{R}$ and

$$\lim_{k\to\infty}\frac{\alpha(2^{kj}x_{a_1},2^{kj}x_{a_2},2^{kj}x_{a_3},2^{kj}x_{a_4},2^{kj}x_{a_5})}{2^{kj}}=0,$$

for all $x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5}\in\chi$. Let $\mu_a:\chi\to\psi$ be an odd function satisfying the inequality

$$\|D\mu_a(x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5})\| \leqslant \alpha(x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5}),$$

for all $x_{a_1},x_{a_2},x_{a_3},x_{a_4},x_{a_5}\in\chi$. There exists a unique additive mapping $A:\chi\to\psi$, which satisfies (1.1) and

$$\|\mu_a(x_a)-A(x_a)\| \leqslant \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha(2^{kj}x_a,0,0,0,0)}{2^{kj}},$$

for all $x_a\in\chi$. The mapping $A(x_a)$ is defined by $A(x_a)=\lim_{k\to\infty}\frac{\mu_a(2^{kj}x_a)}{2^{kj}}$, for all $x_a\in\chi$.

Corollary 3.3. *Let $\lambda, s \geqslant 0$ are real numbers and an odd function $\mu_a : \chi \to \psi$ satisfying the inequality*

$$\|D\mu_a(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6})\| \leqslant \begin{cases} \lambda, \\ \lambda(\|x_{a_1}\|^s + \|x_{a_2}\|^s + \|x_{a_3}\|^s + \|x_{a_4}\|^s + \|x_{a_5}\|^s + \|x_{a_6}\|^s), & s \neq 1, \\ \lambda\{\|x_{a_1}\|^s + \|x_{a_2}\|^s + \|x_{a_3}\|^s + \|x_{a_4}\|^s + \|x_{a_5}\|^s + \|x_{a_6}\|^s) \\ \quad +(\|x_{a_1}\|^{6s}, \|x_{a_2}\|^{6s}, \|x_{a_3}\|^{6s}, \|x_{a_4}\|^{6s}, \|x_{a_5}\|^{6s}, \|x_{a_6}\|^{6s}))\}, & s \neq \frac{1}{6}, \end{cases}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5}, x_{a_6} \in \chi$. Then there exists unique additive function $A : \chi \to \psi$ such that

$$\|\mu_a(x_a) - A(x_a)\| \leqslant \begin{cases} \frac{\lambda}{2}, \\ \frac{\lambda\|x_a\|^s}{2|2-2^s|}, \\ \frac{\lambda\|x_a\|^{6s}}{2|2-2^{6s}|}, \end{cases}$$

for all $x_a \in \chi$.

Remark 3.4. Let $\lambda, s \geqslant 0$ are real numbers and an odd function $\mu_a : \chi \to \psi$ satisfying the inequality

$$\|D\mu_a(x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5})\| \leqslant \begin{cases} \lambda, \\ \lambda(\|x_{a_1}\|^s + \|x_{a_2}\|^s + \|x_{a_3}\|^s + \|x_{a_4}\|^s + \|x_{a_5}\|^s), & s \neq 1, \\ \lambda\{\|x_{a_1}\|^s + \|x_{a_2}\|^s + \|x_{a_3}\|^s + \|x_{a_4}\|^s + \|x_{a_5}\|^s) \\ \quad +(\|x_{a_1}\|^{5s}, \|x_{a_2}\|^{5s}, \|x_{a_3}\|^{5s}, \|x_{a_4}\|^{5s}, \|x_{a_5}\|^{5s}))\}, & s \neq \frac{1}{5}, \end{cases}$$

for all $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}, x_{a_5} \in \chi$. Then there exists unique additive function $A : \chi \to \psi$ such that

$$\|\mu_a(x_a) - A(x_a)\| \leqslant \begin{cases} \frac{\lambda}{2}, \\ \frac{\lambda\|x_a\|^s}{2|2-2^s|}, \\ \frac{\lambda\|x_a\|^{5s}}{2|2-2^{5s}|}, \end{cases}$$

for all $x_a \in \chi$.

4. Conclusion

In this study, we have explored the Hyers-Ulam stability of a-type additive functional equations in Banach spaces using the direct method. By establishing specific conditions under which the stability holds, we have demonstrated that these functional equations exhibit a robust form of stability. This stability is significant as it ensures that small deviations in the initial conditions of the functional equation do not lead to large discrepancies in the solutions, making the functional equation practically applicable in various mathematical and applied contexts. Our findings contribute to the broader understanding of functional equations in Banach spaces, offering a clear framework for analyzing their stability. The direct method employed here provides a straightforward and efficient approach, potentially serving as a useful tool for further research in this area. Future work could extend these results to more complex functional equations and explore the implications of Hyers-Ulam stability in different types of Banach spaces. Overall, this study reaffirms the importance of stability analysis in the study of functional equations and its role in ensuring the reliability and consistency of mathematical models.

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