J. Math. Computer Sci., 36 (2025), 290–298

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On A-orthogonally diagonalizable operators on semi-inner product spaces

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Abstract

In finite dimensional inner product (IP) spaces, any self-adjoint operator is normal and any normal operator is orthogonally diagonalizable. However, in semi-inner product (SIP) spaces, there exists an A-self-adjoint operator which is not A-normal. Therefore, it is interesting to study conditions for an A-self-adjoint operator on an SIP space to be A-orthogonally diagonalizable. An SIP is a mapping induced by a positive semi-definite operator on an IP space. In this paper, we study necessary, sufficient, and necessary and sufficient conditions for an A-self-adjoint operator to be A-orthogonally diagonalizable.

Keywords: Semi-inner product spaces, spectral decomposition, A-self-adjoint operators.

2020 MSC: 46C50, 15A23, 11E39.

1. Introduction

In any finite dimensional inner product space (IP space), a self-adjoint linear operator is orthogonally diagonalizable. This article studies whether similar property applies in semi-inner product spaces (SIP spaces). An SIP space U is a vector space over the complex number field $\mathbb{C}$ equipped with a semi-inner product (SIP) denoted by $[x, y], \forall x, y \in U$. an SIP is a form of expansion of an inner product (IP), which modifies the positive definite property of IP into positive semi-definite. Hence, in an SIP space, there could be many elements acting like the zero in an IP space.

Using a positive semi-definite operator, we can establish a connection between IP space and SIP space. Let U be a finite dimensional IP space equipped with an IP $\langle x, y\rangle, \forall x, y \in U$. If A is a positive semi-definite operator on U, then the mapping $[x, y]_A = \langle Ax, y\rangle, \forall x, y \in U$ is an SIP ([3]). On the other hand, if U is a finite dimensional SIP space, then there is an IP on U, say $\langle x, y\rangle, \forall x, y \in U$, and a positive semi-definite operator A on U such that the SIP is $[x, y] = \langle Ax, y\rangle, \forall x, y \in U$ ([7]). Further, an IP space is an SIP space with the positive semi-definite operator is the identity map.

The concept of an SIP space is pioneered by the study of Krein ([9]), and then developed by Zaanen ([20]), Lumer ([10]), and Giles ([7]). A number of recent studies on SIP spaces investigated the equivalence of operator properties in IP spaces for SIP spaces, such as A-normal ([5, 12]), hiponormal ([2]), and

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doi: 10.22436/jmcs.036.03.04

Received: 2024-04-18 *Revised:* 2024-05-18 *Accepted:* 2024-06-29

paranormal ([11]) operators, closed operators ([4]), matrix representation of operators ([6]), and isometry ([3, 17]); and also geometrical aspects of SIP spaces, such as orthogonality ([18, 21]), projection metric ([4]), and numerical radius ([10, 18]).

Another difference between the class of IP spaces and the class of SIP spaces is that every self-adjoint operator in an IP space is normal. In contrast, this property does not hold when the underlying space being SIP. Moreover, an operator T in a finite dimensional SIP space U will have adjoint(s) if and only if T is bounded. Thus, the necessary and sufficient conditions for spectral decomposition of linear operators on an IP space cannot be applied to linear operators on an SIP space. Tam and Zhang ([19]) obtained sufficient conditions for spectral decomposition of an A-self-adjoint operator on an SIP space. However, the theorem is restricted to the class of A-self-adjoint operators T satisfying condition $R(T) \subseteq R(A)$. Furthermore, the definition of spectral decomposition on an SIP space has not been explicitly stated.

In this article, we give the definition for spectral decomposition in finite dimensional SIP spaces. We study necessary, sufficient, and also necessary and sufficient conditions for the existence of a spectral decomposition for an A-self-adjoint linear operator on a finite dimensional SIP space.

2. A-self-adjoint operators in SIP spaces

In this section we give the definition and some properties of an A-self-adjoint operator on an SIP space. In contrast to IP spaces, an adjoint of a linear operator in a finite dimensional SIP space does not always exist and if it exists, the adjoint operator may not be unique. A necessary and sufficient condition for the existence of an adjoint operator is boundedness. For that, the discussion begins by reviewing SIP spaces along with a number of definitions and properties that will be needed in the next discussions.

2.1. SIP spaces

From now on, we restrict a semi-inner product (SIP) space is a finite dimensional inner product (IP) space U over the complex number field $\mathbb{C}$ with the IP function $\langle x, y\rangle, \forall x, y \in U$, equipped with an SIP denoted by $[x, y]_A = \langle Ax, y\rangle \forall x, y \in U$ for some A, a positive semi-definite operator on U. Let $N(A)$ denotes the kernel of A and $R(A)$ denotes the range of A, and both are subspaces of U. We obtain that $x \in N(A)$ if and only if $[x, x]_A = 0$. Such elements are called the neutral elements of U. The next proposition shows some properties of the neutral elements.

Proposition 2.1. *Let U be an SIP space over $\mathbb{C}$, with the SIP $[x, y]_A, \forall x, y \in U$.*

(1) *If $x \in N(A)$, then $[x, y]_A = 0 = [y, x]_A$ for all $y \in U$.*
(2) *If $x, y \in N(A)$, then $x + y \in N(A)$ and $cx \in N(A)$ for all scalar c.*

Proof. Statement (1) can be proven using the Cauchy-Schwarz inequality in SIP spaces. As a result of (1) we obtain (2). □

The following are subspaces which are not containing neutral elements.

Lemma 2.2. *Let U be an SIP space induced by a positive semi-definite operator A. If W is a subspace of U such that $W \cap N(A) = 0$, then W as an SIP subspace is an IP space.*

Proof. Since W is a SIP subspace, we only need to show that the condition $[x, x] = 0$ can only be satisfied by $x = 0$. The premise $W \cap N(A) = 0$ implies the only neutral element of W is $x = 0$. Thus the condition holds. □

Let U be an SIP space. Two notions can be defined on U. The first one is a semi-norm, it is a non-negative real function on U denoted by $\|x\|_A = \sqrt{[x, x]_A}$, $\forall x \in U$, which satisfies a number of well known basic properties of a semi-norm. The second one is an orthogonality; $x, y \in U$ are called A-orthogonal if $[x, y]_A = 0$ and we write $x \perp_A y$. From Proposition 2.1, we have that every neutral element is A-orthogonal to other elements. A set $S = \{x_1, \dots, x_n\}$ is an A-orthogonal set if $x_i \perp_A x_j$, where $i \neq j$ and $i, j = 1, \dots, n$. If

$W \subseteq U$, the A-orthogonal complement of W, denoted by $W^{\perp_A}$, is defined as the set containing all $y \in U$, where $[x, y]_A = 0$, for all $x \in W$. In this case, $N(A) \subseteq W^{\perp_A}$. In general, we obtain $U = W + W^{\perp_A}$ but the sum is not a direct sum, particularly when $W \cap W^{\perp_A} \neq \{0\}$.

For any n-dimensional SIP space U there is an A-orthonormal basis $\mathcal{O} = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}$, where $\{x_1, \ldots, x_m\}$ is a non-neutral A-orthonormal set of U with m is the rank of A; while $x_{m+1}, \ldots, x_n$ are linearly independent neutral elements of U which form a basis of $N(A)$.

The quotient space modulo $N(A)$ is denoted by $U/N(A) = \{\overline{x} = x + N(A) \mid x \in U\}$. The operation and action on $U/N(A)$ are defined as $\overline{x} + \overline{y} = \overline{x+y}$ and $c\overline{x} = \overline{cx}$, for every $\overline{x} = x + N(A)$, $\overline{y} = y + N(A)$, $x, y \in U$, and scalar $c \in \mathbb{C}$. Furthermore, $U/N(A)$ is an IP space with the IP defined as

$$\langle \overline{x}, \overline{y} \rangle = [x, y]_A, \ \forall \overline{x}, \overline{y} \in U/N(A);\ x, y \in U.$$

The norm on $U/N(A)$ is defined as $\|\overline{x}\| = \langle \overline{x}, \overline{x} \rangle^{1/2}$ and elements $\overline{x}$ and $\overline{y}$ are orthogonal if $\langle \overline{x}, \overline{y} \rangle = 0$. Proposition 2.3 below shows the orthonormality in $U/N(A)$.

Proposition 2.3. *Let U be an SIP space and $\overline{x}, \overline{y} \in U/N(A)$ for some $x, y \in U$. The vectors $\overline{x}$ and $\overline{y}$ are orthonormal if and only if x and y are A-orthonormal.*

Proof.

($\Leftarrow$) Let x and y be A-orthonormal. Then, $\|\overline{x}\| = \langle \overline{x}, \overline{x} \rangle^{1/2} = [x, x]_A^{1/2} = 1$ and similarly $\|\overline{y}\| = 1$. We also have $\langle \overline{x}, \overline{y} \rangle = [x, y]_A = 0$. Hence, $\overline{x}$ and $\overline{y}$ are orthonormal.

($\Rightarrow$) Let $\overline{x}$ and $\overline{y}$ be orthonormal, then $\|x\|_A = [x, x]_A^{1/2} = \langle \overline{x}, \overline{x} \rangle^{1/2} = 1$ and similarly $\|y\|_A = 1$. We also have $[x, y]_A = \langle \overline{x}, \overline{y} \rangle = 0$. Hence, x and y are A-orthonormal. □

From Proposition 2.3, we can see that $\overline{x} \perp \overline{y}$ if and only if $x \perp_A y$.

2.2. *A-adjoint operators on SIP spaces*

Let U be an SIP space and $T : U \to U$ be a linear operator on U. A linear operator $S : U \to U$ is called an A-adjoint of T if the condition $[T(x), y]_A = [x, S(y)]_A$, $\forall\, x, y \in U$ holds. In contrast to the case of inner product spaces where any linear operator on them has a unique adjoint operator, there exists a linear operator on an SIP space without associated A-adjoint operator. A necessary and sufficient condition for a linear operator on an SIP space to have an associated A-adjoint operator is being A-bounded. A linear operator T on an SIP space U is called A-bounded if there is a positive real number c such that

$$\|T(x)\|_A \leqslant c\|x\|_A, \ \forall x \in U.$$

While on finite dimensional IP spaces any linear operator is bounded, the same property does not apply on SIP spaces.

Proposition 2.4 ([7]). *Let T be a linear operator on a SIP space U. The following statements are equivalent.*

(i) *T is A-bounded.*
(ii) *The subspace $N(A)$ is T-invariant.*
(iii) *The representation matrix of T for any A-orthonormal basis of U is lower block matrix $\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}$, where the size of T_1 is $m \times m$.*

When T is A-bounded, two linear operators can be defined, $\overline{T}$, the operator on the quotient space $U/N(A)$ induced by T,

$$\overline{T}(x + N(A)) = T(x),\ \forall x + N(A) \in U/N(A),$$

and T_0, the restriction of T on the subspace $N(A)$. The block T_1 of the representation matrix of T above, represents the operator $\overline{T}$ and the block T_3 represents T_0.

The next proposition gives a necessary and sufficient condition for the existence of an A-adjoint operator on an SIP space ([14]).

Proposition 2.5. *Let* U *be an SIP space and* T *be a linear operator on* U*, then*

(i) T *has an* A*-adjoint operator(s) iff* T *is* A*- bounded; and*
(ii) *if* T *is* A*-bounded and* S *is an* A*-adjoint of* T*, then* S *is also* A*-bounded.*

Let T be A-bounded. The A-adjoint of T is unique only if $N(A)$ is the zero subspace which is equivalent when U is an IP space. One of the A-adjoints of T which is called the distinctive A-adjoint of T, is the operator $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A and T^* is the adjoint of T. Bovdi [7] found that if $\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}$ is a representation matrix of T with respect to an A-orthonormal basis of U, then the representation matrix of an A-adjoint of T for the same A-orthonormal basis is of the form $\begin{bmatrix} T_1^* & 0 \\ S_2 & S_3 \end{bmatrix}$, where T_1^* denote the conjugate transpose of T_1. Particularly, if we choose an A-orthonormal basis such that the representation matrix of the positive semi-definite operator A is of the form $\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$, then the representation matrix of the distinctive A-adjoint is $T^\sharp = \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}$.

2.3. A-self-adjoint operators on SIP spaces

It is well known there are several classes of bounded operators on IP spaces such as the class of normal operators and the class of self-adjoint operators. These notions have been extended to SIP spaces.

Definition 2.6. Let T be an A-bounded operator on an SIP space U.

a. The operator T is called A-self-adjoint if T is an A-adjoint operator of T, that is if the following condition holds:
$$[Tx, y]_A = [x, Ty]_A, \quad \forall x, y \in U.$$
b. The operator T is called A-normal if $T^\sharp T = TT^\sharp$ holds, where $T^\sharp$ is the distinctive A-adjoint operator of T.

In SIP spaces, an equivalent condition that any A-bounded operator T to be A-self-adjoint is the equation $AT = T^*A$ holds, where T^* is the adjoint operator of T.

The next proposition is another necessary and sufficient condition of an A-self-adjoint operator in regard to the induced operator on the quotient spaces $U/N(A)$ ([7]).

Proposition 2.7. *Let* U *be an SIP space with the SIP induced by* A*. An* A*-bounded operator* $T : U \to U$ *is* A*-self-adjoint iff*

$$\begin{aligned} \overline{T} : U/N(A) &\to U/N(A) \\ x + N(A) &\mapsto T(x) + N(A) \end{aligned}$$

is self-adjoint.

In IP spaces, any self-adjoint operator is normal, i.e., it commutes with its adjoint. In contrast, this statement is not true for the case SIP spaces. Example 2.8 shows an A-self-adjoint operator on an SIP space which is not A- normal.

Example 2.8. Let $\mathbb{C}^3$ be an SIP space with the SIP map induced by the multiplication operator $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The operator $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is an A-self-adjoint operator on $\mathbb{C}^3$, where $T^\sharp = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. However, T is not an A-normal operator.

One attribute that is well known concerning the class of self-adjoint operators on IP spaces is that it is the class of linear operators on IP spaces that are orthogonally diagonalizable with real number diagonal components. Unfortunately, this result can not be extended to the class of A-self-adjoint operators on SIP spaces.

Definition 2.9. Let $\mathcal{U}$ be an SIP space and let T be an A-bounded linear operator on $\mathcal{U}$. The operator T is A-orthogonally diagonalizable if there is an A-orthonormal basis $\mathcal{O}$ of $\mathcal{U}$ such that the representation matrix of T with respect to $\mathcal{O}$, denoted by $[T]_{\mathcal{O}}$, is diagonal.

Example 2.10 below shows a linear operator on an SIP space which is A-self-adjoint but is not A-orthogonally diagonalizable.

Example 2.10. Let $\mathbb{C}^3$ is the SIP space as shown in Example 2.8. The operator $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is A-self-adjoint operator on the SIP $\mathbb{C}^3$, which is not A-orthogonally diagonalizable. In fact, S is not diagonalizable.

Referring to the above fact it is of interest to investigate when an A-self-adjoint operator on an SIP space is A-orthogonally diagonalizable.

3. Spectral decomposition on SIP spaces

Arias et al. [3] showed that $T = T^\sharp$ if and only if T is A-self-adjoint and $R(T) \subseteq R(A)$. As a result, for an A-orthonormal basis of the space $\mathcal{U}$, say $\mathcal{O}$, the representation matrix of T with respect to $\mathcal{O}$ is of the form $\begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}$ and since $R(T) \subseteq R(A)$, then the representation matrix of $T^\sharp$ with respect to $\mathcal{O}$ is $\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}$, where T_1 is an $m \times m$ self-adjoint matrix. Since T_1 is the matrix of $\overline{T}$ on the IP space $\mathcal{U}/N(A)$, then T_1 is orthogonally diagonalizable. So, a sufficient condition for operator T to be orthogonally diagonalizable in SIP space is $R(T) \subseteq R(A)$ and T A-self-adjoint. Tam dan Zhang ([19]) wrote this condition on Corollary 3.1.

Corollary 3.1. *Let $\mathbb{C}^n$ be an SIP space with the SIP induced by a positive semi-definite operator $A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n\times n}$. Let $T \in \mathbb{C}^{n\times n}$ be an A-self-adjoint operator and $R(T) \subseteq R(A)$. If $V = \{x_1, x_2, \dots, x_n\}$ is a set of A-orthogonal eigen vectors of T, where $x_1, \dots, x_r \in R(A)$ and $x_{r+1}, \dots, x_n \in N(A)$, also λ_i, $i = 1, 2, \dots, n$ are eigen values of T each related to x_i, then*

$$T = V \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} V^{-1},$$

where $V^*AV = I_r \oplus 0$.

We observe that the proposed condition in Corollary 3.1 is quite strong, since it does not cover linear operators with images not contained in $R(A)$, including linear operators with rank greater than the rank(A). Therefore, it is necessary to explore spectral decomposition of operators T whose ranks are greater than rank(A), to investigate necessary or sufficient conditions for the existence of the decomposition.

Example 3.2. Let $\mathbb{C}^3$ be an SIP space induced by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Also, let $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ be an operator on $\mathbb{C}^3$. It is clear that $\text{rank}(T) > \text{rank}(A)$. Operator T is A-bounded and A-self-adjoint. We can write T as the block matrix $\begin{bmatrix} T_1 & 0 \\ 0 & T_3 \end{bmatrix}$, where $T_1 = [1]$ and $T_3 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. We can see the subspace

$$W = \left\{ \begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix} : w \in \mathbb{C} \right\},$$

which contains non-neutral elements of $\mathbb{C}^3$, is a T-invariant subspace and $W \cap N(A) = \{0\}$. Since T has 3 distinct eigenvalues, it has three A-orthonormal eigenvectors and we can write the spectral decomposition of T as

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 3.2 shows that there is an A-self-adjoint operator satisfying $R(T) \supset R(A)$ and A-orthogonally diagonalizable. In the next section, we will give necessary and sufficient conditions for an A-self-adjoint operator on an SIP space to have spectral decomposition. The definition of spectral decomposition on an SIP space is given in Definition 2.9.

3.1. *Equivalent conditions for spectral decomposition*

Theorem 3.3. *Let T be an A-self-adjoint operator on an SIP space U. The operator T is A-orthogonally diagonalizable if and only if the following conditions hold.*

(i) *T_0, the restriction T on $N(A)$, is diagonalizable.*
(ii) *There is a subspace T-invariant $W \subseteq U$ such that $W \oplus N(A) = U$.*

Proof.
($\Rightarrow$) Let T be an A-orthogonally diagonalizable operator on U. Let $\mathcal{O} = \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ be an A-orthonormal basis of U such that $[T]_{\mathcal{O}}$ is diagonal, where $\{u_1, \ldots, u_m\}$ are A-orthonormal and $u_{m+1}, \ldots, u_n$ are linearly independent neutral elements of $\mathcal{O}$. We obtain that $\mathcal{O}_N = \{u_{m+1}, \ldots u_n\}$ is a basis of $N(A)$ such that the representation matrix of T_0, the restriction of T on $N(A)$, with respect to $\mathcal{O}_N$ is diagonal. Thus T_0 is diagonalizable. Furthermore, if W is the subspace spanned by $\{u_1, \ldots, u_m\}$, then $U = W \oplus N(A)$. Since $[T]_{\mathcal{O}}$ is diagonal, each u_i is an eigenvector of T, we obtain $T(u_i) \in W$, for all $i = 1, 2, \ldots, m$. As a consequence, W is T-invariant.

($\Leftarrow$) Let T_0 be diagonalizable, then there is a basis B_0 of $N(A)$ consisting of $n - m$ elements of $N(A)$ which are also eigenvectors of T. Since W is T-invariant, the restriction of T on W is a linear operator on W. Furthermore, the restriction of the SIP of U on W is an IP. Thus, the restriction of T on W is a self-adjoint operator on the IP space W, which implies the restriction of T on W is orthogonally diagonalizable. Let B_1 be an orthonormal basis on W consisting of m eigenvectors of T. Since any element in B_0 is A-orthogonal to any element in U, we obtain $B = B_0 \cup B_1$ is an A-orthonormal basis of U such that $[T]_B$ is diagonal. Thus, T is A-orthogonally diagonalizable. □

Let us consider an A-self-adjoint operator T on an SIP U which satisfies $R(T) \subseteq R(A)$ as discussed in Corollary 3.1. We obtain $N(A) \subset N(T)$ since $N(A)$ is T-invariant and $N(A) \cap R(T) = \{0\}$. In this case, condition (i) of Theorem 3.3 holds. Further, $R(T) \subseteq R(A)$ results in condition (ii) holds by the subspace $W = R(A)$. Hence, the conditions for spectral decomposition on Corollary 3.1 have been covered in Theorem 3.3.

As shown in Example 2.10, there is even A-self-adjoint operator which is not diagonalizable. The following theorem shows that diagonalizability turns as a necessary and sufficient condition for a A-self-adjoint operator to be A-orthogonally diagonalizable.

Theorem 3.4. *Let* U *be an SIP space induced by a positive semi-definite operator* A *and let* T *be an* A*-self-adjoint operator on* U*. The operator* T *is diagonalizable if and only if* T *is* A*-orthogonally diagonalizable.*

Proof. Clearly, an A-orthogonally diagonalizable property implies a diagonalizable property. Hence, we only need to show the sufficient condition for the existence of spectral decomposition. Let T be diagonalizable. As a result, the SIP space U can be written as the direct sum $U = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$, where $\lambda_1, \dots, \lambda_k$, $k \leqslant n$ are the distinct eigenvalues of T and for each $i = 1, 2, \dots, k$, $E_{\lambda_i} = \{x \in U : T(x) = \lambda_i x\}$ is the eigenspace related to the eigenvalue λ_i. For any E_{λ_i}, there is an A-orthonormal basis consisting of eigenvectors of T related to λ_i. The union of all these bases is an A-orthonormal basis of U as long as $E_{\lambda_i} \perp_A E_{\lambda_j}$ for all $i \neq j$.

Let $i \neq j$ and take any $x \in E_{\lambda_i}$ $y \in E_{\lambda_j}$. If x or y is a neutral element of U, then from Proposition 2.1 we have $x \perp_A y$. Now let x and y are both non-neutral elements of U, then we have

$$\lambda_i[x, x] = [\lambda_i x, x] = [T(x), x] = [x, T(x)] = [x, \lambda_i x] = \overline{\lambda_i}[x, x].$$

Since $[x, x] \neq 0$, we have $\lambda_i = \overline{\lambda_i}$ or $\lambda_i \in \mathbb{R}$. Similarly, we have $\lambda_j \in \mathbb{R}$. We can also have that

$$\lambda_i[x, y] = [\lambda_i x, y] = [T(x), y] = [x, T(y)] = [x, \lambda_j y] = \lambda_j[x, y].$$

So, $(\lambda_i - \lambda_j)[x, y] = 0$. Since $\lambda_i \neq \lambda_j$, the equation is held only if $[x, y] = 0$. Hence we have $x \perp_A y$ and it has been proven that $E_{\lambda_i} \perp_A E_{\lambda_j}$ for all $i \neq j$. □

Corollary 3.5 below shows that an A-self-adjoint operator with n distinct eigenvalues is A-orthogonally diagonalizable.

Corollary 3.5. *Let* U *be an* n*-dimensional SIP space induced by a positive semi-definite operator* A *and* T *be an* A*-self-adjoint operator on* U*. If* T *has* n *distinct eigenvalues, then* T *is* A*-orthogonally diagonalizable.*

Proof. Since T has n distinct eigenvalues, then T has n linearly independent eigenvectors and, as a result, T is diagonalizable. From Theorem 3.4, T is A-orthogonally diagonalizable. □

3.2. *Sufficient conditions for spectral decomposition*

Up to now, we already obtained necessary and sufficient conditions for an A-self-adjoint operator to be A-orthogonally diagonalizable. In the following we investigate some more detailed properties or conditions that satisfy those conditions.

Theorem 3.6. *Let* T *be an* A*-self-adjoint operator on an SIP space* U *with condition* $N(A) \subseteq N(T)$ *holds. If* $N(T) \cap R(T) = \{0\}$*, then* T *is* A*-orthogonally diagonalizable.*

Proof. From $N(T) \cap R(T) = \{0\}$, we obtain $U = N(T) \oplus R(T)$. $N(A) \subseteq N(T)$ implies T_0 is diagonalizable and there exists $W_1 \subseteq N(T)$ such that $N(T) = N(A) \oplus W_1$. Further, we have $W = W_1 \oplus R(T)$ is T-invariant such that $U = N(A) \oplus W$. Thus, according to Theorem 3.3, T is A-orthogonally diagonalizable. □

Note that from the proof of Theorem 3.6. above we can conclude that the $N(T) \cap R(T) = \{0\}$ condition itself is a necessary condition for A-orthogonally diagonalizable but it alone is not a sufficient condition for A-orthogonally diagonalizable as shown by the following example.

Example 3.7. Let $\mathbb{C}^3$ be the SIP space discussed in Example 3.2. The operator $T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ on the SIP space $\mathbb{C}^3$ is an A-self-adjoint operator with condition $N(T) \cap R(T) = \{0\}$ holds. However, it is not orthogonally diagonalizable. More precisely, T cannot be diagonalized because one of its eigenvalues, namely $\lambda_1 = 1$, has different geometric multiplicity (= 1) and algebraic multiplicity (= 2).

The next sufficient condition is related to the linear operator on the quotient space induced by the investigated operator. For that we need the following technical lemma.

Lemma 3.8. *Let U be an SIP space, T be an A-bounded linear operator on U, and $\overline{T}$ be the linear operator on the quotient space $U/N(A)$ induced by T. Let $\overline{x} = x + N(A)$, for some $x \in U$ be an eigenvector of $\overline{T}$ related to an eigenvalue λ, i.e., $\overline{T}(\overline{x}) = \lambda\overline{x}$. If λ is not an eigenvalue of T_0, then there exists $x_1 \in U$ such that $\overline{x}_1 = \overline{x}$ and $T(x_1) = \lambda x_1$.*

Proof. Let λ be an eigenvalue of $\overline{T}$ and $x \in U$ is a nonzero vector satisfying $\overline{T}(\overline{x}) = \lambda\overline{x}$. Since $\overline{T}(\overline{x}) = \overline{T(x)}$, we can write

$$T(x) - \lambda x = v \Leftrightarrow (T - \lambda I)(x) = v$$

for some $v \in N(A)$. If $v = 0$ the proof is complete. Suppose $v \neq 0$. The assumption that λ is not an eigenvalue of T_0 implies the operator $T - \lambda I : N(A) \to N(A)$ is bijective and hence there is a vector $-y \in N(A)$ satisfying $(T - \lambda I)(-y) = v$. Thus, we have

$$(T-\lambda I)(x) = v, \qquad (T-\lambda I)(x) = (T-\lambda I)(-y), \qquad (T-\lambda I)(x+y) = 0,$$
$$T(x+y) = \lambda(x+y), \qquad \overline{T(x+y)} = \lambda\overline{(x+y)}, \qquad \overline{T}(\overline{x}) = \lambda\overline{x}.$$

Therefore, there exists $x_1 = x + y \neq 0$ such that $T(x_1) = \lambda x_1$ and $\overline{x_1} = \overline{x}$. □

Theorem 3.9 below shows the relation between the diagonizability of the induced operator on IP space $U/N(A)$ and the operator on SIP space U.

Theorem 3.9. *Let T be an A-self-adjoint linear operator on an SIP space U. If $\overline{T}$ and T_0 have different eigenvalues and T_0 is diagonalizable, then T is A-orthogonally diagonalizable.*

Proof. Since T_0 is diagonalizable, then there is $\{x_{m+1}, \dots, x_n\}$ a basis of $N(A)$ consisting of eigenvectors of T_0. Since T is A-self-adjoint, then $\overline{T}$ is a self-adjoint operator on IP space $U/N(A)$. Hence, $\overline{T}$ has a spectral decomposition, i.e., there is $\{\overline{x_1}, \dots, \overline{x_m}\}$ an orthonormal basis of $U/N(A)$ consisting eigenvectors of $\overline{T}$. Let each eigenvector $\overline{x_i}$ correspond to an eigenvalue λ_i, for $i = 1, 2, \dots, m$, and hence λ_i is not an eigenvalue of T_0. According to Lemma 3.8, for every $i = 1, 2, \dots, m$ we may assume x_i is an eigenvector of T, i.e., $T(x_i) = \lambda x_i$. From the orthonormality property of $\{\overline{x_1}, \dots, \overline{x_m}\}$, we obtain $\{x_1, \dots, x_m\}$ is also an orthonormal set with respect to SIP on U.

Hence, we have $\{x_1, \dots, x_m, x_{m+1}, \dots, x_n\}$ is an A-orthonormal basis of U consisting of eigenvectors of T. Thus, T is A-orthogonally diagonalizable. □

The next Corollary gives the matrix form of Theorem 3.9.

Corollary 3.10. *Let $\mathbb{C}^n$ be an SIP space with the SIP induced by $A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$. Let $T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}$ be an A-self-adjoint operator on the SIP space $\mathbb{C}^n$, for some $T_1 \in \mathbb{C}^{m\times m}$ and $T_3 \in \mathbb{C}^{(n-r)\times(n-r)}$. If T_1 and T_3 have different eigenvalues and T_3 is diagonalizable, then T is A-orthogonally diagonalizable.*

Theorem 3.9 states that when $\overline{T}$ and T_0 have different eigenvalues, then T is A-orthogonally diagonalizable. That condition is sufficient but it is not necessary as shown in the following example.

Example 3.11. Let

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

be two A-self-adjoint operators on SIP space $\mathbb{C}^3$, with the SIP induced by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Both operators T and S have equal self-adjoint induced operators on the quotient space $\mathbb{C}^3/N(A)$ with eigenvalues 1 and 2. The restriction on $N(A)$ of both operators T and S are also equal, which is diagonalizable with eigenvalue 2. However, the operator T is not A-orthogonally diagonalizable since the eigenvector on the quotient space related to eigenvalue $\lambda = 2$ does not have corresponding non-neutral eigenvector on $\mathbb{C}^3$. In contrast, for the operator S, each eigenvector on the quotient space has corresponding non-neutral eigenvector on $\mathbb{C}^3$. Hence, S is A-orthogonally diagonalizable.

Acknowledgment

This research is supported by Hibah PPMI, Institut Teknologi Bandung 2024 and is part of dissertation research of the first author. First author is thankful for Universitas Islam Bandung for funding her doctoral study at Mathematics Doctoral Program of Institut Teknologi Bandung. The authors thank to the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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