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Scaled consensus of hybrid multi-agent systems via impulsive protocols

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Abstract

This paper investigates scaled consensus problems of hybrid multi-agent systems(HMASs), which consist of a group of continuous-time(CT) and discrete-time(DT) dynamics agents. Three kinds of consensus protocols have been proposed to solve scaled consensus problems, respectively. The first two consensus protocols are designed for solving the scaled consensus problems where the CT-agents can only communicate with their neighbors in the sampling time t_k. In addition, the impulsive consensus protocols are designed for solving scaled consensus problems of HMASs. Finally, some numerical examples are provided to illustrate the effectiveness of the theoretical results.

Keywords: Hybrid multi-agent system, spanning tree, consensus problem, scaled consensus, impulsive control.

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1. Introduction

Multi-Agent Systems (MAS) refer to a group of autonomous agents that interact with each other to achieve common goals or tasks. These agents can be software entities, robots, humans, or any other intelligent entities capable of perception, decision-making, and action.

Over the past decades, multi-agent coordination problems encompass a wide range of challenges that arise when multiple autonomous agents need to work together to achieve common objectives. These problems can occur in various domains, including robotics, artificial intelligence, distributed systems, and social networks. Some common multi-agent coordination problems include: consensus, task allocation, cooperative control, formation control, resource sharing, path planning, synchronization problems (see [2, 4, 9, 11, 14]). In the recent years, consensus problems, which refer to the process of reaching an agreement among a group of agents regarding a certain quantity of interest that depends on the states of all agents involved, have been extensively studied since it can apply in various fields, including synchronization of coupled oscillators, flocking behavior, formation control, distributed sensor fusion in sensor networks, and belief propagation. The goal of consensus algorithms is to ensure that all agents converge to a common value or state, despite starting with potentially different initial values. This agreement is achieved through iterative interactions and communication between agents, leading to a collective decision or alignment on

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a specific outcome. As a result, many consensus protocols have been proposed to solve consensus problem (see examples in [12, 15, 18, 19]).

One of the most powerful consensus protocols is an impulsive consensus protocol, which is a control strategy used in multi-agent systems to achieve consensus among the agents' states through intermittent and instantaneous state updates at specific time instants, known as impulses. Unlike continuous control methods where information exchange and state adjustments occur continuously, impulsive control involves discrete-time updates that can lead to efficient and robust consensus in dynamic systems. At each impulse, agents communicate their current states to their neighbors and update their states based on the received information, allowing for rapid decision-making and coordination. Impulsive control methods can be particularly useful in various real-world scenarios where instantaneous state updates and intermittent communication between agents are more practical or efficient than continuous control. For example, in a network of robotic agents collaborating on a task, impulsive control can facilitate quick decision-making and coordination by allowing robots to exchange information and adjust their actions at specific time instants, leading to faster convergence to a common goal. In wireless sensor networks, impulsive control can help in achieving consensus among sensor nodes for tasks such as environmental monitoring or target tracking. Impulsive updates can reduce energy consumption and improve network efficiency. As a consequence, impulsive consensus protocols have been widely investigated in the recent years (see examples in [5, 10]).

In many real-world applications, systems may need to achieve consensus on ratios rather than absolute values. Examples include compartmental mass-action systems, water distribution networks, and coordination control between spacecraft and ground vehicles. In order to solve this problem, the scaled consensus was introduced by Roy [13]. Scaled consensus problems refer to a specific type of consensus problem in multi-agent systems where the objective is for the states of the agents to converge to prescribed ratios, rather than a common value as in traditional consensus problems. In scaled consensus, each agent aims to reach a specific ratio relative to the other agents in the system, leading to a more generalized form of consensus. In the recent years, scaled consensus problems have been widely investigated, for example see [1, 3, 5, 13, 16, 17]).

Motivated by the above discussion, this paper aims to investigate scaled consensus of hybrid multi-agent systems. Our main contributions are summarized as follows.

(1) Compared with the usual consensus problems focusing on reaching an agreement on a common quantity, the scaled consensus problem means that the state of each agent will converge to a prescribed ratio in the asymptote, which implies the generalization of consensus. Furthermore, by selecting appropriate scalar scales, the scaled consensus problem can solve the group consensus problems, bipartite consensus problems, etc.
(2) The impulsive control is a powerful control method, in particular, when an agent exchanges information instantaneously at discrete times. This work proposes the impulsive consensus protocols to solve scaled consensus problems.
(3) Sufficient and necessary conditions under directed fixed topology are derived, it is found that the scaled consensus in multi-agent systems can be reached if and only if the network is balanced and contains a spanning tree.

The rest of this paper is organized as follows. Some preliminaries and the problem formulation are provided in Section 2. In Section 3, the scaled consensus problems of hybrid multi-agent systems are solved under some necessary and sufficient conditions. In Section 4, numerical examples are provided to demonstrate the effectiveness of our main results. Finally, some conclusions are drawn in Section 5.

2. Preliminaries and problem formulations

2.1. Preliminaries

In this section, we review the basic concepts of algebraic graph, basic matrix theroy, some definitions and lemmas that will be used in this work. For more details, refer to [6, 8].

Throughout this paper, an interaction among n agents is described as a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that consists of a set of nodes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and a set of edges $\mathcal{E} = \{e_{ij} = (v_i, v_j)\} \subseteq \mathcal{V} \times \mathcal{V}$ and a nonnegative symmetric matrix $\mathcal{A} = [a_{ij}]_{n\times n}$. The set of all neighbours of an agent i is denoted by $N_i = \{k : a_{ik} > 0\}$. A vertex v_i corresponds to agent i. An edge of $\mathcal{G}$ is denoted by $(v_i, v_j) \in \mathcal{E} \iff (v_j, v_i) \in \mathcal{E}$. $\mathcal{A}$ is called the weight matrix and a_{ij} is the weight of edge (v_i, v_j) such that $a_{ij} > 0$ if and only if agents i and j are adjacent, i.e., they can communicate with each other. Moreover, $a_{ii} = 0, \forall i$. The degree of node v_i is denoted by $\deg(v_i) = d_i = \sum_{j=1}^{n} a_{ij}$, which is the number of edges that connect to v_i. Then $D = \mathrm{diag}\{d_1, \dots, d_n\}$ is the degree matrix of graph $\mathcal{G}$. The Laplacian matrix is $\mathcal{L} = D - A$. A path of $\mathcal{G}$ is a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}), \dots$ in a digraph $\mathcal{G}$. A graph $\mathcal{G}$ is called strongly connected if there is a path connecting any two arbitrary nodes in $\mathcal{G}$. Moreover, we denote by $\mathbb{R}$ the real number set, $\mathbb{N}$ the positive integer set, and $\mathbb{R}^n$ the n-dimensional real vector space. For a given vector or matrix X, X^T denotes its transpose and $\|X\|$ denotes the Euclidean norm of a vector X. A vector is nonnegative if all its elements are nonnegative and the column vector with all entries equal to one or zeroes are denoted by 1_n and 0_n, respectively. I_n is an n-dimensional identity matrix and the diagonal matrix with diagonal elements being $a_1, a_2, \dots, a_n$ is denoted by $\mathrm{diag}\{a_1, a_2, \dots, a_n\}$. Moereover, $[a_{ij}]_{n\times n}$ is an n by n matrix with a_{ij} representing its (i, j)th entry. A matrix $B = [b_{ij}]_{n\times n}$ is said to be nonnegative, denoted by $B \geqslant 0$, if all its entries are nonnegative. For the set of nonnegative matrices, we define an order as follows: if A and B are nonnegative matrices, then $A \geqslant B$ implies $A - B$ is a nonnegative matrix. A is a stochastic matrix if A is nonnegative and all its row sums are 1. A stochastic matrix P is called indecomposable and aperiodic (SIA) if there exists a column vector y such that $\lim_{n\to\infty} P^n = 1_n y^T$, where $1 = (1, 1, \dots, 1)^T$ is an $n \times 1$ vector.

Furthermore, some useful definitions, lemmas, and properties are provided as follows.

Definition 2.1 ([8]). For an undirected graph $\mathcal{G}$ with the Laplacian matrix $\mathcal{L}$ the algebraic connectivity is defined as $\lambda_2(\mathcal{L})$, the second smallest eigenvalue of $\mathcal{L}$,

$$\lambda_2(\mathcal{L}) = \min_{x\neq 0,\ 1^T x = 0} \frac{x^T \mathcal{L} x}{x^T x}.$$

Definition 2.2 ([8, Balanced graphs]). A node v_i of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is balanced if its in-degree and out-degree are equal. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is called balanced if all of its nodes are balanced

Definition 2.3 ([20, Mirror Graphs]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph. Let $\tilde{\mathcal{E}}$ be the set of reverse edges of $\mathcal{G}$ obtained by reversing the order of nodes of all the pair in $\mathcal{E}$. The mirror of $\mathcal{G}$ is denoted by $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ with the same set of nodes as $\mathcal{G}$, the set of edges $\hat{\mathcal{E}} = \mathcal{E} \bigcup \tilde{\mathcal{E}}$, and the symmetric adjacency matrix $\hat{\mathcal{A}} = [\hat{a}_{ij}]_{n\times n}$ with elements

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \geqslant 0.$$

Proposition 2.4 ([12]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a digraph with an adjacency matrix $\mathcal{A} = [a_{ij}]$ satisfying $a_{ii} = 0$, for all i. Then all the following statements are equivalent:*

i) *$\mathcal{G}$ is balanced;*
ii) *$1^T \mathcal{L} = 0$;*
iii) *$\sum_{i=1}^{n} u_i = 0$, $\forall x \in \mathbb{R}^n$ with $u = -\mathcal{L}x$.*

Lemma 2.5 ([12]). *Let $\mathcal{L}$ be the Laplacian matrix of a directed graph $\mathcal{G}$ and $\hat{\mathcal{G}}$ be the mirror graph of $\mathcal{G}$. Then $\hat{\mathcal{L}} = \mathrm{Sym}(\mathcal{L}) = (\mathcal{L} + \mathcal{L}^T)/2$ is a valid Laplacian matrix for $\hat{\mathcal{G}}$ if and only if $\mathcal{G}$ is balanced.*

Lemma 2.6 ([12]). *A stochastic matrix has algebraic multiplicity equal to one for eigenvalue $\lambda = 1$ if and only if the graph associated with matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.*

Lemma 2.7 ([12]). *Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{k\to\infty} A^k = \mathbf{1}_n y^T$, where y is nonnegative and satisfies $A^T y = y$, $\mathbf{1}_n^T y = 1$.*

Lemma 2.8. *Let $\mathcal{L}$ be the Laplacian matrix of a directed network $\mathcal{G}$ and $\beta_i \neq 0$ be a scalar scale of agent i. Define $\beta_{max} = \max_{1\leqslant i\leqslant n} |\beta_i|$, $H = \text{diag}\{h_1, h_2, \dots, h_n\}$ such that $0 < h_i < \frac{1}{d_{max}\beta_{max}}$, $i \in \mathcal{I}_n$, and $|\mathcal{B}| = \text{diag}(|\beta_1|, |\beta_2|, \dots, |\beta_n|)$. Then $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is SIA, i.e., $\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n y^T$ if and only if $\mathcal{G}$ has a spanning tree. Furthermore, $[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^T y = y$, $\mathbf{1}_n^T y = 1$, where each element of y is nonnegative.*

Proof.

Sufficiency. Since $0 < h < (d_{max}\beta_{max})^{-1}$, one obtains $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L} = (\mathbf{I}_n - H|\mathcal{B}|\mathcal{D}) + H|\mathcal{B}|\mathcal{A}$ is a stochastic matrix with positive diagonal entries, where $\mathcal{D} = \text{diag}(d_1, \dots, d_n)$ and $\mathcal{A}$ are the degree matrix and adjacency matrix of $\mathcal{G}$, respectively. Obviously, for all $i, j \in \mathcal{I}_n$; $i \neq j$, the (i, j)-th entry of $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is positive if and only if $a_{ij} > 0$. Then, $\mathcal{G}$ is the graph associated with $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$. Combining Lemmas 2.6 and 2.7, gives $\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n y^T$, when $\mathcal{G}$ has a spanning tree, where y is nonnegative vector. Moreover, y satisfies $[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^T y = y$, $\mathbf{1}_n^T y = 1$.

Necessity. From Lemma 2.6, if $\mathcal{G}$ does not have a spanning tree, the algebraic multiplicity of eigenvalue $\lambda = 1$ of $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is $m > 1$. Then, the rank of $\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k$ is not equal to 1, which is not equal to the rank of $\mathbf{1}_n y^T$. This implies that $\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k \neq \mathbf{1}_n y^T$. Therefore, $I_n - H|\mathcal{B}|\mathcal{L}$ is not SIA. This completes the proof. □

2.2. *Problem formulation*

In this work, we assume that the hybrid multi-agent system consists of N agents which are continuous-time and discrete-time dynamic agents, labelled 1 through N, where the number of continuous-time dynamic agents is M, $M < N$. Without loss of generality, we assume that agent 1 through M are continuous-time dynamic agents. Moreover, $\mathcal{I}_M = \{1, 2, 3, \dots, M\}$, $\mathcal{I}_N/\mathcal{I}_M = \{M+1, M+2, M+3, \dots, N\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = u_i(t), & \text{for } i \in \mathcal{I}_M, \\ x_l(t_{k+1}) = x_l(t_k) + u_l(t_k),\ t_k = kh, & \text{for } l \in \mathcal{I}_N/\mathcal{I}_M, \end{cases} \tag{2.1}$$

where h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$ and $x(0) = [x_{10}, x_{20}, \dots, x_{N0}]^T$.

Definition 2.9. Given any scalar scale $\beta_i \neq 0$ for the agent i, the hybrid multi-agent system (2.1) is said to reach scaled consensus to $(\beta_1, \dots, \beta_N)$ if for any initial conditions, we have

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0, \quad \text{for } i, j \in \mathcal{I}_N, \tag{2.2}$$

and

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_M. \tag{2.3}$$

Remark 2.10. If a scalar scale $\beta_i = 1$ for all i, then it is easy to see that the scaled consensus can reduce to the standard consensus, that is, it is more general than the standard consensus problems.

3. Consensus results

3.1. *Case I*

All agents communicate with their neighbours and update their control inputs in a sampling time t_k.

Then, the consensus protocol for hybrid multi-agent system (2.1) is defined as

$$\begin{cases} u_i(t) = \mathrm{sgn}(\beta_i)\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } t\in(t_k,t_{k+1}],\ i\in\mathcal{I}_M,\\ u_i(t_k) = h\cdot \mathrm{sgn}(\beta_i)\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i\in\mathcal{I}_N/\mathcal{I}_M, \end{cases} \tag{3.1}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$ and $h = h_i = t_{k+1} - t_k$ is the sampling period.

Theorem 3.1. *Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (2.1) and $\beta_i \neq 0$ be any scalar scale of agent i. Assume that $0 < h < \frac{1}{d_{max}\beta_{max}}$. Then, the hybrid multi-agent system (2.1) under the protocol (3.1) reaches scaled consensus to $(\beta_1,\dots,\beta_N)$ if and only if $\mathcal{G}$ has a spanning tree.*

Proof.

Sufficiency. Let $\beta_i \neq 0$ be any scalar scale of agent i, we first show that equation (2.2) holds. From (3.1) we have, for $t\in(t_k,t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) + (t-t_k)|\beta_i|\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i\in\mathcal{I}_M,\\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i|\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i\in\mathcal{I}_N/\mathcal{I}_M. \end{cases} \tag{3.2}$$

Therefore, it follows that

$$\beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i|\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } i\in\mathcal{I}_N. \tag{3.3}$$

Let $x(t_k) = (x_1(t_k), x_2(t_k),\dots,x_N(t_k))^T \in \mathbb{R}^N$, $\mathcal{B} = \mathrm{diag}(\beta_1,\beta_2,\dots,\beta_N)\in\mathbb{R}^{N\times N}$, $|\mathcal{B}| = \mathrm{diag}(|\beta_1|,|\beta_2|,\dots,|\beta_N|)\in\mathbb{R}^{N\times N}$ and $H = \mathrm{diag}(h_1,h_2,\dots,h_N)$. Then, equation (3.3) can be written as

$$\mathcal{B}x(t_{k+1}) = [I - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k).$$

Since $\mathcal{G}$ has a directed spanning tree and $0 < h < \frac{1}{d_{max}\beta_{max}}$, by Lemma 2.8, we have $\lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}]^k = 1_N y^T$, where y is nonnegative and satisfies $[I - H|\mathcal{B}|\mathcal{L}]^T y = y$. Thus

$$\lim_{k\to\infty}\mathcal{B}x(t_k) = \lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}]^k\mathcal{B}x(0) = 1_n y^T\mathcal{B}x(0).$$

As a consequence, equation (2.2) holds. Furthermore,

$$\lim_{t_k\to\infty}\beta_i x_i(t_k) = y^T\mathcal{B}x(0) \ \text{ for } \ i\in\mathcal{I}_N.$$

Now, we will show that

$$\lim_{t\to\infty}\|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \ \text{ for } \ i,j\in\mathcal{I}_M.$$

Consider, for $i,j\in\mathcal{I}_M$ and any $\beta_i\neq 0$,

$$\|\beta_i x_i(t) - \beta_j x_j(t)\| \leqslant \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|. \tag{3.4}$$

From equation (3.2), one obtains, for $t\in(t_k,t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leqslant h|\beta_i|\sum_{j\in\mathcal{N}_i} a_{ij}\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|.$$

As $t\to\infty$, we have $t_k\to\infty$. Thus,

$$\lim_{t\to\infty}\|\beta_i x_i(t) - \beta_i x_i(t_k)\| = 0 \ \text{ for } \ i,j\in\mathcal{I}_M.$$

Taking the limit as $t \to \infty$ on both sides of equation (3.4), one obtains

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_M.$$

Furthermore,

$$\lim_{t\to\infty} \beta_i x_i(t) = \lim_{t_k\to\infty} \beta_i x_i(t_k) = y^T \mathcal{B} x(0) \ \text{ for } \ i \in \mathcal{I}_M,$$

which implies that equation (2.3) holds. Therefore, the hybrid multi-agent system (2.1) with protocol (3.1) reaches scaled consensus.

Necessity. Suppose that $\mathcal{G}$ does not contain a spanning tree. Then, by Lemma 2.8, we have $\lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}]^k \neq 1y^T$. Hence,

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \neq 0 \ \text{ for some } \ i, j \in \mathcal{I}_N.$$

This implies that the hybrid multi-agent system (2.1) cannot achieve scaled consensus. □

3.2. Case II

All agents communicate with their neighbours and update their control inputs in a sampling time t_k. However, different from Case I, we assume that each continuous-time dynamic agent can observe its own state in real time. Then, the consensus protocol for hybrid multi-agent system (2.1) is defined by:

$$\begin{cases} u_i(t) = |\beta_i| \sum_{j\in\mathcal{N}} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t)], & \text{for } t \in (t_k, t_{k+1}], \ \ i \in \mathcal{I}_M, \\ u_i(t_k) = h \cdot |\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N/\mathcal{I}_M, \end{cases} \tag{3.5}$$

where all variables are defined as in the previous section.

Theorem 3.2. *Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (2.1) and $\beta_i \neq 0$ be any scalar scale of agent i. Assume that $0 < h < \frac{1}{\bar{d}_{max}\alpha_{max}}$. Then, the hybrid multi-agent system (2.1) with the protocol (3.5) achieves scaled consensus to $(\beta_1, \ldots, \beta_N)$ if and only if $\mathcal{G}$ has a spanning tree. Moreover, the scaled consensus state is $y^T\mathcal{B}x(0)$, where $|\mathcal{B}|\mathcal{L}^T H y = 0$ and*

$$H = \operatorname{diag}\left\{ \frac{1 - e^{-\sum_{j=1}^N a_{1j}|\beta_1| h}}{\sum_{j=1}^N a_{1j}|\beta_1|}, \ldots, \frac{1 - e^{-\sum_{j=1}^N a_{Mj}|\beta_M| h}}{\sum_{j=1}^N a_{Mj}|\beta_M|}, h, \ldots, h \right\}.$$

Proof.

Sufficiency. We first show that equation (2.2) holds. From (3.5) we know that for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) + |\beta_i| \left(\frac{1 - e^{-\sum_{j=1}^N a_{ij}|\beta_i|(t-t_k)}}{\sum_{j=1}^N a_{ij}|\beta_i|} \right) \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_M, \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N/\mathcal{I}_M. \end{cases} \tag{3.6}$$

Accordingly, at time t_{k+1}, the states of agents are

$$\begin{cases} \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + |\beta_i| \left(\frac{1 - e^{-\sum_{j=1}^N a_{ij}|\beta_i| h}}{\sum_{j=1}^N a_{ij}|\beta_i|} \right) \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_M, \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } i \in \mathcal{I}_N/\mathcal{I}_M. \end{cases} \tag{3.7}$$

By letting $x(t_k) = (x_1(t_k), x_2(t_k), \ldots, x_N(t_k))^T \in \mathbb{R}^N$, $\mathcal{B} = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{R}^{N\times N}$, $|\mathcal{B}| = \operatorname{diag}(|\beta_1|, |\beta_2|, \ldots, |\beta_N|) \in \mathbb{R}^{N\times N}$, equation (3.7) can be written as $\mathcal{B}x(t_{k+1}) = [I - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k)$, where $H =$

$\text{diag}\left\{\frac{1-e^{-\sum_{j=1}^{N}a_{1j}|\beta_1|h}}{\sum_{j=1}^{N}a_{1j}|\beta_1|}\dots,\frac{1-e^{-\sum_{j=1}^{N}a_{Mj}|\beta_M|h}}{\sum_{j=1}^{N}a_{Mj}|\beta_M|},h,\dots,h\right\}$. Since $\frac{1-e^{-\sum_{j=1}^{N}a_{ij}|\beta_i|h}}{\sum_{j=1}^{N}a_{ij}|\beta_i|} < \frac{1}{d_{ii}|\beta_i|}$ for $i \in \mathcal{I}_M$, and $h < \frac{1}{d_{\max}\beta_{\max}}$, one obtains $0 < h_i < \frac{1}{d_{\max}\beta_{\max}}$ for H. Since $\mathcal{G}$ has a spanning tree, by Lemma 2.8, $I - H|\mathcal{B}|\mathcal{L}$ is an SIA, i.e., $\lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}]^k = 1_N y^T$, where y is non-negative and satisfies $[I - H|\mathcal{B}|\mathcal{L}]^T y = y$. Thus

$$\lim_{k\to\infty}\mathcal{B}x(t_k) = \lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}]^k\mathcal{B}x(0) = 1_N y^T\mathcal{B}x(0).$$

As a consequence, equation (2.2) holds. Moreover,

$$\lim_{t_k\to\infty}\beta_i x_i(t_k) = y^T\mathcal{B}x(0) \ \text{ for } \ i \in \mathcal{I}_N.$$

Now, we will show that

$$\lim_{t\to\infty}\|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \ \text{ for } \ i,j \in \mathcal{I}_M.$$

From equation (3.6), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leqslant \frac{|\beta_i|}{\sum_{j=1}^{n}a_{ij}|\beta_i|}\sum_{j\in\mathcal{N}_i}a_{ij}\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|.$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t\to\infty}\|\beta_i x_i(t) - \beta_i x_i(t_k)\| = 0 \ \text{ for } \ i,j \in \mathcal{I}_M. \tag{3.8}$$

Consider, for $i,j \in \mathcal{I}_M$ and any $\beta_i \neq 0$,

$$\|\beta_i x_i(t) - \beta_j x_j(t)\| \leqslant \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|.$$

Thus, by (3.8), we get

$$\lim_{t\to\infty}\|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \ \text{ for } \ i,j \in \mathcal{I}_M.$$

Furthermore,

$$\lim_{t\to\infty}\beta_i x_i(t) = \lim_{t_k\to\infty}\beta_i x_i(t_k) = y^T\mathcal{B}x(0) \ \text{ for } \ i \in \mathcal{I}_M,$$

which implies that equation (2.3) holds. Therefore, the hybrid multi-agent system (2.1) with protocol (3.5) reaches scaled consensus.

Necessity. Using the same methodology as in the proof of necessity part of Theorem 3.1, this completes the proof. □

3.3. *Case III: impulsive consensus protocol*

Assume that all continuous-time dynamic agents communicate with their neighbours and update their control inputs in real time, while all discrete-time dynamic agents communicate with their neighbours and update their control inputs at a sampling time t_k. In addition, the interactions between the discrete-time dynamic agents and the continuous-time dynamic agents happen at the impulsive time $t = t_k$.

Assuming that the hybrid multi-agent system (2.1) has been modelled as a connected digraph $\mathcal{G} = \mathcal{G}_c \cup \mathcal{G}_d \cup \mathcal{G}'$, where $\mathcal{G}_c$, $\mathcal{G}_d$, $\mathcal{G}'$ are the communication networks of continuous-time agents, discrete-time agents, and the interactions between each other, respectively. Then the consensus protocol for hybrid multi-agent system (2.1) is defined as: for $t \in (t_{k-1}, t_k]$,

$$\begin{cases} u_i(t) = \text{sgn}(\beta_i) \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)] \\ \qquad\qquad + \text{sgn}(\beta_i) \sum_{k=1}^{\infty} \sum_{s\in\mathcal{N}_i'} a_{is}'[\beta_s x_s(t) - \beta_i x_i(t)]\delta(t-t_k), \quad \text{for } i \in \mathcal{I}_M, \\ u_l(t_k) = h \cdot \text{sgn}(\beta_i) \sum_{j\in\mathcal{N}_l} b_{lj}[\beta_j x_j(t_k) - \beta_l x_l(t_k)], \quad \text{for } l \in \mathcal{I}_N/\mathcal{I}_M, \end{cases} \tag{3.9}$$

where β_i is a nonzero scalar of an agent i, $\mathcal{A} = [a_{ij}]$ and $\mathcal{B} = [b_{lj}]$ are the weighted adjacency matrices associated with the graph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$, respectively. Moreover, $h = t_k - t_{k-1}$ is the sampling period, $\mathcal{N}_i$ and $\mathcal{N}_i'$ are the neighbor sets of i in $\mathcal{G}_c \cup \mathcal{G}'$ at time $t \neq t_k$ and $t = t_k$, respectively. $\mathcal{N}_l$ is a neighbor set of agent l in $\mathcal{G}_d \cup \mathcal{G}'$ at time t_k and $\delta(\cdot)$ is the Dirac delta function, i.e.,

$$\delta(t-t_k) = \begin{cases} 1, & t = t_k, \\ 0, & t \neq t_k, \end{cases}$$

and the signum function, noted by $\text{sgn}(\beta_i)$, is defined as

$$\text{sgn}(\beta_i) = \begin{cases} -1, & \beta_i < 0, \\ 0, & \beta_i = 0, \\ 1, & \beta_i > 0. \end{cases}$$

To establish our main results, some assumptions are provided as follows:

(A1) $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n}\{d_{ii}\}}$;

(A2) there exists constant $0 < \alpha \leqslant 1$ such that

$$(1-\alpha)I - \mathcal{L}'^T|\mathcal{B}| - |\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T|\mathcal{B}|^2\mathcal{L}' \leqslant 0,$$

where $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t = t_k$.

Now, we are in the position to introduce our main result.

Theorem 3.3. *Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (2.1). Assume that the assumptions* **(A1)** *and* **(A2)** *hold. Then, the hybrid multi-agent system (2.1) with the protocol (3.9) reaches scaled consensus $(\beta_1, \beta_2, \dots, \beta_N)$ if and only if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are both balanced and contain a spanning tree.*

Proof.

Sufficiency. Consider $\mathcal{G} = \mathcal{G}_c \cup \mathcal{G}_d \cup \mathcal{G}'$ be a directed connected communication network of the hybrid multi-agent system (2.1), where $\mathcal{G}_c$, $\mathcal{G}_d$, and $\mathcal{G}'$ are the communication networks of continuous-time agents, discrete-time agent and interactions between continuous-time and discrete-time agents.

First of all, consider for each $i \in \mathcal{I}_M$. Without loss of generality, we assume that all discrete-time dynamic agents have interacted with some continuous-time dynamic agents. Hence, the system (2.1) with the protocol (3.9) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with N nodes, where $N = |\mathcal{G}_c \cup \mathcal{G}'|$.

For simplicity of presentation, agents which maintain communication with agent i for a period of time are called as regular neighbors of agent i, while the agents which maintain information exchange at impulsive time are called impulsive neighbors of agent i. The sets of regular neighbors and impulsive neighbors of agent i are denoted by $\mathcal{N}_i$ and $\mathcal{N}_i'$, respectively. In addition, let D be diagonal matrix with the out-degree of each vertex along the diagonal, where the out-degree of node i is denoted by $\sum_{j\in\mathcal{N}_i} a_{ij}$. Then, the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t \neq t_k$ is denoted by $\mathcal{L} = [l_{ij}]_{n\times n}$, defined as $\mathcal{L} = D - A$, where

$$l_{ij} = \begin{cases} \sum_{j\in\mathcal{N}_i} a_{ij}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

On the other hand, for $t = t_k$, the out-degree of node i is denoted by $\sum_{j\in\mathcal{N}_i'} a_{ij}'$. Then, the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t = t_k$ is denoted by $\mathcal{L}' = [l_{ij}']_{n\times n}$, where

$$l_{ij}' = \begin{cases} \sum_{j\in\mathcal{N}_i'} a_{ij}', & i = j, \\ -a_{ij}', & i \neq j. \end{cases}$$

Hence, for $i \in \mathcal{I}_M$, the system (2.1) with the protocol (3.9) can be written as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with n nodes as follows:

$$\begin{cases} \dot{x}_i(t) = \operatorname{sgn}(\beta_i) \sum_{(\nu_i,\nu_j)\in\mathcal{E}} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)], \ t \in (t_{k-1}, t_k], \\ \Delta x_i(t_k) = \operatorname{sgn}(\beta_i) \sum_{(\nu_i,\nu_j)\in\mathcal{E}'} a_{ij}'[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \end{cases} \tag{3.10}$$

where $t \in \mathbb{R}^+$, $x_i(t) \in \mathbb{R}$ is the state of agent i at time t, $i = 1, 2, \ldots, n$. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$: $x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k + h)$ and $x_i(t_k^-) = \lim_{h\to 0^+} x_i(t_k - h)$. This implies that an agent i can intermittently update its state on the basis of the state information of itself and its neighbours at time t_k. Without loss of generality, we assume that $\lim_{h\to 0^+} x_i(t_k - h) = x_i(t_k)$, that is, $x_i(t_k)$ is left-continuous. The sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. It follows from (3.10) that

$$\begin{cases} \beta_i \dot{x}_i(t) = |\beta_i| \sum_{(\nu_i,\nu_j)\in\mathcal{E}} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)], \ t \in (t_{k-1}, t_k], \\ \Delta\beta_i x_i(t_k) = |\beta_i| \sum_{(\nu_i,\nu_j)\in\mathcal{E}'} a_{ij}'[\beta_j x_j(t_k) - \beta_i x_i(t_k)]. \end{cases} \tag{3.11}$$

By letting $x = (x_1, x_2, \ldots, x_N^T \in \mathbb{R}^N$, $\mathcal{B} = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{R}^{N\times N}$, $|\mathcal{B}| = \operatorname{diag}(|\beta_1|, |\beta_2|, \ldots, |\beta_N|) \in \mathbb{R}^{N\times N}$, and $H = \operatorname{diag}(h_1, h_2, \ldots, h_N)$, the system (3.11) can be written as the form:

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}\mathcal{B}x(t), & t \neq t_k, \\ \Delta\mathcal{B}x(t) = -|\mathcal{B}|\mathcal{L}'\mathcal{B}x(t), & t = t_k, \ k \in \mathbb{N}, \end{cases}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively. Since $\mathcal{G}_c \cup \mathcal{G}'$ is balanced, $\bar{x} = \frac{1}{N}\sum_{j=1}^N \beta_j x_j$ is an invariant quantity,

$$\bar{x}(t) = \bar{x}(0) = \frac{1}{N}\sum_{j=1}^N \beta_j x_j(0),$$

which is not true for an arbitrary digraph. The invariant of $\bar{x}$ allows decomposition of x_i for $i = 1, 2, \ldots, n$ as in the following equation:

$$\delta_i(t) = \beta_i x_i(t) - \bar{x}, \ t \in (t_{k-1}, t_k],$$

$\delta_i(t_k^+) = \beta_i x_i(t_k^+) - \bar{x}$ and $\delta_i(t_k^-) = \delta_i(t_k)$, $i = 1, 2, 3, \ldots, n$, with initial conditions $x(t_0) = x(0) = [x_{10}, x_{20}, \ldots, x_{N0}]^T$, where $\delta = (\delta_1, \ldots, \delta_N)^T$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \dot{\delta}(t) = -|\mathcal{B}|\mathcal{L}\delta(t), & t \neq t_k, \\ \delta(t_k^+) = [I - |\mathcal{B}|\mathcal{L}']\delta(t_k), & t = t_k, \ k \in \mathbb{N}. \end{cases} \tag{3.12}$$

Consider the Lyapunov function candidate as $V(\delta) = \delta^T\delta$. Let $V(\delta) =: V(\delta(t))$. Since $\mathcal{G}_c \cup \mathcal{G}'$ is balanced, $\hat{\mathcal{L}} = \operatorname{Sym}(\mathcal{L}) = (|\mathcal{B}|\mathcal{L} + \mathcal{L}^T|\mathcal{B}|)/2$, the total derivation of $V(\delta)$ with respect to (3.12) is

$$\dot{V}(t) = \dot{\delta}^T(t)\delta(t) + \delta^T(t)\dot{\delta}(t) = -\delta^T(t)(\mathcal{L}^T|\mathcal{B}| + |\mathcal{B}|\mathcal{L})\delta(t) = -2\delta^T(t)(\hat{\mathcal{L}})\delta(t), \ t \in (t_{k-1}, t_k).$$

It can be seen that $\lambda_2(\hat{\mathcal{L}}) = \min_{x\neq 0,\ 1^Tx=0} \frac{x^T\hat{\mathcal{L}}x}{x^Tx}$ since the mirror graph $\hat{\mathcal{G}}$ of $\mathcal{G}_c \cup \mathcal{G}'$ is a connected undirected graph. Hence,

$$\dot{V}(t) \leqslant -2\lambda_2(\hat{\mathcal{L}})V(t) \ \text{ for } \ t \in (t_{k-1}, t_k),$$

which implies that, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leqslant e^{-2\lambda_2(\hat{\mathcal{L}})(t-t_{k-1})} V(t_{k-1}^+).$$

On the other hand, when $t = t_{k-1}$, using $(1-\alpha)I - \mathcal{L}'^T|\mathcal{B}| - |\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T|\mathcal{B}|^2\mathcal{L}' \leqslant 0$, for $0 < \alpha \leqslant 1$, we have

$$\begin{aligned} V(t_{k-1}^+) &= \delta^T(t_{k-1})(I - |\mathcal{B}|\mathcal{L}')^T(I - |\mathcal{B}|\mathcal{L}')\delta(t_{k-1}) \\ &= \delta^T(t_{k-1})[I - \mathcal{L}'^T|\mathcal{B}| - |\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T|\mathcal{B}|^2\mathcal{L}' - \alpha I + \alpha I]\delta(t_{k-1}) \\ &= \delta^T(t_{k-1})[(1-\alpha)I - \mathcal{L}'^T|\mathcal{B}| - |\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T|\mathcal{B}|^2\mathcal{L}']\delta(t_{k-1}) + \alpha\delta^T(t_{k-1})\delta(t_{k-1}) \\ &\leqslant \alpha\delta^T(t_{k-1})\delta(t_{k-1}) = \alpha V(t_{k-1}). \end{aligned}$$

In general, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leqslant \alpha^{k-1} e^{-2\lambda_2(\hat{\mathcal{L}})(t-t_0^+)} V(t_0^+).$$

Hence, for $t \in (t_{k-1}, t_k]$,

$$|\delta(t)| \leqslant \alpha^{(k-1)/2} e^{-\lambda_2(\hat{\mathcal{L}})(t-t_0)} |\delta(t_0^+)|.$$

Thus,

$$\|\beta_i x_i(t) - \bar{x}\| \to 0 \ \text{ as } \ t \to \infty \ \text{ or } \ \lim_{t\to\infty} \beta_i x_i(t) = \bar{x}, \ \ \forall i \in \mathcal{I}_M.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_M. \tag{3.13}$$

Now, we will show that

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_N.$$

Consider, for $i, j \in \mathcal{I}_N$,

$$\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \leqslant \|\beta_i x_i(t_k) - \beta_i x_i(t)\| + \|\beta_i x_i(t) - \beta_j x_j(t)\| + \|\beta_j x_j(t) - \beta_j x_j(t_k)\|.$$

The proof can be separated into three cases as follows.

Case 1. If $i, j \in \mathcal{I}_M$, the above discussion gives

$$\lim_{t_k\to\infty} x_i(t_k) = \lim_{t\to\infty} x_i(t) = \bar{x}, \ \ \forall i \in \mathcal{I}_M.$$

This implies that

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_M \subset \mathcal{I}_N.$$

Case 2. If $i, j \in \mathcal{I}_N/\mathcal{I}_M = \{M+1, M+2, \ldots, N\}$, the problem can be simplified by considering the communication network of $\mathcal{G}_d \cup \mathcal{G}'$. Since the discrete-time dynamic agents interact with their neighbours at time $t = t_k$, one obtains

$$\beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{(i,j)\in\mathcal{E}'} b_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \tag{3.14}$$

where $h = t_k - t_{k-1}$ is a sampling period, $\mathcal{E}'$ is the set of edges and $B = [b_{ij}]_{r\times r}$ is the adjacency matrix of $\mathcal{G}_d \cup \mathcal{G}'$, where $r = |\mathcal{G}_d \cup \mathcal{G}'|$ is the number of the discrete-time dynamic agents and continuous-time dynamic agents that interact with them. By letting $x(t_k) = [x_1(t_k), x_2(t_k), \dots, x_r(t_k)]^T$, the equation (3.14) can be written as

$$\mathcal{B}x(t_{k+1}) = [I_r - H|\mathcal{B}|\mathcal{L}_d]\mathcal{B}x(t_k),$$

where $H = \text{diag}\{h, h, \dots, h\}, \mathcal{B} = \text{diag}\{\beta_1, \beta_2, \dots, \beta_i\}$, I_r is an identity matrix and $\mathcal{L}_d$ is the Laplacian matrix of $\mathcal{G}_d \cup \mathcal{G}'$. According to Lemma 2.8, since $\mathcal{G}_d \cup \mathcal{G}'$ has a directed spanning tree and $h < \frac{1}{d_{\max}\beta_{\max}}$, there exists a column vector y such that

$$\lim_{k\to\infty} [I_r - H|\mathcal{B}|\mathcal{L}_d]^k = 1y^T, \quad \text{where} \quad [I_r - H|\mathcal{B}|\mathcal{L}_d]^T y = y.$$

Thus,

$$\lim_{t_k\to\infty} x(t_k) = \lim_{k\to\infty} [I_r - H|\mathcal{B}|\mathcal{L}_d]^k x(0) = 1y^T x(0) \ \text{ and } \ \mathcal{L}_d^T|\mathcal{B}|y = 0.$$

This implies that

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_N/\mathcal{I}_M.$$

Moreover, there exists a column vector y such that

$$\lim_{t_k\to\infty} \beta_i x_i(t_k) = y^T x(0) \ \text{ for all } \ i \in \mathcal{I}_N/\mathcal{I}_M.$$

Case 3. If $j \in \mathcal{I}_N/\mathcal{I}_M$ and $i \in \mathcal{I}_M$ (or $i \in \mathcal{I}_N/\mathcal{I}_M$ and $j \in \mathcal{I}_M$), we consider, for $i, l \in \mathcal{I}_M$ and $j \in \mathcal{I}_N/\mathcal{I}_M$,

$$\begin{aligned}\|\beta_i x_i(t) - \beta_l x_l(t)\| \leqslant{} & \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ & + \|\beta_j x_j(t_k) - \beta_l x_l(t_k)\| + \|\beta_l x_l(t_k) - \beta_l x_l(t)\|.\end{aligned}$$

For $i, l \in \mathcal{I}_M$,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_l x_l(t)\| = 0 \ \text{ and } \ \lim_{t\to\infty} \beta_i x_i(t) = \bar{x}, \ \ \forall i \in \mathcal{I}_M.$$

When $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t_k\to\infty} \|\beta_i x_i(t) - \beta_i x_i(t_k)\| = 0 \ \text{ and } \ \lim_{t_k\to\infty} \|\beta_l x_l(t_k) - \beta_l x_l(t)\| = 0.$$

This implies that $\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0$ and $\lim_{t_k\to\infty} \|\beta_j x_j(t_k) - \beta_l x_l(t_k)\| = 0$. Hence,

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0, \quad \text{for } \ j \in \mathcal{I}_N/\mathcal{I}_M \ \text{ and } \ i \in \mathcal{I}_M.$$

From Cases 1, 2, and 3, we can conclude that

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0 \ \text{ for } \ i, j \in \mathcal{I}_N. \tag{3.15}$$

Therefore, from (3.13) and (3.15), the hybrid multi-agent system (2.1) with protocol (3.9) reaches consensus.

Necessity. Suppose that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are not balanced or do not contain a spanning tree. Then, by Lemma 2.8, we have $\lim_{k\to\infty}[I - H|\mathcal{B}|\mathcal{L}_d]^k \neq 1y^T$. Hence,

$$\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \neq 0 \ \text{ for } \ i, j \in \mathcal{I}_N.$$

This implies that the hybrid multi-agent system (2.1) cannot achieve consensus. □

Remark 3.4. It can be seen that if $M = N$, then the hybrid multi-agent systems can reduce as a continuous time dynamic system. On the other hand if $M = 0$, the hybrid multi-agent systems is a discrete-time dynamic system.

Remark 3.5. It is easy to see that the results from Theorem 3.3 are more general than the results of Zheng et al. [20], which assumes that the interactions among agents occur only in the sampling time t_k.

4. Numerical examples

In order to show the effectiveness of the theoretical results in this work, two examples have been investigated as following.

Example 4.1. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. Let $x(0) = [-6\,4\,-2\,1\,-1\,2\,-4\,6]^T$. The communication network $\mathcal{G}$ with $0-1$ weights is shown in Figure 1, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

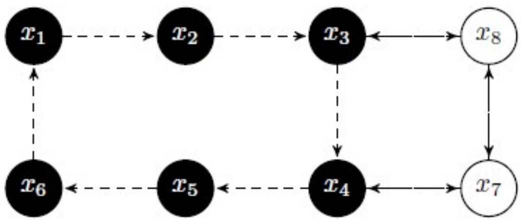

Figure 1: A connected directed network $\mathcal{G}$.

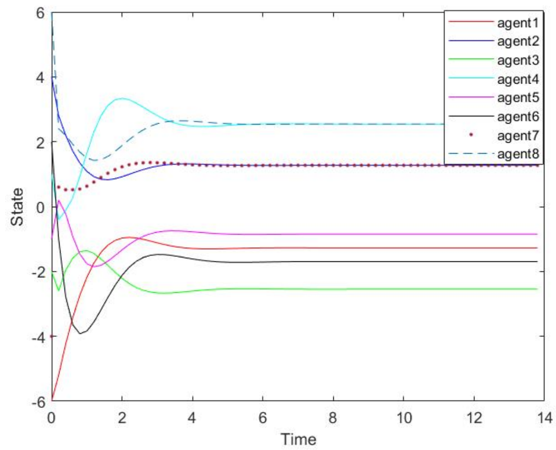

Figure 2: The state trajectories of all using the consensus protocol (3.1) with $h = 0.2$.

Consider a communication network $\mathcal{G}$ in Figure 1, it can be seen that $\mathcal{G}$ is balanced and contains a directed spanning tree with $d_{max} = 2$ and the Laplacian matrix of a network $\mathcal{G}$ is as

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Let the scalar scales be $(2, -2, 1, -1, 3, 1.5, -2, -1)$, one obtains that $\beta_{max} = 3$. Clearly, $h = 0.2 < 0.33 = (d_{max}\beta_{max})^{-1}$. By using the consensus protocol (3.1), the state trajectories of all agents are shown as in Figure 2, which is consistent with the sufficiency of Theorem 3.1. Furthermore, by selecting $h = 0.2$ and scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$, then the state trajectories of all agents under consensus protocol (3.1) can be described as in the Figure 3. Moreover, if the scalar scale $\beta_i = 1$ for all i, then all state trajectories under consensus protocol (3.1) can be shown as in Figure 4. In addition, if the sampling period $h = 0.4 > 0.33 = (d_{max}\beta_{max})^{-1}$ the state trajectories of all agent under the consensus protocol (3.1) are divergent as in Figure 5.

Example 4.2. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. The communication network $\mathcal{G}$ with $0-1$ weights is shown in Figure 6, where the continuous-time dynamic agents can observe their own state in real time, while the interactions among agents happen in the sampling time t_k.

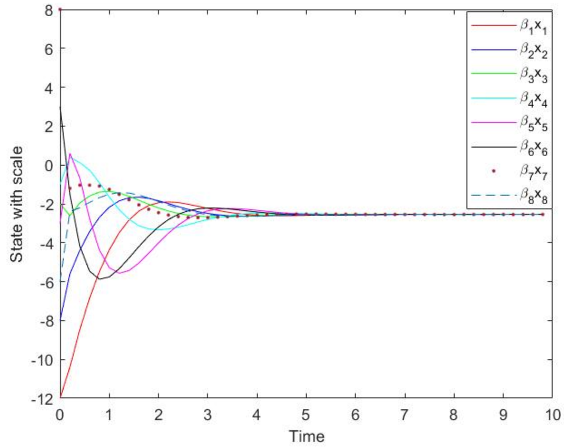

Figure 3: The state trajectories scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ with $h = 0.2$.

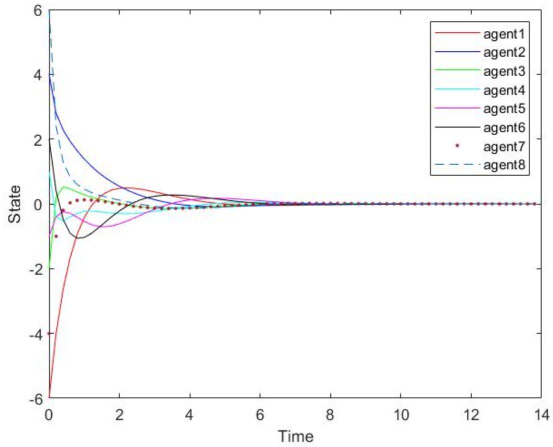

Figure 4: The state trajectories of all agents with scalar scales $(1, 1, 1, 1, 1, 1, 1, 1)$ with $h = 0.2$.

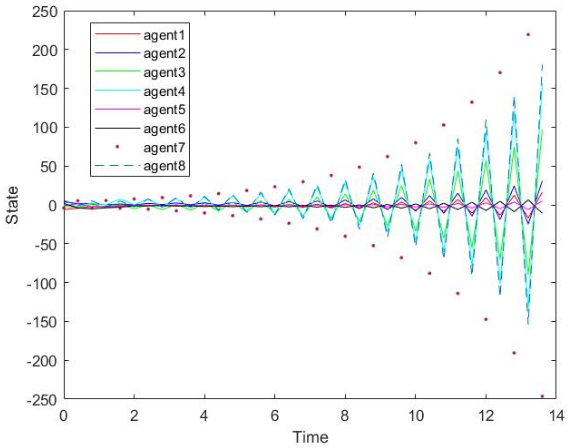

Figure 5: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ with $h = 0.4$.

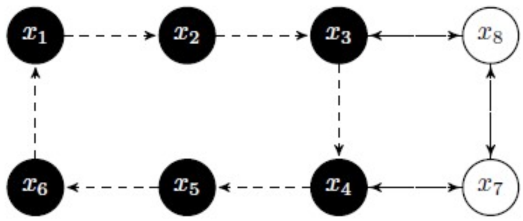

Figure 6: A connected directed network $\mathcal{G}$.

It can be seen in Figure 6 that a network $\mathcal{G}$ is balanced and contains a spanning tree with $d_{max} = 2$. Moreover, the Laplacian matrix of $\mathcal{G}$ can be described as

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Let the initial states of all agents be $x(0) = [-6\ 4\ -2\ 1\ -1\ 2\ -4\ 6]^T$ and the scalar scales be $(2, -2, 1, -1, 3, 1.5, -2, -1)$. Thus, $\beta_{max} = 3$ and by selecting the sampling period $h = 0.2 < 0.33 = (d_{max}\beta_{max})^{-1}$, all the conditions of Theorem 3.2 are satisfied. Hence, the consensus protocol (3.5) can guarantee reaching scaled consensus of the system and the state trajectories of all agents are shown in

Figure 7, which is consistent with the sufficiency of Theorem 3.2.

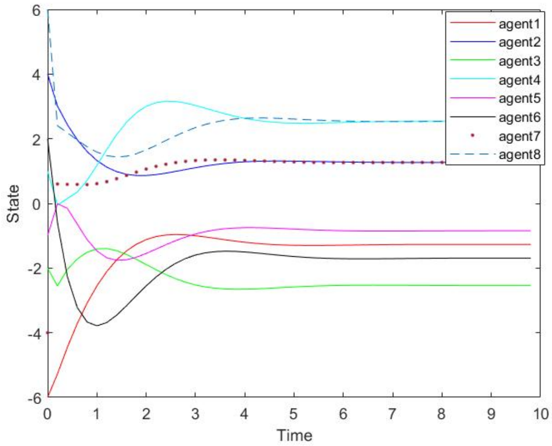

Figure 7: The state trajectories of all agents using the consensus protocol (3.5) and communication network $\mathcal{G}$ with $h = 0.2$.

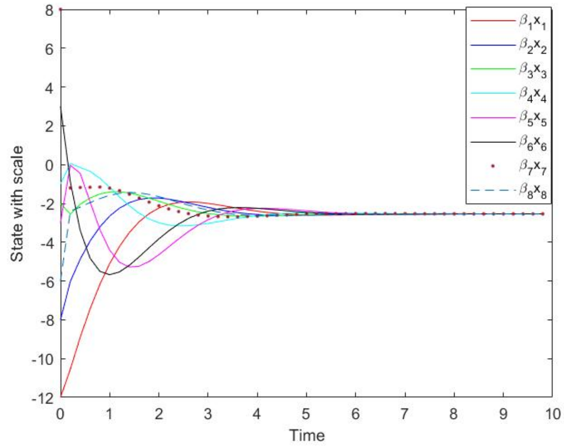

Figure 8: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$.

In addition, the state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ using the consensus protocol (3.5) and communication network $\mathcal{G}$ with $h = 0.2$ are described as in Figure 8. Furthermore, if the scalar scales $\beta_i = 1$ for all i, the state trajectories of all agents under the consensus protocol (3.5) with $h = 0.2$ can be described as in Figure 9.

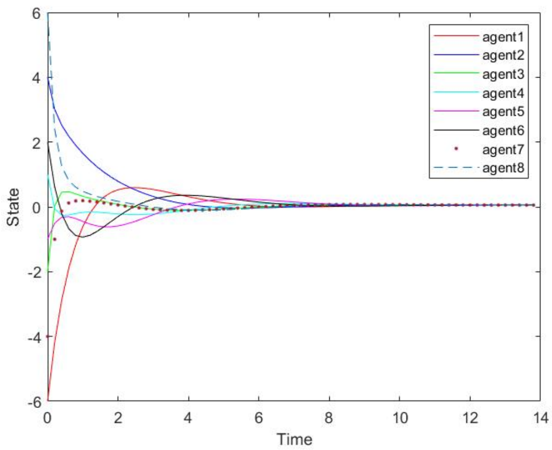

Figure 9: The state trajectories of all agents with scalar scales $(1, 1, 1, 1, 1, 1, 1, 1)$ and $h = 0.2$.

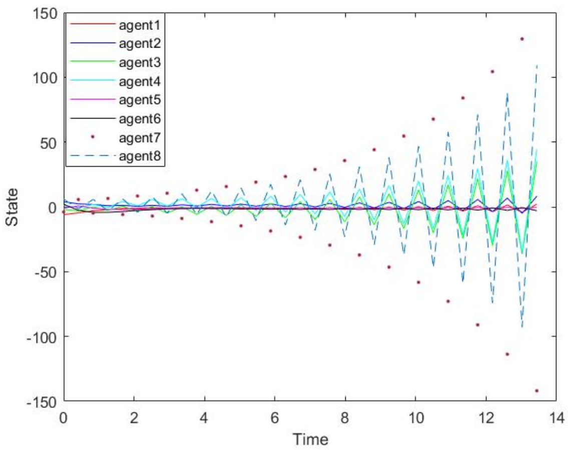

Figure 10: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ with $h = 0.42$.

Moreover, if the sampling period $h = 0.42 > 0.33 = (d_{max}\beta_{max})^{-1}$ the state trajectories of all agent under the consensus protocol (3.5) are divergent as in Figure 10.

5. Conclusion

In this work, scaled consensus problems for the hybrid multi-agent system (2.1) consisting of CT and DT dynamics agents have been studied. Three consensus protocols are proposed based on the interactions among agents. Firstly, we assume that the directed communication networks $\mathcal{G}$ contains a spanning tree with $0 < h < (d_{max}\beta_{max})^{-1}$ and interactions among agents occur in the sampling time t_k. Hence, by Theorem 3.1 and protocol (3.1), the HMASs (2.1) achieves scaled consensus to $(\beta_1, \ldots, \beta_N)$; $\beta_i \neq 0$

for all i. Secondly, assume that the directed communication network $\mathcal{G}$ contains a spanning tree with $0 < h < (d_{max}\beta_{max})^{-1}$ and interactions among agents occur in the sampling time t_k but the continuous-time dynamic agents can observe their own states in real time. By Theorem 3.2 and protocol (3.5), we show that the hybrid multi-agent system (2.1) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$; $\beta_i \neq 0$. Thirdly, the impulsive consensus protocols are investigated to solve the scaled consensus of HMASs. Obviously, if there is no DT dynamic agent the HMASs can be reduced as the MASs, studied by Mana and Liu [5], which show the generalization of our theorem. Moreover, under the consensus protocols (3.1) and (3.5), we see that if $\beta_i = 1$ for all i, the state trajectories of all agents are as in Figures 4 and 9. This shows that our scaled consensus results are more general than the consensus results of Zheng [20]. In addition, if the sampling period $0 < h < (d_{max}\beta_{max})^{-1}$, our results in Theorems 3.1 and 3.2 can guarantee reaching scaled consensus to $(\beta_1, \ldots, \beta_n)$ as shown in Figures 2 and 7. However, if $h > (d_{max}\beta_{max})^{-1}$, the HMASs (2.1) cannot achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$ under protocols (3.1) and (3.5) as shown in Figures 5 and 10.

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