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Stability and bifurcation analysis in a discrete-time logistic model under harvested and feedback control

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Abstract

We discuss a discrete-time logistic model with harvesting and feedback control. The discrete model is obtained by applying Euler's method to its continuous model. We first determine the equilibrium points, including their existence conditions and their local stability properties. We then apply the central manifold theorem and bifurcation theory to establish conditions for the existence of both period-doubling bifurcation and Neimark-Sacker bifurcation around the positive equilibrium point. Finally, we provide some numerical simulations to verify the feasibility of the theoretical results and demonstrate the complex dynamic behavior. Moreover, the presence of chaos in the system is justified numerically by the computed maximum Lyapunov exponent.

Keywords: Logistic model, feedback control, period-doubling bifurcation, Neimark-Sacker bifurcation, Lyapunov exponent, chaotic behavior.

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1. Introduction

One of the well-known models of single-species population is the logistic model

$$\frac{dx(t)}{dt} = x(t)\Big(1 - x(t)\Big), \tag{1.1}$$

where $x(t)$ denotes the size of population at time t. Here the intrinsic growth rate and carrying capacity are normalized to 1. Equation (1.1) has an unstable extinction equilibrium point $x^0 = 0$ and a unique positive equilibrium point $x^* = 1$ which is globally asymptotically stable [21]. It is also known that the environment can be subjected to a feedback control. Feedback control is a process in which a change in a particular component of an ecosystem necessarily generates a corresponding set of changes in other components, which in turn affects the component that was changed in the first place [29]. Species can be affected by negative feedbacks in their habitats, such as accumulation of toxic residues and anthropogenic control adaptations. Over the past decade, logistic model with feedback control has become one of the

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most important topics in population dynamics research. In this regard, Fan dan Wang [5] has discussed the global stability of a logistic model with feedback control. The influence of various types of Allee effects on the dynamics of a logistic model with feedback control has been studied in [14, 17, 33]. The generalization of feedback control on a logistic model has been introduced by Hoang [9], namely by introducing a control parameter such that the feedback can be positive or negative. Recently, logistic models with feedback control have also been considered at fractional orders [10, 11] where the authors have applied the Grünwald-Letnikov method in combination with a nonstandard finite difference scheme to solve fractional logic models.

In addition to feedback control, many scientists also studied the effects of harvesting. For instance, the impact of species harvesting on predator-prey interactions was studied in[2, 3, 18, 23, 25]. The effects of species harvesting have also been studied in single population models, please refer to [13, 15, 31, 32]. Very recently, the authors have introduced a harvested logistic model with feedback control and studied its dynamics [27]. In the normalized variable, the model is given by

$$\frac{dx(t)}{dt} = x(t)\Big(1 - x(t) - ay(t)\Big) - bx(t),\ x(0) \geqslant 0, \quad \frac{dy(t)}{dt} = -ey(t) + cx(t),\ y(0) \geqslant 0, \tag{1.2}$$

where a, c, and e are positive constant and b is the harvesting coefficient. It was shown in [27] that the model (1.2) has the following properties.

1. The model (1.2) always has an extinction point $E^0(x^0, y^0) = (0, 0)$, which is asymptotically stable if $b > 1$.
2. If $b < 1$, then besides the extinction point E^0, the model (1.2) has also a unique positive equilibrium $E^*(x^*, y^*)$, where
$$x^* = \frac{e(1-b)}{e+ac}, \quad y^* = \frac{c(1-b)}{e+ac}. \tag{1.3}$$
The positive equilibrium point is asymptotically stable only if $b < 1$.

We notice that the model (1.2) takes the form of a system of ordinary differential equations. This is typically used in populations with overlapping generations, i.e., where the reproductive process occurs continuously. However, in many species, generations do not overlap or are born during the normal breeding season. In this case, the dynamical populations can be described using difference equations or discrete time maps [16, 22]. Discrete models are well-known to exhibit richer dynamics than continuous models, see e.g. [1, 19, 20, 24, 30], and can also provide efficient computational models of continuous models for numerical simulations [8, 12, 28].

Motivated by the above discussion, this article considers the discrete-time version of (1.2). The discrete-time model is obtained by the forward Euler scheme, namely

$$x_{n+1} = x_n + h(1 - x_n - ay_n - b)x_n,\ x_0 \geqslant 0, \quad y_{n+1} = y_n + h(-ey_n + cx_n),\ y_0 \geqslant 0, \tag{1.4}$$

where $h > 0$ is the step-size of integration. Since the scheme (1.4) is derived using the forward Euler method, the convergence of this scheme is of first-order. Detailed proof of the convergence of error estimates can be seen in [6]. In this article, we aim to study the dynamics behavior of the discrete-time model (1.4), including the existence and stability properties of equilibrium points, as well as the period-doubling bifurcation and Neimark-Sacker bifurcation.

The organization of this article is as follows. In Section 2 we discuss the existence and stability of equilibrium points. We next show in Section 3 that under suitable parameters values, the model (1.4) undergoes period-doubling bifurcation and Neimark-Sacker bifurcation. To illustrate the analytical results, we present some numerical simulations in Section 4. A brief conclusion will be given in Section 5.

2. The existence and stability of equilibrium points

The equilibrium points of discrete-time model (1.4) can be determined by solving the following equation

$$x = x + h(1 - x - ay - b)x, \quad y = y + h(-ey + cx).$$

It is easy to show that model (1.4) has always an extinction equilibrium point $E^0(x^0, y^0) = (0, 0)$. Moreover, if $b < 1$, then the discrete-time model (1.4) has, additionally, a unique positive equilibrium point $E^*(x^*, y^*)$, where x^* and y^* satisfy (1.3). Thus, the discrete-time model (1.4) has exactly the same equilibrium points as those of continuous model (1.2).

Based on the stability theory, the local stability of an equilibrium point $(\hat{x}, \hat{y})$ is determined by the eigenvalues of the Jacobian matrix. The Jacobian matrix of (1.2) at the equilibrium point $(\hat{x}, \hat{y})$ is

$$J(\hat{x}, \hat{y}) = \begin{bmatrix} 1 + h(1 - 2\hat{x} - a\hat{y} - b) & -ah\hat{x} \\ ch & 1 - eh \end{bmatrix}.$$

We know that if μ_1 and μ_2 are eigenvalues of $J(\hat{x}, \hat{y})$, then $(\hat{x}, \hat{y})$ is asymptotically stable if $\max\{|\mu_1|, |\mu_2|\} < 1$. The equilibrium point $(\hat{x}, \hat{y})$ is unstable when $\max\{|\mu_1|, |\mu_2|\} > 1$. Moreover, the equilibrium point $(\hat{x}, \hat{y})$ is non-hyperbolic if either $|\mu_1| = 1$ or $|\mu_2| = 1$. The local stability analysis for each equilibrium point is shown below.

If the Jacobian matrix is evaluated at the extinction point $E^0(x^0, y^0)$, then we have

$$J(x^0, y^0) = \begin{bmatrix} 1 + (1 - b)h & 0 \\ ch & 1 - eh \end{bmatrix}.$$

The eigenvalues of $J(x^0, y^0)$ are $\mu_1 = 1 + (1 - b)h$ and $\mu_2 = 1 - eh$. Then, the stability properties of E^0 can be summarized as in the following theorem.

Theorem 2.1. *The extinction point* $E^0(x^0, y^0)$ *is*

1. *non-hyperbolic if* $b = 1$ *or* $h = \frac{2}{e}$;
2. *asymptotically stable if* $b > 1$ *and* $h < \min\{\frac{2}{b-1}, \frac{2}{e}\}$;
3. *unstable if either*
   (a) $b < 1$ *or* $h > \frac{2}{e}$; *or*
   (b) $b > 1$ *and* $h > \frac{2}{b-1}$.

The Jacobian matrix evaluated at the positive equilibrium point $E^*(x^*, y^*)$ is given by

$$J(x^*, y^*) = \begin{bmatrix} 1 - hx^* & -ahx^* \\ ch & 1 - eh \end{bmatrix}. \tag{2.1}$$

The characteristic equation of (2.1) can be written as

$$F(\mu) = \mu^2 - \text{tr}(J((x^*, y^*)))\mu + \det(J(x^*, y^*)) = 0, \tag{2.2}$$

where

$$\text{tr}(J((x^*, y^*))) = 2 - (e + x^*)h,$$
$$\det(J(x^*, y^*)) = (1 - eh)(1 - hx^*) + ach^2x^* = (ac + e)x^*h^2 - (e + x^*)h + 1 = e(1 - b)h^2 - (e + x^*)h + 1.$$

It can be shown that $F(1) = (e + ac)x^*h^2 > 0$. Hence, to analyze the eigenvalues of matrix (2.1), we apply the following lemma.

Lemma 2.2 ([4]). *Consider the characteristic equation* (2.2), $F(\mu) = 0$, *where* $F(1) > 0$. *Let* μ_1 *and* μ_2 *be two roots of* $F(\mu) = 0$. *Then*

1. $|\mu_1| < 1$ *and* $|\mu_2| < 1$ *if and only if* $F(0) < 1$ *and* $F(-1) > 0$;
2. $|\mu_1| < 1$ *and* $|\mu_2| > 1$ *(or* $|\mu_1| > 1$ *and* $|\mu_2| < 1$ *) if and only if* $F(-1) < 0$;
3. $|\mu_1| > 1$ *and* $|\mu_2| > 1$ *if and only if* $F(0) > 1$ *and* $F(-1) > 0$;
4. $\mu_1 = -1$ *and* $|\mu_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $-\text{tr}(J((x^*, y^*))) \neq 0, 2$;
5. μ_1 *and* μ_2 *are complex and* $|\mu_1| = |\mu_2| = 1$ *if and only if* $\text{tr}(J((x^*, y^*)))^2 - 4\det(J((x^*, y^*))) < 0$ *and* $F(0) = 1$.

Based on Lemma 2.2, it is seen that the properties of roots of (2.2) depend on $F(0)$ and $F(-1)$. It can be verified that $F(0) = J(x^*, y^*)$, where

1. $F(0) = 1$ if $h = \frac{e+x^*}{e(1-b)}$;
2. $F(0) > 1$ if $h > \frac{e+x^*}{e(1-b)}$;
3. $F(0) < 1$ if $h < \frac{e+x^*}{e(1-b)}$.

The value of $F(\mu)$ evaluated at $\mu = -1$ can be written as $F(-1) = (ac + e)x^*h^2 - 2(e + x^*)h + 4$. Let $\Delta = (e - x^*)^2 - 4acx^*$, $\hat{h} = \frac{e+x^*}{e(1-b)}$, $h_1 = \frac{(e+x^*)-\sqrt{\Delta}}{e(1-b)}$, and $h_2 = \frac{(e+x^*)+\sqrt{\Delta}}{e(1-b)}$, then we have

1. if $\Delta = 0$, then
   (a) $F(-1) = 0$ if $h = \hat{h}$;
   (b) $F(-1) > 0$ if $h \neq \hat{h}$;
2. if $\Delta < 0$, then $F(-1) > 0$;
3. if $\Delta > 0$, then
   (a) $F(-1) < 0$ if $h_1 < h < h_2$;
   (b) $F(-1) > 0$ if $h < h_1$ or $h > h_2$;
   (c) $F(-1) = 0$ if $h = h_1$ or $h = h_2$.

Using Lemma 2.2 and the results of discussion above, the stability of positive equilibrium point E^* can be stated as in the following theorem.

Theorem 2.3. *The stability properties of the positive equilibrium point* $E^*(x^*, y^*)$ *is as follows.*

1. *If* $\Delta > 0$, *then*
   (a) E^* *is asymptotically stable (sink) if* $0 < h < h_1$;
   (b) E^* *is a source point (unstable) if* $h > h_2$;
   (c) E^* *is a saddle (unstable) if* $h_1 < h < h_2$;
   (d) E^* *is a non-hyperbolic if* $h = h_1$ *or* $h = h_2$.
2. *If* $\Delta \leqslant 0$, *then*
   (a) E^* *is asymptotically stable (sink) if* $0 < h < \hat{h}$;
   (b) E^* *is a source point (unstable) if* $h > \hat{h}$;
   (c) E^* *is a non-hyperbolic if* $h = h_1$ *or* $h = \hat{h}$.

Remark 2.4. In Section 1 we mentioned that the extinction point E^0 (the positive equilibrium E^*) of the continuous-time model (1.2) is asymptotically stable if $b > 1$ ($b < 1$). The stability of the extinction point E^0 (the positive equilibrium E^*) of the discrete-time model (1.4) also requires that $b > 1$ ($b < 1$). Note that if $b > 1$, then $\hat{h} < 0$ and $h_1 < 0$, which means that E^* will not be stable. But the dependence on parameter b is not the only condition for both equilibrium points to be stable. This is because the stability of the two equilibrium points also requires a relatively small value of h. Moreover, as shown in the following Section, the use of relatively large h may lead to a period-doubling bifurcation or Neimark-Sacker bifurcation. This shows that the discrete-time model (1.4) has richer dynamics than its continuous version (1.2).

3. Bifurcation analysis

3.1. Period-doubling bifurcation

Let $\Omega_j = \Big\{(a, b, c, e, h) \mid \Delta > 0, h = h_j, a > 0, b > 0, c > 0, e > 0\Big\}, j = 1, 2$. Based on the previous analysis, it is found that for all parameters in Ω_j, one of eigenvalues of the Jacobian matrix at the positive equilibrium E^* is $\mu_1 = -1$, and the other eigenvalue (μ_2) is neither $\mu_2 = 1$ nor $\mu_2 = -1$. Then, the positive equilibrium E^* may undergo period-doubling bifurcation. Hence, we discuss the period-doubling bifurcation of discrete-time model (1.4) at E^* when parameter changes in the small neighborhood of Ω_1. For values of parameters in Ω_2, it can be investigated analogously.

By taking any parameters $(a, b, c, e, h) \in \Omega_1$, we have $h = h_1$. By choosing h as the bifurcation parameter, we introduce a small perturbation $\bar{h}$ into h such that $h = h_1 + \bar{h}$, where $|\bar{h}| \ll 1$. Next we transform the positive equilibrium point $E^*(x^*, y^*)$ into the origin by taking $u_n = x_n - x^*$ and $v_n = y_n - y^*$. The model (1.4) can be now turned into

$$\begin{aligned} u_{n+1} &= u_n + (h_1 + \bar{h})(1 - (u_n + x^*) - a(v_n + y^*) - b)(u_n + x^*) \\ &= \alpha_{11}u_n + \alpha_{12}v_n + \alpha_{13}u_nv_n + \alpha_{14}u_n^2 + \beta_1u_n\bar{h} + \beta_2v_n\bar{h} + \beta_3u_nv_n\bar{h} + \beta_4u_n^2\bar{h}, \\ v_{n+1} &= v_n + (h_1 + \bar{h})(-e(v_n + y^*) + c(u_n + x^*)) = f_2(u_n, v_n, \bar{h}) = \alpha_{21}u_n + \alpha_{22} + c_1u_n\bar{h} + c_2v_n\bar{h}, \end{aligned} \tag{3.1}$$

where $\alpha_{11} = (1 - h_1x^*)$, $\alpha_{12} = -ah_1x^*$, $\alpha_{13} = -ah_1$, $\alpha_{14} = -h_1$, $\beta_1 = -x^*$, $\beta_2 = -ax^*$, $\beta_3 = -a$, $\beta_4 = -1$, $\alpha_{21} = ch_1$, $\alpha_{22} = (1 - eh_1)$, $c_1 = c$, and $c_2 = e$. System (3.1) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = A \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} f_1(u_n, v_n, \bar{h}) \\ f_2(u_n, v_n, \bar{h}) \end{bmatrix}, \tag{3.2}$$

where $f_1(u_n, v_n, \bar{h}) = \alpha_{13}u_nv_n + \alpha_{14}u_n^2 + \beta_1u_n\bar{h} + \beta_2v_n\bar{h} + \beta_3u_nv_n\bar{h} + \beta_4u_n^2\bar{h}$, $f_2(u_n, v_n, \bar{h}) = c_1u_n\bar{h} + c_2v_n\bar{h}$, and

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

The eigenvalues of A are $\mu_1 = -1$ and $\mu_2 = 3 - h_1(e + x^*) \neq 0$ with the corresponding eigenvectors are, respectively, $\vec{v}_1 = \begin{bmatrix} \alpha_{12} \\ -1 - \alpha_{11} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \alpha_{12} \\ \mu_2 - \alpha_{11} \end{bmatrix}$. Hence, by applying the translation

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = P \begin{bmatrix} U_n \\ V_n \end{bmatrix}, \quad \text{with } P = \begin{bmatrix} \alpha_{12} & \alpha_{12} \\ -1 - \alpha_{11} & \mu_2 - \alpha_{11} \end{bmatrix},$$

system (3.2) can be transformed into

$$\begin{bmatrix} U_{n+1} \\ V_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} + \begin{bmatrix} \tilde{f}_1(u_n, v_n, \bar{h}) \\ \tilde{f}_2(u_n, v_n, \bar{h}) \end{bmatrix}, \tag{3.3}$$

where $u_n = \alpha_{12}(U_n + V_n), v_n = -(1 + \alpha_{11})U_n + (\mu_2 - \alpha_{11})V_n$, and

$$\begin{aligned} \tilde{f}_1(u_n, v_n, \bar{h}) &= \frac{\mu_2 - \alpha_{11}}{\alpha_{12}(1 + \mu_2)} f_1(u_n, v_n, \bar{h}) - \frac{1}{(1 + \mu_2)} f_2(u_n, v_n, \bar{h}), \\ \tilde{f}_2(u_n, v_n, \bar{h}) &= \frac{1 + \alpha_{11}}{\alpha_{12}(1 + \mu_2)} f_1(u_n, v_n, \bar{h}) + \frac{1}{(1 + \mu_2)} f_2(u_n, v_n, \bar{h}). \end{aligned}$$

Then, there exists a center manifold $W^c(0, 0)$ of system (3.3) at the equilibrium point $(0, 0)$ in a small neighborhood of $\bar{h} = 0$. The center manifold $W^c(0, 0)$ can be represented as

$$W^c(0, 0) = \Big\{(U_n, V_n, \bar{h}) \in \mathbb{R}^3 \mid V_n = \varphi(U_n, \bar{h}) = \alpha_1U_n^2 + \alpha_2U_n\bar{h} + \alpha_3\bar{h}^2 + \mathcal{O}((|U_n| + |V_n|)^3)\Big\},$$

where

$$\alpha_1 = \frac{(1+\alpha_{11})(\alpha_{12}\alpha_{14}-\alpha_{13}(1+\alpha_{11}))}{1-\mu_2^2}, \quad \alpha_2 = \frac{-\beta_1(1+\alpha_{11})-c_1\alpha_{12}+c_2(1+\alpha_{11})}{(1+\mu_2)^2} + \frac{\beta_2(1+\alpha_{11})}{\alpha_{12}(1+\mu_2)^2}, \quad \alpha_3 = 0.$$

The center maniford must satisfy

$$\varphi(-U_n+\tilde{f}_1(U_n,\varphi(U_n,\bar{h}),\bar{h}),\bar{h}) - \mu_2\varphi(U_n,\bar{h}) - \tilde{f}_2(U_n,\varphi(U_n,\bar{h}),\bar{h}) = 0.$$

Now, system (3.3) restricted to the center manifold $W^c(0,0)$ can be written as

$$\psi(U_n,\bar{h}) = -U_n + \psi_1 U_n^2 + \psi_2 U_n\bar{h} + \psi_3 U_n^2\bar{h} + \psi_4 U_n\bar{h}^2 + \psi_5 U_n^3 + \mathcal{O}((|U_n|+|\bar{h}|)^4), \tag{3.4}$$

where

$$\begin{aligned}
\psi_1 &= \frac{\mu_2-\alpha_{11}}{1+\mu_2}\Big(\alpha_{12}\alpha_{14}-\alpha_{13}(1+\alpha_{11})\Big),\\
\psi_2 &= \frac{\mu_2-\alpha_{11}}{\alpha_{12}(1+\mu_2)}\Big(\beta_1\alpha_{12}-\beta_2(1+\alpha_{11})\Big) - \frac{\alpha_{12}c_1-c_2(1+\alpha_{11})}{1+\mu_2},\\
\psi_3 &= \frac{\mu_2-\alpha_{11}}{\alpha_{12}(1+\mu_2)}\Big(\alpha_{13}\alpha_{12}\alpha_2(\mu_2-1-2\alpha_{11}) + 2\alpha_{14}\alpha_{12}^2\alpha_2 + \beta_1\alpha_1\alpha_{12}\\
&\quad + \beta_2(\mu_2-\alpha_{11})\alpha_1 - \beta_3\alpha_{12}(1+\alpha_{11}) + \beta_4\alpha_{12}^2\Big) - \frac{\alpha_1}{1+\mu_2}(c_1\alpha_{12}+c_2(\mu_2-\alpha_{11})),\\
\psi_4 &= \frac{\mu_2-\alpha_{11}}{\alpha_{12}(1+\mu_2)}\Big(\alpha_2\beta_2(\mu_2-\alpha_{11}) + \alpha_{12}\alpha_2\beta_1 - c_1\alpha_2\alpha_{12}\Big) - \frac{\alpha_2\alpha_{12}c_2}{1+\mu_2},\\
\psi_5 &= \frac{\mu_2-\alpha_{11}}{1+\mu_2}\Big(\alpha_1\alpha_{13}(\mu_2-\alpha_{11}) - (\alpha_1\alpha_{13}(1+\alpha_{11}) + 2\alpha_1\alpha_{12}\alpha_{14})\Big).
\end{aligned}$$

System (3.4) undergoes a period-doubling bifurcation if the two discriminatory quantities χ_1 and χ_2 are not zero, where

$$\chi_1 = \left(\frac{\partial^2\psi}{\partial U_n\partial\bar{h}} + \frac{1}{2}\frac{\partial\psi}{\partial\bar{h}}\frac{\partial^2\psi}{\partial U_n^2}\right)\Big|_{(0,0)} = \psi_2, \qquad \chi_2 = \left(\frac{1}{6}\frac{\partial^3\psi}{\partial U_n^3} + \left(\frac{1}{2}\frac{\partial^2\psi}{\partial U_n^2}\right)^2\right)\Big|_{(0,0)} = \psi_1^2 + \psi_5.$$

By direct calculation we obtain that

$$\chi_1 = \frac{1}{h_1(1+\mu_2)}\Big((ac-e)h_1x^* - 4\Big) \neq 0.$$

Based on the above results and using the theorem in [7, 26], we get the following theorem.

Theorem 3.1. *If $\chi_2 \neq 0$, then system (1.4) undergoes a period-doubling bifurcation at the equilibrium point $E^* = (x^*, y^*)$, when the parameter $\bar{h}$ varies in a small neighborhood of the origin. Furthermore, if $\chi_2 > 0$ (respectively, $\chi_2 < 0$), the period-2 points are stable (respectively, unstable).*

3.2. Neimark-Sacker bifurcation

From the previous section, we know that for all parameters in Ω, where

$$\Omega = \Big\{(a,b,c,e,h) \mid \Delta < 0, h = \hat{h}, a > 0, b > 0, c > 0, e > 0\Big\},$$

the Jacobian matrix at the positive equilibrium $E^* = (x^*, y^*)$ has two complex conjugate roots with modulus one. Thus, the system (1.4) will undergoes a Neimark-Sacker bifurcation if parameters vary in the small neighborhood of Ω. To investigate the Neimark-Sacker bifurcation with the bifurcation parameter

h, we introduce a small perturbation $h^* \ll 1$ into the parameter h around $h = \hat{h}$ such that system (1.4) leads to

$$x_{n+1} = x_n + (\hat{h} + h^*)(1 - x_n - a y_n - b)x_n, \quad y_{n+1} = y_n + (\hat{h} + h^*)(-e y_n + c x_n). \tag{3.5}$$

By assuming $u_n = x_n - x^*$ and $v_n = y_n - y^*$, the positive equilibrium point E^* is translated to the origin $(0, 0)$, and system (3.5) can be written as

$$\begin{aligned} u_{n+1} &= u_n + (\hat{h} + h^*)(1 - (u_n + x^*) - a(v_n + y^*) - b)(u_n + x^*), \\ v_{n+1} &= v_n + (\hat{h} + h^*)(-e(v_n + y^*) + c(u_n + x^*)). \end{aligned} \tag{3.6}$$

It can be easily demonstrated that the linearized system of (3.6) at $(u_n, v_n) = (0, 0)$ has a characteristic equation

$$\mu^2 + p(h^*)\mu + q(h^*) = 0,$$

where

$$p(h^*) = (e + x^*)(\hat{h} + h^*) - 2, \quad q(h^*) = (1 - x^*(\hat{h} + h^*))(1 - e(\hat{h} + h^*)) + acx^*(\hat{h} + h^*)^2.$$

The corresponding characteristic roots are

$$\mu_{1,2} = \frac{1}{2}(-p(h^*) \pm \sqrt{p(h^*)^2 - 4q(h^*)}).$$

After some algebraic manipulation, we can show that

$$|\mu_{1,2}| = q(h^*)^{\frac{1}{2}}, \quad \frac{d|\mu_{1,2}|}{dh^*}\Big|_{h^*=0} = \frac{e + x^*}{2} > 0.$$

Moreover, we also require that when $h^* = 0$, $\mu_{1,2}^m \neq 1$, $m = 1, 2, 3, 4$, which is equivalent to $p(0) \neq -2, 0, 1, 2$. We notice that for $(a, b, c, e, h) \in \Omega$, we have $p(0)^2 - 4q(0) < 0$ or $p(0)^2 < 4q(0) = 4$. It is now seen that $p(0) \neq \pm 2$. Hence, we only need that $p(0) \neq 0, 1$. It is satisfied by condition

$$\frac{(e + x^*)^2}{e(1 - b)} \neq 2, 3. \tag{3.7}$$

We next consider the normal form of (3.6) when $h^* = 0$. Let $\sigma = \mathrm{Re}(\mu_{1,2}|_{h^*=0})$ and $\gamma = \mathrm{Im}(\mu_{1,2}|_{h^*=0})$, namely

$$\sigma = -\frac{p(0)}{2}, \quad \gamma = \frac{\sqrt{4q(0) - p(0)^2}}{2}.$$

By applying the translation

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = Q \begin{bmatrix} U_n \\ V_n \end{bmatrix},$$

where Q is an invertible matrix defined by

$$Q = \begin{bmatrix} 0 & 1 \\ \gamma & \sigma \end{bmatrix},$$

system (3.5) can be transformed into

$$\begin{bmatrix} U_{n+1} \\ V_{n+1} \end{bmatrix} = \begin{bmatrix} \sigma & -\gamma \\ \gamma & \sigma \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} + \begin{bmatrix} \tilde{g}_1(U_n, V_n) \\ \tilde{g}_2(U_n, V_n) \end{bmatrix},$$

where

$$\tilde{g}_1(U_n, V_n) = \frac{\sigma\hat{h}}{\gamma}(a\gamma U_n V_n + (a\sigma+1)V_n^2), \quad \tilde{g}_1(U_n, V_n) = -\hat{h}(a\gamma U_n V_n + (a\sigma+1)V_n^2). \tag{3.8}$$

From (3.8), we have that

$$\tilde{g}_{1U_nU_n} = 0, \quad \tilde{g}_{1U_nV_n} = a\sigma\hat{h}, \quad \tilde{g}_{1V_nV_n} = \frac{2\sigma(a\sigma+1)\hat{h}}{\gamma}, \quad \tilde{g}_{1U_nU_nU_n} = \tilde{g}_{1V_nV_nV_n} = \tilde{g}_{1U_nU_nV_n} = \tilde{g}_{1U_nV_nV_n} = 0,$$
$$\tilde{g}_{2U_nU_n} = 0, \quad \tilde{g}_{2U_nV_n} = -a\gamma\hat{h}, \quad \tilde{g}_{2V_nV_n} = -2(a\sigma+1)\hat{h}, \quad \tilde{g}_{2U_nU_nU_n} = \tilde{g}_{2V_nV_nV_n} = \tilde{g}_{2U_nU_nV_n} = \tilde{g}_{2U_nV_nV_n} = 0.$$

System (1.4) undergoes a Neimark-Sacker bifurcation if the discriminatory quantity χ^* is not zero

$$\chi^* = -\text{Re}\left(\frac{(1-2\bar{\mu})\bar{\mu}^2}{1-\mu}\zeta_{11}\zeta_{20}\right) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + \text{Re}(\bar{\mu}\zeta_{21}),$$

where $\mu = \sigma + i\gamma$, $\bar{\mu} = \sigma - i\gamma$, and

$$\begin{aligned}
\zeta_{11} &= \frac{1}{4}\left[\tilde{g}_{1U_nU_n} + \tilde{g}_{1V_nV_n} + i(\tilde{g}_{2U_nU_n} + \tilde{g}_{2V_nV_n})\right],\\
\zeta_{20} &= \frac{1}{8}\left[\tilde{g}_{1U_nU_n} - \tilde{g}_{1V_nV_n} + 2\tilde{g}_{2U_nV_n} + i(\tilde{g}_{2U_nU_n} - \tilde{g}_{2V_nV_n} - 2\tilde{g}_{1U_nV_n})\right],\\
\zeta_{02} &= \frac{1}{8}\left[\tilde{g}_{1U_nU_n} - \tilde{g}_{1V_nV_n} + 2\tilde{g}_{2U_nV_n} + i(\tilde{g}_{2U_nU_n} - \tilde{g}_{2V_nV_n} + 2\tilde{g}_{1U_nV_n})\right],\\
\zeta_{21} &= \frac{1}{16}\Big[\tilde{g}_{1U_nU_nU_n} + \tilde{g}_{1U_nV_nV_n} + \tilde{g}_{2U_nU_nV_n} + \tilde{g}_{2V_nV_nV_n}\\
&\qquad + i(\tilde{g}_{2U_nU_nU_n} + \tilde{g}_{2U_nV_nV_n} - \tilde{g}_{1U_nU_nV_n} - \tilde{g}_{1V_nV_nV_n})\Big].
\end{aligned}$$

By direct calculation, we can show that

$$\zeta_{11} = \frac{1}{4}\left(\frac{2\hat{h}\sigma(a\sigma+1)}{\gamma} + 2\hat{h}(a\sigma+1)i\right), \qquad \zeta_{21} = 0,$$
$$\zeta_{20} = \frac{1}{8}\left(-\frac{2\hat{h}}{\gamma}(\sigma(a\sigma+1) + a\gamma^2) + 2\hat{h}i\right), \qquad \zeta_{02} = \frac{1}{8}\left(-\frac{2\hat{h}}{\gamma}(\sigma(a\sigma+1) + a\gamma^2) + 2\hat{h}(1+2a\sigma)i\right),$$
$$|\zeta_{11}|^2 = \frac{1}{4\gamma^2}\hat{h}^2(\sigma^2+\gamma^2)(a\sigma+1)^2, \qquad |\zeta_{02}|^2 = \frac{1\hat{h}^2}{16}\left(\frac{(\sigma(a\sigma+1)+a\gamma^2)^2}{\gamma^2} + (1+2a\sigma)^2\right),$$
$$\zeta_{11}\zeta_{20} = -\frac{\hat{h}^2(a\sigma+1)}{8}\left(\frac{\sigma}{\gamma^2}(\sigma(a\sigma+1)+a\gamma) + 1 + a(\sigma^2+\gamma^2)i\right),$$

and

$$\frac{(1-2\bar{\mu})\bar{\mu}^2}{1-\mu} = \frac{2(1-\sigma^2)(1+\sigma-4\sigma^2) + (1-\sigma)(-8\sigma^3+2\sigma^2+6\sigma-1)}{2(1-\sigma)} + \frac{\gamma(-16\sigma^3+12\sigma^2+6\sigma-3)}{2(1-\sigma)}i.$$

Using the results of above discussion and theorem in [7, 26], we get the following theorem.

Theorem 3.2. *If the condition (3.7) is satisfied and $\chi^* \neq 0$, then the discrete-time system (1.4) undergoes Neimark-Sacker bifurcation at the equilibrium point $E^* = (x^*, y^*)$, when the parameter h^* changes in the small neighborhood of the origin. Furthermore, if $\chi^* < 0$ (respectively $\chi^* > 0$), then an attracting (respectively, repelling) closed invariant curve bifurcates from E^* for $h^* > 0$ (respectively, $h^* < 0$).*

4. Numerical simulations

We perform numerical simulations to illustrate and confirm the previous analytical results. For the first simulation, we take values of parameters $a = 0.4, b = 0.01, c = 0.01, e = 0.1$, and vary the value of time-step in the range $2 \leqslant h \leqslant 3$. Under these parameters values, the discrete-time system (1.4) has a positive equilibrium point $E^*(0.9519, 0.0952)$. Our calculations lead to $h_1 = 2.11097, \Delta = 0.7105 > 0, \chi_1 = -0.0408 \neq 0$, and $\chi_2 = 3.0823$. Hence, our parameters $(a, b, c, e, h_1) \in \Omega_1$. Based on Theorem 2.3, the point $E^*(0.9519, 0.0952)$ is asymptotically stable for $h < h_1$ and loses its stability at $h = h_1$. Since $\chi_2 > 0$, Theorem 3.1 states that system (1.4) undergoes a period-doubling at $E^*(0.9519, 0.0952)$ driven by parameter h. Furthermore, the bifurcation point is at $h = h_1$ and the period-2 points are stable. This behavior is clearly seen from the bifurcation diagram with respect to parameter h in Figure 1 (a) and the associated maximum Lyapunov exponent in Figure 1 (b). Figure 1 shows the existence of period-doubling bifurcation in system (1.4). Here we observe that $E^*(0.9519, 0.0952)$ is stable if $h < h_1$, the period-2-points are stable, and there is a cascade of period-doubling. Moreover, we also observe that there is a range of parameter h in which the system (1.4) exhibits chaotic behavior. The appearance of chaotic dynamics is confirmed by the existence of positive maximum Lyapunov exponents in some range of h, see Figure 1 (b). Notice that this simulation is taken using initial values $x(0) = 0.5, y(0) = 0.2$. To see more detail, in Figure 2 we plot the phase portraits of (1.4) which correspond to Figure 1. Using $h = 2, 2.4, 2.65, 2.68$, the solutions of system (1.4) are convergent to the positive equilibrium point $E^*(0.9519, 0.0952)$, period-2 points, period-4 points, and period-8 points, respectively. If we take a larger value of h, we may have a solution with chaotic behavior; the plot is not shown in this article.

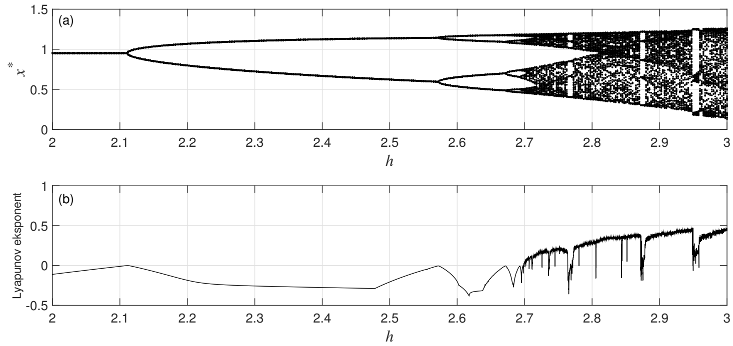

Figure 1: Bifurcation diagram and the associated maximum Lyapunov exponent of the system (1.4) with $a = 0.4$, $b = 0.01$, $c = 0.01$, $e = 0.1$, and h in the range $2 \leqslant h \leqslant 3$.

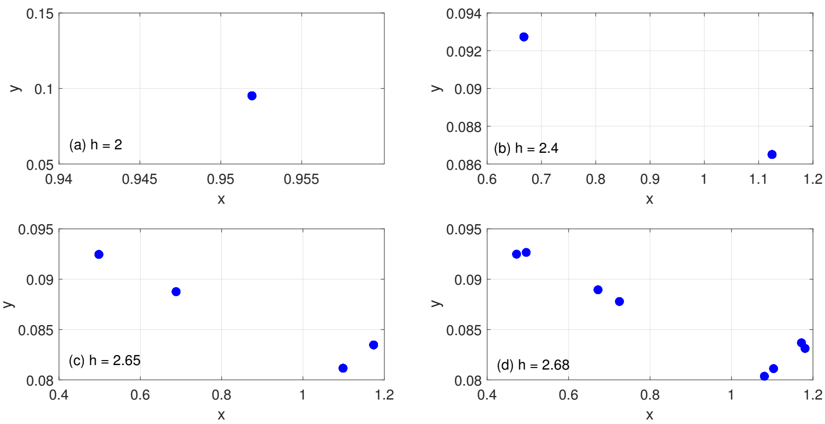

Figure 2: The phase portraits of system (1.4) with $a = 0.4$, $b = 0.01$, $c = 0.01$, $e = 0.1$, and (a) $h = 2$; (b) $h = 2.4$; (c) $h = 2.65$; and (d) $h = 2.68$.

In the second simulation, we apply parameters values $a = 0.8, b = 0.25, c = 0.5, e = 0.5$, and h in the range $2.3 \leqslant h \leqslant 2.7$. In this case, system (1.4) has a unique positive equilibrium point $E^*(0.4167, 0.4167)$. The equilibrium point $E^*(0.4167, 0.4167)$ is asymptotically stable for $h < \hat{h} = 2.4444$. We also have $\Delta = -0.6597$, and therefore our parameters $(a, b, c, e, \hat{h}) \in \Omega$. The eigenvalue of the linearized system (1.4) is complex and given by $\mu = -0.1204 + 0.9927i$, where $|\mu| = 1$. We can show that

$$\frac{(e + x^*)^2}{e(1-b)} = 2.2407 \neq 2, 3,$$

and thus the condition (3.7) is satisfied. Because $\chi^* = -0.1126$, Theorem 3.2 states that the system (1.4) undergoes Neimark-Sacker bifurcation around the equilibrium point $E^* = (0.4167, 0.4167)$ at $h = \hat{h} = 2.4444$. This behavior is confirmed by the bifurcation diagram depicted in Figure 3. For more details, we also plot in Figure 4 the phase portraits of system (1.4) associated with Figure 3 using four different values of h. It is seen from Figures 3 and 4 that when $h < \hat{h}$, then the solution is attracting to the equilibrium point $E^* = (0.4167, 0.4167)$, while if $h > \hat{h}$, then the solution is convergent to a closed invariant orbit.

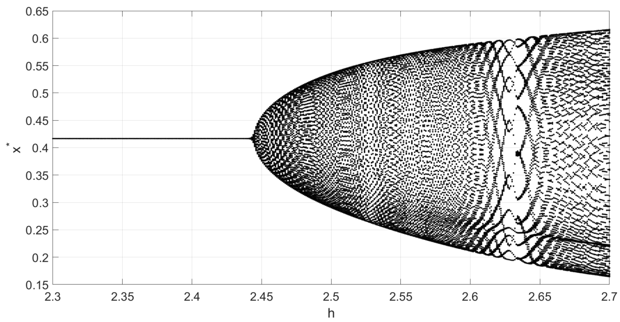

Figure 3: Bifurcation diagram of system (1.4) with $a = 0.8, b = 0.25, c = 0.5, e = 0.5$, and h in the range $2.3 \leqslant h \leqslant 2.7$.

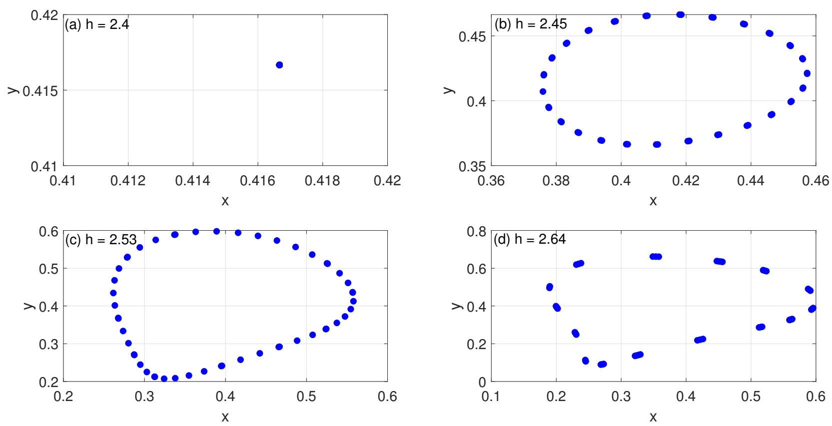

Figure 4: The phase portraits of system (1.4) with $a = 0.8$, $b = 0.25$, $c = 0.5$, $e = 0.5$, and (a) $h = 2.4$; (b) $h = 2.45$; (c) $h = 2.53$; and (d) $h = 2.64$.

5. Conclusions

A discrete-time harvested logistic model with feedback control has been constructed using the Euler method. It is proven that the proposed discrete-time model has only an extinction equilibrium point when the harvesting constant (b) is larger than or equal to one. However, if $b < 1$, then our discrete-time model also has a positive equilibrium point. The local stability of all equilibrium points is completely studied and is shown to depend not only on the model parameters but also on the time-step of numerical integration (h). We also prove analytically that the proposed discrete-time model may exhibit a period-doubling bifurcation as well as a Neimark-Sacker bifurcation. This dynamical behavior is confirmed by our numerical simulations. Furthermore, our numerical simulations also show that our discrete-time model may undergo to chaotic dynamic for suitable parameters values. Hence, the proposed discrete-time model certainly has richer dynamics than its continuous counterpart.

We have shown that harvesting has a significant effect on the dynamics of logistic model with feedback control. The proposed model can be used as a basis for developing models to describe various real-world situations, for purposes such as disease spread control and ecological and economic stability and sustainability. Additionally, we are able to describe the chaotic behavior that might occur in the proposed discrete-time system, suggesting its potential application in image encryption/decryption. These open problems will be the focus of our future works.

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