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Stability and bifurcation analysis of predator-prey model with Allee effect using conformable derivatives

M. B. Almatrafi[a,*], Messaoud Berkal[b]

[a]Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia.

[b]Department of Applied Mathematics, University of Alicante, 03690 Alicante, Spain.

Abstract

Some organisms coexist on the expense of others. This coexistence is called predation which has been successfully investigated using differential equations. In this work, we aim to analyse a fractional order predator-prey dynamical system with Allee effect using bifurcation theory. The Allee effect is a density-dependent phenomenon where the population growth and individual fitness increase as population density increases. Several mechanisms, such as cooperative feeding, mate limitation, and predator satiation, can cause Allee effects. The piecewise-constant approximation method and the conformable derivatives are utilized to discretise the propose model. We explore equilibrium points, the local stability, the Neimark-Sacker bifurcation, periodic-doubling bifurcation, chaos control, and numerical simulations of the proposed model. The linear theory of stability is used to examine the local attractivity of the fixed points. Our findings include that the coexistence equilibrium point is locally stable, source, unstable under certain constraints. We also prove that the considered discrete model goes through Neimark-Sacker and periodic-doubling bifurcations according to specific conditions. The used techniques can be applied for other nonlinear discrete systems.

Keywords: Stability, Allee effect, prey-predator model, Neimark-Sacker bifurcation, period-doubling bifurcation, fractional derivatives, chaos control.

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1. Introduction

Some species usually connect with each other to be alive. This contact can be beneficial to both species or beneficial to one population at the expense of the other. For instance, when two organisms engage antagonistically, one benefits at the expense of the other. In particular, this type of interactions can be obviously seen in predation, parasitism, and herbivory. The predation depends on two or more species where one organism feeds on the other. For example, tigers hunt buffaloes, wolves hunt deer, hyenas and lions hunt zebras, foxes hunt rabbits, shrews hunt worms and insects, etc. The predation has a strong impact on both species. More specifically, the number of predators sharply grows if the number of prey is high whereas the number of the predators reduces if the number of prey is low. The predation can occur in plants. For example, the Venus flytrap and the pitcher plants consume some insects. It is important

*Corresponding author
Email addresses: mmutrafi@taibahu.edu.sa (M. B. Almatrafi), mb299@gcloud.ua.es (Messaoud Berkal)

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to emphasize that the relationships between predators and prey are crucial for preserving the ecological balance among various kinds of organisms. In the absence of predators, food resources for prey may finish due to the competition of prey on food. In the other hand, predators cannot still alive without prey. More information about biological interaction between populations can be found in refs. [11, 13, 14].

Mathematics have been successfully used to explain the behavior of some biological phenomena. In fact, the concepts of differential equations and difference equations have been widely invoked to describe such phenomena. This can be clearly seen in many discrete and continuous dynamical systems such as Lotka-Volterra system, which was separately written by Lotka (1925) and Volterra (1926). This model, which consists of two differential equations, investigates the interactions between predators and prey populations (parasite-host or herbivore-plant). This dynamical system has been widely investigated by several scholars to give more logical justification about the interaction between these species. Some pf the previous studies are listed as follows. Leslie [16, 17] wrote a useful Leslie predator-prey dynamical system where the carrying capacity of the predator species is proportional to the number of prey species. The behavior of the predation when a prey species is frightened by predators was nicely studied in [24]. Kumar and Kharbanda [15] explored the occurrence of the equilibrium points, stability, and the bifurcation of the parasite-host dynamical model in the presence of defense. The local behavior of the fixed points and the Hopf bifurcations of a predator-prey dynamical model with Holling-II type functional response were successfully discussed in [31]. Moreover, finite difference schemes were used in [2] to obtain the solutions of a continuous-time Leslie prey-predator system and to analyse the Neimark-Sacker bifurcation of the considered model. In [2], the authors utilized the hybrid control approach to show the chaos and the bifurcation of a predator-prey dynamical model. Furthermore, Moustafa et al. [22] formulated and discussed a fractional-order eco-epidemiological model with disease in the prey species. The occurrence of non-negativity and boundedness of the solutions of the considered model were shown in [22]. Finally, Arif et al. [3] studied a fractional order system which describes a predator and two types of prey populations using the Caputo fractional derivative. The authors in [3] obtained the equilibrium points of the proposed fractional order system and examined the stability of this model. In addition, Din [7] discussed a discrete-time prey-predator system in terms of its complex behavior and chaos control. The existence and uniqueness of the fixed points of the discrete-time Leslie-Gower prey-predator model were also successfully investigated in [7]. In [8], the authors examined the dynamical behavior of a host-parasitoid system with a strong Allee effect. In particular, they studied local asymptotic stability, Neimark-Sacker bifurcation of the discretized model presented in [8]. Furthermore, Khan et al. [12] considered a discrete time plant-herbivore system. They presented the Neimark-Sacker bifurcation, invariant closed curve, and the numerical simulations of the proposed model in [12].

Some researchers have discovered that some natural events can be precisely investigated by using fractional-order systems. Fractional calculus, which was discovered in 1695, is involved in some biological models to control the performance of these systems. Scientists established some definitions for fractional derivatives. For instance, Laplace developed a notable definition for fractional derivatives of functions by utilizing integrals in 1812. Then, Lacroix wrote the n-th derivative of a given power function in 1812 [27]. Liouville developed his first concept for fractional derivatives in 1832 [21]. After that, Riemann published his beneficial definition for fractional derivatives [26]. In addition, the Riemann-Liouville definition for fractional derivatives of a given function was nicely produced in the 19th century. Regrettably, some of these concepts do not lead to the same outcomes for fractional derivatives. As a result, the most modern concept of fractional derivative which is the comfortable fractional derivative [10] is utilized. The main motivation of this work comes from the above mentioned literature reviews in which discrete predator-prey models with Allee effect have not received enough investigations. Moreover, the motivation comes from useful applications of this model in the real life problems where some biological phenomena, such as the relationships between lions and buffaloes, can be studied using this model. In this article, we aim to investigate various qualitative properties including the occurrence of equilibrium points and the stability of the model

$$\begin{cases} T^{\gamma}u(t) = u(t)\left(1 - u(t) - \frac{u(t)v(t)}{u(t)+C}\right), \\ T^{\gamma}v(t) = v(t)(Nu(t) - M), \end{cases} \tag{1.1}$$

where the variables $u(t) > 0$ and $v(t) > 0$ represent population densities of prey and predator at time t, respectively. The parameter M is the intrinsic death rates of predator and N is the prey capture. The term $\left(\frac{u(t)}{u(t)+C}\right)$ denotes the Allee effect and $C > 0$ is the constant of the proposed effect. The fractional-order parameter is $0 < \gamma < 1$. Moreover, T_p^{γ} is the fractional derivative of the conformable-type. The piecewise-constant numerical technique is utilized to find the discretization of system (1.1). We also discuss the conditions under which system (1.1) goes through Neimark-Sacker and period-doubling bifurcations. The chaos control is also analysed using hybrid control method. The derived theoretical results are then supported by numerical computations.

The structure of this work is shown as follows. Section 3 illustrates the discretization of our model. The equilibrium points and the stability of these points are given in Section 4. Section 5 is devoted to analyse the Neimark-Sacker and period-doubling bifurcations. In Section 6, we show the chaos control while Section 7 illustrates some numerical simulations for the obtained outcomes. This article is finally concluded in Section 8.

2. Preliminaries

In this part, we define the most important terminology in this paper.

Definition 2.1 ([18])**.** The Neimark-Sacker bifurcation occurs when an equilibrium point changes stability by a pair of complex eigenvalues with unit modulus in a dynamical system. This type of bifurcation can be subcritical or supercritical in an unstable or stable closed invariant curve.

Definition 2.2 ([18])**.** The Period doubling bifurcation (flip bifurcation) takes place when a small change in the bifurcation parameter leads to a new behavior with twice the period of the original system. Mathematically, we can define a flip bifurcation as follows. Let $h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a one-parameter family of C^3 maps satisfying the following conditions:

$$h(0,0) = 0, \qquad \left(\frac{\partial h}{\partial u}\right)_{p=0,u=0} = -1, \qquad \left(\frac{\partial^2 h}{\partial u^2}\right)_{p=0,u=0} < 0, \qquad \left(\frac{\partial^3 h}{\partial u^3}\right)_{p=0,u=0} < 0.$$

Then, there are intervals $(p_1, 0)$, $(0, p_2)$, and $\epsilon > 0$ such that

1. if $p \in (0, p_2)$, then $h_p(u)$ has one stable orbit of period two for $u \in (-\epsilon, \epsilon)$ and one unstable equilibrium point;
2. if $p \in (p_1, 0)$, then $h_p(u)$ has a unique stable equilibrium point for $u \in (-\epsilon, \epsilon)$.

A flip bifurcation is the term utilized to describe this kind of bifurcation.

Definition 2.3 ([29, 30])**.** Let the discrete dynamical system $u_{k+1} = h(u_k) = h^{k+1}(u_0)$, where $u \in \mathbf{R}^n$. Consider a minor change Du_0 in the initial values u_0, the sensitivity to initial conditions can be measured as

$$||Du_k|| \approx ||Du_0||e^{k\lambda},$$

where λ is the maximum Lyapunov exponent (MLE), which can be calculated by

$$L = \lim_{k\to\infty} \frac{1}{k} \ln \frac{||Du_k||}{||Du_0||}.$$

For $n = 1$ (one dimensional system), the Lyapunov exponents is described as

$$L = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} \ln |\, h'(u_i) \,| \,.$$

Definition 2.4 ([9, 10])**.** Let $h : (0, \infty) \to R$ be a function. Then, the conformable fractional derivative of order $0 < \gamma \leqslant 1$ of h at $t > 0$ is described as

$$T_p^{\gamma} h(t) = \lim_{\epsilon \to 0} \frac{h(t + \epsilon(t-p)^{1-\gamma}) - h(t)}{\epsilon}, \quad 0 < \gamma < 1, \tag{2.1}$$

where T_p^{γ} is a fractional derivative of the conformable-type and $p > 0$ is the discretization parameter. It was presented in [1], that the derivative of Eq. (2.1) is given by $T_p^{\gamma} h(t) = (t-p)^{1-\gamma} h'(t)$.

3. Discretization technique

In this part, system (1.1) is discretized by employing the piecewise-constant numerical method [19, 28]. Applying this method leads to

$$\begin{cases} \frac{\mathcal{T}^{\gamma} u(t)}{u(t)} = \left(1 - u(\left[\frac{t}{p}\right] p) - \frac{u([\frac{t}{p}]p) v([\frac{t}{p}]p)}{u([\frac{t}{p}]p) + C}\right), \\ \frac{\mathcal{T}^{\gamma} v(t)}{v(t)} = (N u(\left[\frac{t}{p}\right] p) - M), \end{cases} \tag{3.1}$$

where $[\frac{t}{p}]$ represents the integer part of $t \in [np, (n+1)p), n = 0, 1, \ldots$, and $p > 0$ is a discretization parameter. The conformable fractional derivative is then applied on the first equation of system (3.1) to end up with the equation

$$\frac{du(t)}{u(t)dt} = \left(1 - u(np) - \frac{u(np)v(np)}{u(np) + C}\right)(t - np)^{\gamma - 1},$$

whose solution over the interval $[np, t)$ is given by

$$\ln(u(t)) - \ln(u(np)) = \left(1 - u(np) - \frac{u(np)v(np)}{u(np) + C}\right)\frac{(t-np)^{\gamma}}{\gamma}. \tag{3.2}$$

Let $t \to (n+1)p$ and replace $v(np)$ and $u(np)$ by v_n and u_n, respectively, then Eq. (3.2) becomes

$$u_{n+1} = u_n e^{(1 - u_n - \frac{u_n v_n}{u_n + C})\frac{p^{\gamma}}{\gamma}}.$$

In an analogous way, one can obtain the solution of the second equation of system (3.1) as follows:

$$v_{n+1} = v_n e^{(N u_n - M)\frac{p^{\gamma}}{\gamma}}.$$

The discrete version of the model (1.1) is so presented as

$$\begin{cases} u_{n+1} = u_n e^{(1 - u_n - \frac{u_n v_n}{u_n + C})\frac{p^{\gamma}}{\gamma}}, \\ v_{n+1} = v_n e^{(N u_n - M)\frac{p^{\gamma}}{\gamma}}. \end{cases} \tag{3.3}$$

4. Local stability of equilibrium point

This section discusses the occurrence of the equilibrium points of model (3.3). We also investigate the local stability of the obtained points. To find the equilibrium points of system (3.3), we obtain the solution of the following equations:

$$\begin{cases} u = u e^{(1 - u - \frac{uv}{u + C})\frac{p^{\gamma}}{\gamma}}, \\ v = v e^{(Nu - M)\frac{p^{\gamma}}{\gamma}}. \end{cases}$$

System (3.3) has only three equilibrium points which are $O = (0, 0)$, $E = (1, 0)$ (the point O represents the extinction point of both prey and predator populations, and the point E represents the extinction point of predator population), and the positive equilibrium point $P = \left(\frac{M}{N}, \frac{(N-M)(M+NC)}{MN}\right)$ for $N > M$ (the point P represents the coexistence equilibrium point).

Next, the stability of the equilibrium points is examined with the help of the following Lemma.

Lemma 4.1 ([4, 5, 19])**.** *Let (u, v) be an equilibrium point of system (3.3) with multipliers (eigenvalues of Jacobian matrix) δ_1 and δ_2. Then,*

1. *the equilibrium point (u, v) is a sink (locally asymptotic stable) if $|\delta_1| < 1$ and $|\delta_2| < 1$;*
2. *the equilibrium point (u, v) is a source if $|\delta_1| > 1$ and $|\delta_2| > 1$;*
3. *the equilibrium point (u, v) is a saddle if $|\delta_1| < 1$ and $|\delta_2| > 1$, or if $|\delta_1| > 1$ and $|\delta_2| < 1$;*
4. *the equilibrium point (u, v) is a non-hyperbolic if $|\delta_1| = 1$ or $|\delta_2| = 1$.*

Lemma 4.2 ([4, 6, 19])**.** *Assume that the polynomial $P(\delta) = \delta^2 - \mathcal{T}\delta + \mathcal{D}$, where $P(1) > 0$, and δ_1 and δ_2 are the two roots of $P(\delta) = 0$. Then,*

1. *$|\delta_1| < 1$ and $|\delta_2| < 1$ if and only if $P(-1) > 0$ and $P(0) < 1$;*
2. *$|\delta_1| > 1$ and $|\delta_2| > 1$ if and only if $P(-1) > 0$ and $P(0) > 1$;*
3. *$|\delta_1| < 1$ and $|\delta_2| > 1$ (or $|\delta_1| > 1$ and $|\delta_2| < 1$) if and only if $P(-1) < 0$;*
4. *$\delta_1 = -1$ and $\delta_2 \neq 1$ if and only if $P(-1) = 0$ and $\mathcal{T} \neq 0, 2$;*
5. *δ_1 and δ_2 are complex numbers and $|\delta_1| = |\delta_2| = 1$ if and only if $|\mathcal{T}| < 2$ and $P(0) = 1$.*

Next, we attempt to analyse the stability of the proposed system. The Jacobian matrix of system (3.3) at any equilibrium point (u, v) can be written as

$$\mathcal{J}((u,v)) = \begin{pmatrix} \left(1 - \frac{up^{\gamma}}{\gamma}\left(1 + \frac{Cv}{(u+C)^2}\right)\right) e^{\left(1-u_n-\frac{u_n v_n}{u_n+C}\right)\frac{p^{\gamma}}{\gamma}} & -\frac{u^2 p^{\gamma} e^{\left(1-u_n-\frac{u_n v_n}{u_n+C}\right)\frac{p^{\gamma}}{\gamma}}}{\gamma(u+C)} \\ \frac{Nvp^{\gamma}e^{(Nu_n-M)\frac{p^{\gamma}}{\gamma}}}{\gamma} & e^{(Nu_n-M)\frac{p^{\gamma}}{\gamma}} \end{pmatrix}. \tag{4.1}$$

Consequently, the auxiliary equation of the previous matrix is $\rho(\mu) = \mu^2 - \mathcal{T}\mu + \mathcal{D}$. Here, we have

$$\mathcal{T} = \left(1 - \frac{up^{\gamma}}{\gamma}\left(1 + \frac{Cv}{(u+C)^2}\right)\right) e^{\left(1-u_n-\frac{u_n v_n}{u_n+C}\right)\frac{p^{\gamma}}{\gamma}} + e^{(Nu_n-M)\frac{p^{\gamma}}{\gamma}},$$
$$\mathcal{D} = \left(\left(1 - \frac{up^{\gamma}}{\gamma}\left(1 + \frac{Cv}{(u+C)^2}\right)\right) + \frac{Nvu^2p^{2\gamma}}{\gamma^2(u+C)}\right) e^{\left(1-u_n-\frac{u_n v_n}{u_n+C}+Nu_n-M\right)\frac{p^{\gamma}}{\gamma}}.$$

Next, we present some theorems on the stability of the fixed points.

Theorem 4.3. *The equilibrium point $O = (0, 0)$ of model (3.3) is a saddle point.*

Proof. The Jacobian matrix of system (3.3) at the fixed point $O = (0, 0)$ can be written as

$$J(O) = \begin{pmatrix} e^{\frac{p^{\gamma}}{\gamma}} & 0 \\ 0 & e^{\frac{-Mp^{\gamma}}{\gamma}} \end{pmatrix},$$

whose eigenvalues are $\mu_1 = e^{\frac{p^{\gamma}}{\gamma}} > 1$, $\mu_2 = e^{\frac{-Mp^{\gamma}}{\gamma}} < 1$. Applying Lemma 4.1 we can conclude that the fixed point $O = (0, 0)$ is a saddle point. □

Theorem 4.4. *Let E be an equilibrium point for system (3.3). Then, the following outcomes hold.*

1. *If $N < M$, then*
   - *the equilibrium point E is a sink, if $0 < p < (2\gamma)^{\frac{1}{\gamma}}$;*
   - *the equilibrium point E is non-hyperbolic, if $p = (2\gamma)^{\frac{1}{\gamma}}$;*
   - *the equilibrium point E is a saddle, if $p > (2\gamma)^{\frac{1}{\gamma}}$.*
2. *If $N = M$, then the equilibrium point E is non-hyperbolic.*
3. *If $N > M$, then*
   - *the equilibrium point E is a saddle, if $0 < p < (2\gamma)^{\frac{1}{\gamma}}$;*

- *the equilibrium point* E *is non-hyperbolic, if* $p = (2\gamma)^{\frac{1}{\gamma}}$;
- *the equilibrium point* E *is a source, if* $p > (2\gamma)^{\frac{1}{\gamma}}$.

Proof. The Jacobian matrix of model (3.3) at $E = (1, 0)$ can be expressed as

$$J(O) = \begin{pmatrix} 1 - \frac{p^\gamma}{\gamma} & -\frac{p^\gamma}{\gamma(1+C)} \\ 0 & e^{\frac{(N-M)p^\gamma}{\gamma}} \end{pmatrix},$$

whose eigenvalues are $\mu_1 = 1 - \frac{p^\gamma}{\gamma}$, $\mu_2 = e^{\frac{(N-M)p^\gamma}{\gamma}}$. Using Lemma 4.1, it is easy to prove the results of this theorem. □

Theorem 4.5. *Assume that* $N > M$ *and let*

$$\Delta = (M^2 + CN^2)^2 - 4MN(N-M)(M+CN)^2.$$

Consequently, the subsequent claims are true.

1. *The point* P *is locally asymptotically stable (sink) if one of the following requirements holds:*
   (i) $\Delta < 0$ *and* $0 < p < \left(\frac{\gamma(M^2+CN^2)}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}$;
   (ii) $\Delta \geqslant 0$ *and* $0 < p < \left(\frac{\gamma((M^2+CN^2)-\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}$.
2. *The point* P *is unstable (source) if one of the following requirements holds:*
   (i) $\Delta < 0$ *and* $p > \left(\frac{\gamma(M^2+CN^2)}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}$;
   (ii) $\Delta \geqslant 0$ *and* $p > \left(\frac{\gamma((M^2+CN^2)-\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}$.
3. *The equilibrium point* P *is unstable (saddle) if*
$$\Delta > 0 \quad \text{and} \quad \left(\frac{\gamma((M^2+CN^2)-\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}} < p < \left(\frac{\gamma((M^2+CN^2)+\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}.$$
4. *The point* P *is non-hyperbolic and the solutions of the characteristic polynomial at this point are* $\delta_1 = -1$ *and* $|\delta_2| \neq 1$ *if*
$$\Delta \geqslant 0, \quad p = \left(\frac{\gamma((M^2+CN^2)\mp\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}, \quad \text{and} \quad p \neq \left(\frac{2\gamma N(M+CN)}{M^2+CN}\right)^{\frac{1}{\gamma}}.$$
5. *The point* P *is non-hyperbolic and the solutions of the characteristic polynomial at this point are complex numbers with modulus one if*
$$\Delta < 0, \quad p = \left(\frac{\gamma(M^2+CN^2)}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}.$$

Proof. Evaluating the Jacobian matrix (4.1) at the fixed point $P = \left(\frac{M}{N}, \frac{(N-M)(M+NC)}{MN}\right)$ leads to

$$J(P) = \begin{pmatrix} 1 - \frac{p^\gamma}{\gamma}\left(\frac{M^2+CN^2}{N(M+CN)}\right) & -\frac{M^2p^\gamma}{\gamma N(M+CN)} \\ \frac{p^\gamma(N-M)(M+CN)}{\gamma M} & 1 \end{pmatrix},$$

with characteristic polynomial

$$\rho(\mu) = \mu^2 - \mathcal{T}\mu + \mathcal{D}. \tag{4.2}$$

Here, we have

$$\mathcal{T} = \left(2 - \frac{p^\gamma}{\gamma}\left(\frac{M^2+CN^2}{N(M+CN)}\right)\right), \quad \mathcal{D} = \left(1 - \frac{p^\gamma}{\gamma}\left(\frac{M^2+CN^2}{N(M+CN)}\right) + \frac{p^{2\gamma}}{\gamma^2}\left(\frac{M(N-M)}{N}\right)\right).$$

Hence,

$$\rho(1) = \frac{p^{2\gamma}}{\gamma^2}\left(\frac{M(N-M)}{N}\right), \quad \rho(-1) = 4 - \frac{2p^{\gamma}}{\gamma}\left(\frac{M^2+CN^2}{N(M+CN)}\right) + \frac{p^{2\gamma}}{\gamma^2}\left(\frac{M(N-M)}{N}\right),$$

and

$$\rho(0) = 1 - \frac{p^{\gamma}}{\gamma}\left(\frac{M^2+CN^2}{N(M+CN)}\right) + \frac{p^{2\gamma}}{\gamma^2}\left(\frac{M(N-M)}{N}\right).$$

Now, applying Lemma 4.2, it easy to prove the results presented in the theorem. □

5. Bifurcation analysis

This section investigates some certain types of bifurcations and presents them under some specific constraints.

Lemma 5.1 ([20, 23, 25]). *Assume that $U_{k+1} = F_\mu(U_k)$ is a n-dimensional discrete dynamical system where $\mu \in R$ is a bifurcation parameter. Let U^* be an equilibrium point of F_μ and suppose that the characteristic polynomial of the Jacobian matrix $J(U^*) = (b_{ij})_{n\times n}$ of n-dimensional map $F_\mu(U_k)$ is given by*

$$P_\mu(\delta) = \delta^n + b_1\delta^{n-1} + \cdots + b_{n-1}\delta + b_n, \tag{5.1}$$

where $b_i = b_i(\mu, u)$, $i = 1, 2, 3, \ldots, n$ and u is a control parameter or another parameter to be deduced. Let $D_0^{\pm}(\mu, u) = 1$, $D_1^{\pm}(\mu, u), \ldots, D_n^{\pm}(\mu, u)$ be a sequence of the determinants defined by

$$D_i^{\pm}(\mu, u) = \det(A_1 \pm A_2), \quad i = 1, 2, \ldots, n, \tag{5.2}$$

where

$$A_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & 0 & 0 \\ b_n & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.3}$$

Moreover, suppose that the following conditions hold.

(P1) *Eigenvalue criterion: $P_{\mu_0}(-1) = 0$, $D_{n-1}^{\pm}(\mu_0, u) > 0$, $P_{\mu_0}(1) > 0$, $D_i^{\pm}(\mu_0, u) > 0$, $i = n-2, n-4, \ldots, 1$ (or 1), when n is even (or odd), respectively.*

(P2) *Transversality criterion: $\frac{\sum_{i=1}^{n}(-1)^{n-i}b_i'}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)b_{i-1}} \neq 0$, where b_i' denotes derivative of $b(\mu)$ at $\mu = \mu_0$.*

Then, a period-doubling bifurcation occurs at critical value μ_0.

Lemma 5.2 ([20, 25]). *Consider the following n-dimensional discrete system $U_{k+1} = F_\mu(U_k)$, where $U_k \in R^n$ and $\mu \in R$ denotes the bifurcation parameter. Furthermore, suppose that the requirements (5.1)-(5.3) of Lemma 5.1 are satisfied and suppose that the following conditions hold.*

(H1) *Eigenvalue assignment $D_{n-1}^{-}(\mu_0, u) = 0$, $D_{n-1}^{+}(\mu_0, u) > 0$, $P_{\mu_0}(1) > 0$, $(-1)^n P_{\mu_0}(-1) > 0$, $D_i^{\pm}(\mu, u) > 0$, for $i = n-3, n-5, \ldots, 2$ (or 1), when n is odd (or even), respectively.*

(H2) *Transversality condition: $\left[\frac{dD_{n-1}^{-}(\mu, u)}{d\mu}\right]_{\mu=\mu_0} \neq 0$.*

(H3) *Non-resonance condition: $\cos(2\pi/j) \neq \phi$, or resonance condition $\cos(2\pi/j) = \phi$, where $j = 3, 4, 5, \ldots$, and $\phi = 1 - 0.5P_{\mu_0}(1)D_{n-3}^{-}(\mu_0, u)/D_{n-2}^{+}(\mu_0, u)$.*

Then, a Neimark-Sacker bifurcation occurs at μ_0.

Theorem 5.3. *Let us assume* $N > M$. *Then, model* (3.3) *goes through a period-doubling bifurcation at the unique positive equilibrium point* P, *if the following constraints hold:*

$$1 + \mathcal{D} > 0, \qquad 1 + \mathcal{T} + \mathcal{D} = 0, \qquad 1 - \mathcal{T} + \mathcal{D} > 0.$$

Thus, the period-doubling bifurcation occurs at p *if the parameters* (C, M, N, γ, p) *vary in a neighborhood of the set*

$$\mathcal{B}_1 = \left\{ \begin{array}{l} (C, M, N, \gamma, p) \in R^5 \,\middle|\, (M^2 + CN^2)^2 \geqslant 4MN(N - M)(M + CN)^2, \\ p \neq \left(\frac{2\gamma N(M+CN)}{M^2+CN}\right)^{\frac{1}{\gamma}} p = p_1 = \left(\frac{\gamma((M^2+CN^2)-\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}, \gamma \in (0, 1] \end{array} \right\},$$

or,

$$\mathcal{B}_2 = \left\{ \begin{array}{l} (C, M, N, \gamma, p) \in R^5 \,\middle|\, (M^2 + CN^2)^2 \geqslant 4MN(N - M)(M + CN)^2, \\ p \neq \left(\frac{2\gamma N(M+CN)}{M^2+CN}\right)^{\frac{1}{\gamma}} p = p_2 = \left(\frac{\gamma((M^2+CN^2)+\sqrt{\Delta})}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}}, \gamma \in (0, 1] \end{array} \right\}.$$

Proof. Based on condition 4 presented in Theorem 4.5, using Lemma 5.1, and from the evaluation of Eq. (4.2) of system (3.3) at P, we have

$$D_0^{\mp}(p) = 1 > 0, \quad D_1^{+}(p) = 1 + \mathcal{D} > 0, \quad (-1)^2 P(-1) = 1 + \mathcal{T} + \mathcal{D} = 0, \quad P(1) = 1 - \mathcal{T} + \mathcal{D} > 0,$$

if and only if

$$p = p_{1,2} = \left[\frac{\gamma((M^2 + CN^2) \mp \sqrt{\Delta})}{M(M + CN)(N - M)}\right]^{\frac{1}{\gamma}} \quad \text{and} \quad \Delta = (M^2 + CN^2)^2 - 4MN(N - M)(M + CN)^2 \geqslant 0.$$

In addition, the transversality condition is

$$\frac{\mathcal{T}' + \mathcal{D}'}{\mathcal{T} + 2} = \frac{2p_{1,2}^{\gamma-1} M(N - M)(M + CN)}{M^2 + CN^2} = \frac{2M(N - M)(M + CN)}{M^2 + CN^2} \left[\frac{\gamma((M^2 + CN^2) \mp \sqrt{\Delta})}{M(M + CN)(N - M)}\right]^{\frac{\gamma-1}{\gamma}} \neq 0,$$

with $\mathcal{T}' = \left.\frac{d\mathcal{T}}{dp}\right|_{p=p_{1,2}}$ and $\mathcal{D}' = \left.\frac{d\mathcal{D}}{dp}\right|_{p=p_{1,2}}$. Then, the period-doubling bifurcation occurs at p. □

Theorem 5.4. *Let us assume* $N > M$. *Then, model* (3.3) *goes through a Neimark-Sacker bifurcation at the point* P, *if the following conditions hold*

$$1 - \mathcal{D} = 0, \quad 1 + \mathcal{D} > 0, \quad 1 - \mathcal{T} + \mathcal{D} > 0, \quad 1 + \mathcal{T} + \mathcal{D} > 0.$$

Thus, it can be said that a Neimark-Sacker bifurcation occurs at p *if the parameters* (C, M, N, γ, p) *vary in a neighborhood of the set*

$$\mathcal{B}_3 = \left\{ \begin{array}{l} (C, M, N, \gamma, p) \in R^5 \,\middle|\, p = p_3 = \left(\frac{\gamma(M^2+CN^2)}{M(M+CN)(N-M)}\right)^{\frac{1}{\gamma}} \\ (M^2 + CN^2)^2 < 4MN(N - M)(M + CN)^2, \gamma \in (0, 1] \end{array} \right\}.$$

Proof. Based on condition 5 presented in Theorem 4.5, using Lemma 5.2, and from the evaluation of Eq. (4.2) of system (3.3) at P, we find

$$\begin{array}{ccc} D_0^{\mp}(p) = 1 > 0, & D_1^{-}(p) = 1 - \mathcal{D} = 0, & D_1^{+}(p) = 1 + \mathcal{D} > 0, \\ P(1) = 1 - \mathcal{T} + \mathcal{D} > 0, & (-1)^2 P(-1) = 1 + \mathcal{T} + \mathcal{D} > 0, & \end{array}$$

if and only if

$$p = p_3 = \left(\frac{\gamma(M^2 + CN^2)}{M(M + CN)(N - M)}\right)^{\frac{1}{\gamma}} \quad \text{and} \quad (M^2 + CN^2)^2 < 4MN(N - M)(M + CN)^2.$$

Moreover, the transversality condition is

$$\left[\frac{dD_1^-(p)}{dp}\right]_{p=p_3} = \left[\frac{d(1 - \mathcal{D})}{dp}\right]_{p=p_3} = -\left(\frac{(M^2 + CN^2)}{N(M + CN)}\right)\left(\frac{\gamma(M^2 + CN^2)}{M(M + CN)(N - M)}\right)^{\frac{\gamma-1}{\gamma}} < 0.$$

Then, the Neimark-Sacker bifurcation takes place at p_1. Hence, the proof is complete. □

6. Chaos control

The chaos control approach aims to stabilize chaotic systems towards desirable behaviors. This technique applies external perturbations and manipulates system parameters. It is worth noting that the effectiveness of this strategy can be sometimes limited to control chaotic behavior in a given model. This part investigates the chaos control of system (3.3) by utilizing the hybrid control technique. To employ this technique to system (3.3), we write the corresponding control system in the following form:

$$\begin{cases} u_{n+1} = \epsilon u_n e^{\left(1 - u_n - \frac{u_n v_n}{u_n + B}\right)\frac{p^\gamma}{\gamma}} + (1 - \epsilon)u_n, \\ v_{n+1} = \epsilon v_n e^{(N u_n - M)\frac{p^\gamma}{\gamma}} + (1 - \epsilon)v_n. \end{cases} \tag{6.1}$$

Here, $0 < \epsilon < 1$ is considered as a control parameter for the hybrid control strategy. The Jacobian matrix of system (6.1) can be written as

$$J(P) = \begin{pmatrix} 1 - \frac{\epsilon p^\gamma}{\gamma}\left(\frac{M^2 + CN^2}{N(M + CN)}\right) & -\frac{\epsilon M^2 p^\gamma}{\gamma N(M + CN)} \\ \frac{\epsilon p^\gamma (N - M)(M + CN)}{\gamma M} & 1 \end{pmatrix},$$

from which we have

$$\delta^2 - \mathcal{T}\delta + \mathcal{D} = 0. \tag{6.2}$$

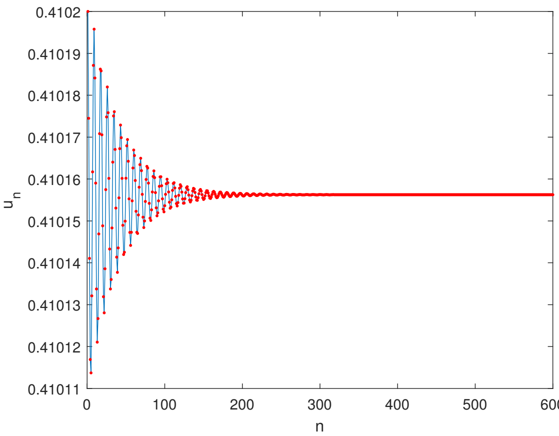
Here, we have

$$\mathcal{T} = 2 - \frac{\epsilon p^\gamma}{\gamma}\left(\frac{M^2 + CN^2}{N(M + CN)}\right), \quad \mathcal{D} = 1 - \frac{\epsilon p^\gamma}{\gamma}\left(\frac{M^2 + CN^2}{N(M + CN)}\right) + \frac{\epsilon^2 M p^\gamma (N - M)}{\gamma^2 N}.$$

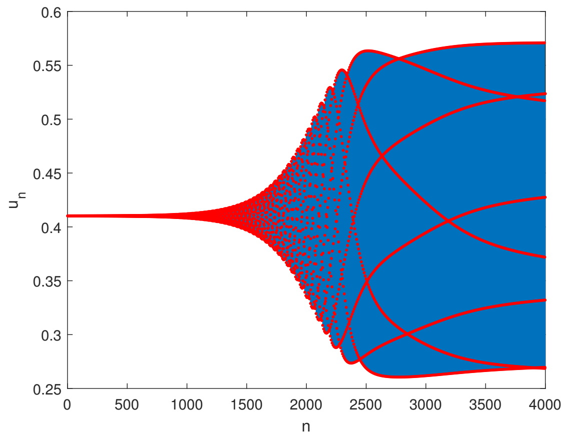
Lemma 6.1. *The point* $P = \left(\frac{M}{N}, \frac{(N-M)(M+NC)}{MN}\right)$ *of model* (6.1) *is locally asymptotically stable, if the following condition satisfies:*

$$\left|2 - \frac{\epsilon p^\gamma}{\gamma}\left(\frac{M^2 + CN^2}{N(M + CN)}\right)\right| < 2 - \frac{\epsilon p^\gamma}{\gamma}\left(\frac{M^2 + CN^2}{N(M + CN)}\right) + \frac{\epsilon^2 M p^\gamma (N - M)}{\gamma^2 N} < 2.$$

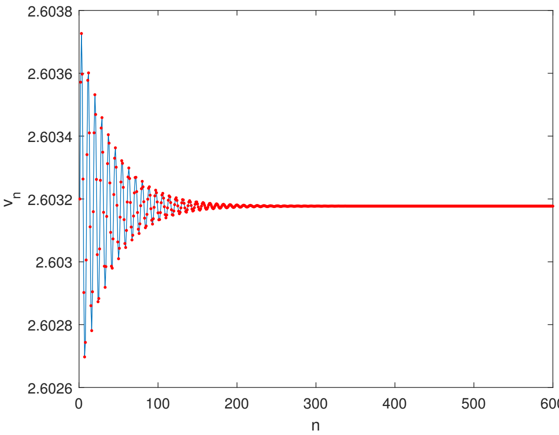
Example 6.2. This example shows the effectiveness of the hybrid control method on the stability of the considered system. We assume that $C = 1.4$, $M = 2.1$, $N = 5.12$, $\gamma = 0.75$, $p = 0.43$ and the initial condition $(0.4102, 2.6032)$. In Figure 1, we can observe that at the previous parameter values, the equilibrium point P is unstable (Figures 1b 1d) and all orbits tend towards a closed invariant curve (Figure 1f). After applying the hybrid control method, the equilibrium point P becomes stable (Figures 1a and 1c) and the closed invariant curve disappears (Figure 1e). This surely verifies that controlling chaos nicely improves the behavior of the system.



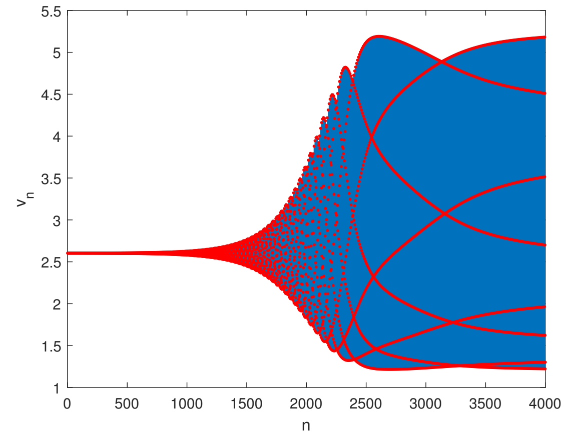
(a) Stable solution for u_n in system (6.1) when $p = 0.43$ and $\epsilon = 0.9$.



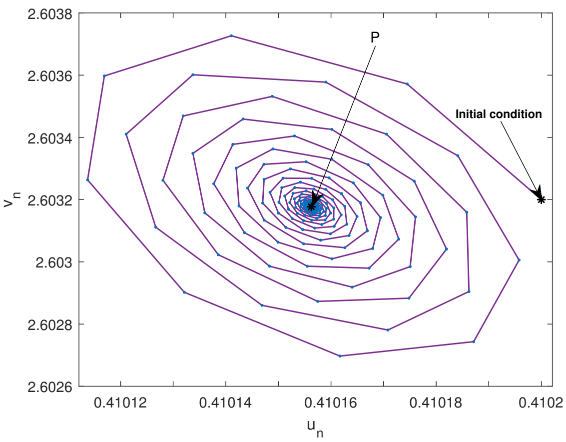
(b) Unstable solution for u_n in system (3.3) when $p = 0.43$.



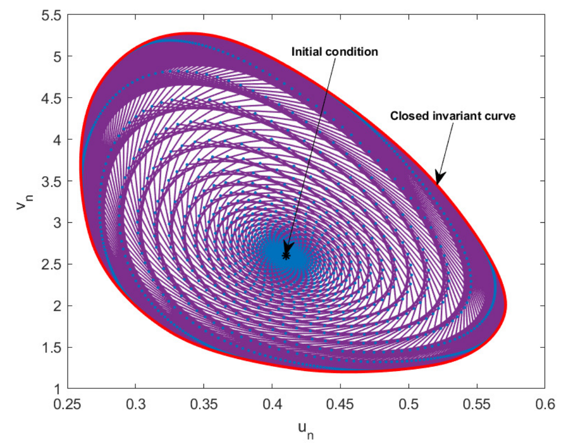
(c) Stable solution for v_n in model (6.1) when $p = 0.43$ and $\epsilon = 0.9$.



(d) Unstable solution for v_n in model (3.3) when $p = 0.43$.



(e) Phase portrait of model (6.1) when $p = 0.43$ and $\epsilon = 0.9$.



(f) Phase portrait of model (3.3) when $p = 0.43$.

Figure 1: The impact of the chaos control in the stability of systems (6.1) and (3.3).

7. Numerical simulations and discussion

This part illustrates some numerical examples to show the behavior of the considered model under certain conditions. We also verify the constructed theoretical results using these numerical examples. Table 1 shows the main initial conditions, parameters, and their constraints.

Table 1: Used parameters and the initial conditions.

Examples	Parameters	Initial conditions
Example 7.1	$C = 2.3$, $M = 2.5$, $N = 2.59$, $\gamma = 0.5$, $p \in [0, 3]$	$M_0 = (0.961, 0.173)$, $p \in [0, 1.42]$
Example 7.2	$C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$	$M_1 = (0.4839, 0.9428)$, $M_2 = (0.42, 1.15)$

Example 7.1. This example considers the values $C = 2.3$, $M = 2.5$, $N = 2.59$, $\gamma = 0.5$, $p \in [0, 3]$, and the initial condition $M_0 = (0.961, 0.173)$. As a result, a period-doubling bifurcation occurs for system (3.3) when the bifurcation parameter p varies in a small neighbourhood of $p_1 = 1.2557$. In order to verify the existence of this type of bifurcation, we calculate the Jacobean matrix at this point as follows:

$$\mathcal{J}(E_+) = \begin{pmatrix} -1.2182 & -0.6395 \\ 0.6823 & 1.0000 \end{pmatrix},$$

whose characteristic equation is shown as $Q(\delta) = \delta^2 + 0.2182\delta - 0.7818$. The roots of this equation are $\delta_1 = -1$, $\delta_2 = 0.7818$ where $|\delta_2| \neq 1$. We also have

$$D_0^{\mp}(p) = 1 > 0, \qquad D_1^{+}(p) = 1 + \mathcal{D} = 0.2182 > 0,$$
$$(-1)^2 P_p(-1) = 1 + \mathcal{T} + \mathcal{D} = 0, \qquad P_p(1) = 1 - \mathcal{T} + \mathcal{D} = 0.4364 > 0.$$

Therefore, the period-doubling bifurcation takes place at $p_1 = 1.2557$. It can be noted from the bifurcation diagrams of u_n and v_n (shown in Figures 2a and 2b, respectively) that the positive equilibrium point P of system (3.3) is stable for $0 < p < 1.2557$, while this point loses its stability through a period doubling bifurcation when $p \geqslant 1.2557$. Moreover, there is a period doubling cascade in orbits of periods 2, 4, 8, 16 (Figures 2a and Figure 2b) and chaotic set for different values of p. The maximum Lyapunov exponents associated with bifurcation diagrams are obviously drawn in Figure 2c. This surely verifies the existence of the chaotic behavior and period orbits in the parametric space. From Figure 2c we note that some "Maximal LE" values are positive and some of them are negative. Thus, there exists a stable equilibrium point or stable period orbits in the chaotic region. Biologically, the period-doubling bifurcation signifies that the discrete-time system changes from a equilibrium point to a cycle of period-2 when the parameter varies, i.e., both species (prey and predator) may coexist in period-2 cycles under some conditions.

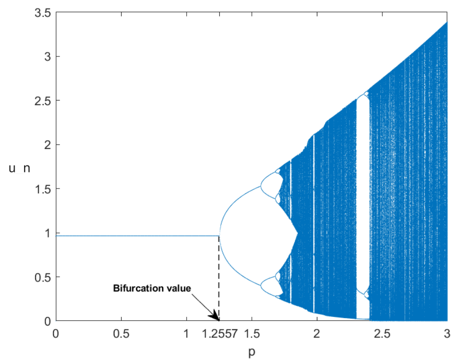
Example 7.2. We now let $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, and $p \in [0, 1.42]$ and the initial conditions $M_1 = (0.4839, 0.9428)$ and $M_2 = (0.42, 1.15)$. Consequently, system (3.3) encounters a Neimark-Sacker bifurcation when $p = 0.2443$ at the point P. In order to prove this behavior, we assume that $(C, M, N, \gamma, p) = (0.4, 3, 6.2, 0.75, 2.2443)$, then the Jacobian matrix of system (3.3) is shown as follows:

$$J(P) = \begin{pmatrix} 0.6676 & -0.1227 \\ 2.7084 & 1.0000 \end{pmatrix}.$$

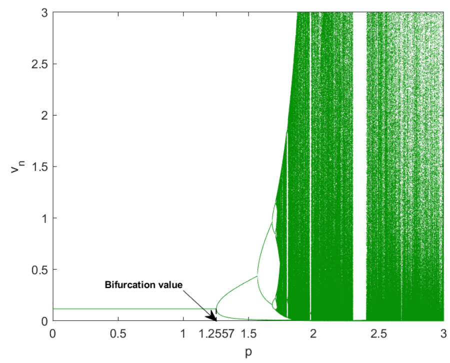
Hence, the eigenvalues are $\delta_1 = 0.8338 - 0.5521i$ and $\delta_2 = 0.8338 + 0.5521i$ with $|\delta| = |\delta'| \neq 1$. Moreover, we notice that

$$D_1^{-}(p) = 1 - \mathcal{D} = 1 - 1 = 0, \qquad D_1^{+}(p) = 1 + \mathcal{D} = 2 > 0,$$
$$P(1) = 1 - \mathcal{T} + \mathcal{D} = 0.3324 > 0, \qquad (-1)^2 P(-1) = 1 + \mathcal{T} + \mathcal{D} = 3.6676 > 0.$$

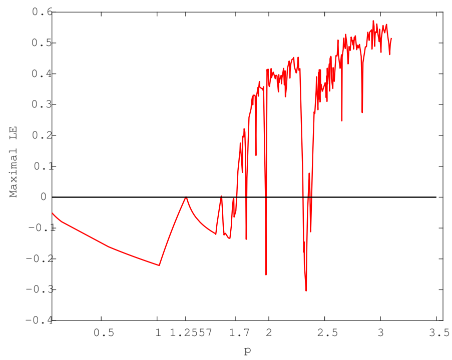
Consequently, the constraints of the Neimark-Sacker bifurcation shown in Theorem 5.3 are satisfied. Figures 3a and 3b illustrate the diagrams of the Neimark-Sacker bifurcation of u_n and v_n presented in system (3.3). For $0 < p < 0.2443$, the equilibrium point P is local asymptotically stable while this point loses its stability at $p = 0.2443$. Therefore, a limit cycle is created around the fixed point. It is worth noting that the diameter of this curve increases with the increment in the value of p. We also observe that the dense chaos takes place from 0.2443 to 1.42 with intermittent quasi-periodic window at $p \in [0.3186, 0.414]$. Biologically, the invariant curve is an indication on the existence of prey and predator populations. In addition, the Allee effect improves the dynamics of the proposed model and balances the density of populations. Figure 3c shows the maximum Lyapunov exponent which is related to Figures 3a and 3b. Figure 3c confirms the existence of stable periodic windows or stable equilibrium point. In Figures 5 and 6, we depict some phase portraits for system (3.3) and the evolution of u_n under the values of the bifurcation parameter $p \in [0, 1.42]$. Moreover, Figure 5 presents the stability of the equilibrium point when $0 < p < 0.2443$. It is important to notice that when $0.2443 \leqslant p < 0.3186$, an attracting closed invariant curve around the equilibrium point occurs as can be seen in Figure 6. Note that we use various values for the parameter γ and we find that using different values for γ gives different behaviours. In Figure 9, we use different values for γ and we obtain different behaviours where the stability moves along the bifurcation parameter axis which is p-axis. We also examine the bifurcation for system (3.3) at the coexistence point P, where $\gamma \in [0, 1]$ is a bifurcation parameter and $(C, M, N, p) = (0.34, 9, 16.2, 0.07)$. Figure 11 demonstrates that the coexistence point P of system (3.3) is unstable for $\gamma \leqslant 0.7535$ and it is stable when $0.7535 \leqslant \gamma \leqslant 1$.



(a) Bifurcation graph for u_n with $p \in [0, 3]$.

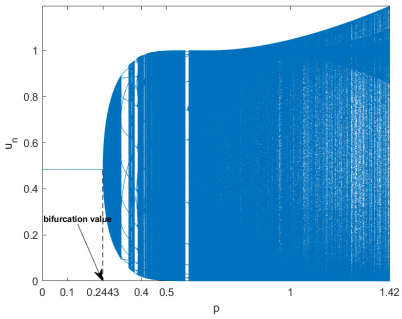


(b) Bifurcation graph for v_n with $p \in [0, 3]$.

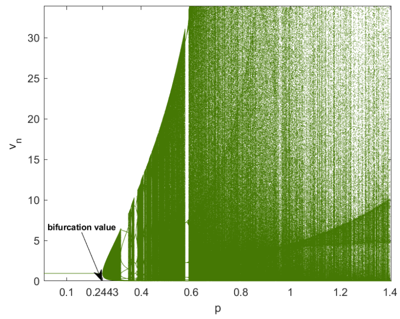


(c) Maximum Lyapunov exponent (MLE) with $p \in [0, 3]$.

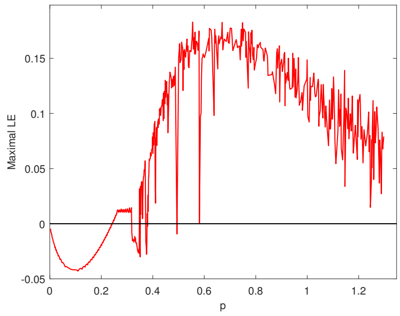
Figure 2: Graphs (a) and (b) show the period doubling bifurcation of system (3.3) while diagram (c) depicts the maximum Lyapunov exponent for model (3.3) when $C = 2.3$, $M = 2.5$, $N = 2.59$, $\gamma = 0.5$, and $p \in [0, 3]$.



(a) Bifurcation diagram for u_n with $p \in [0, 1.42]$.

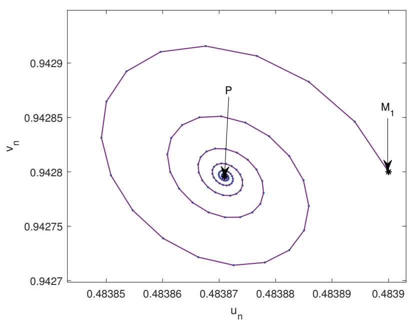


(b) Bifurcation diagram for v_n with $p \in [0, 1.42]$.

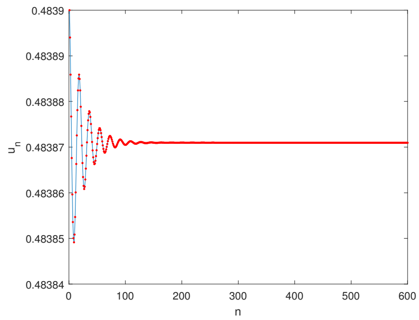


(c) Maximum Lyapunov exponent (MLE) with $p \in [0, 1.42]$.

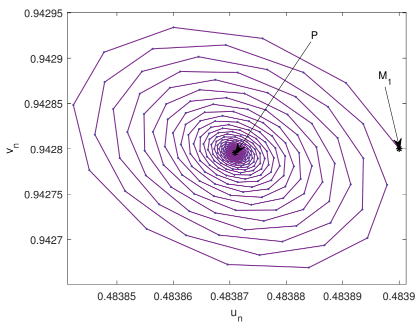
Figure 3: Graphs (a) and (b) illustrate the Neimark-Sacker bifurcation of system (3.3) while graph (c) presents the maximum Lyapunov exponent for system (3.3) when $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, and $p \in [0, 1.42]$.



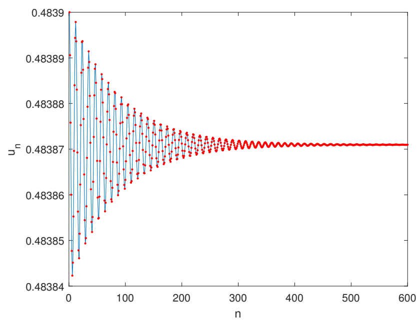
(a) Phase portrait of system (3.3) when $p = 0.12$.



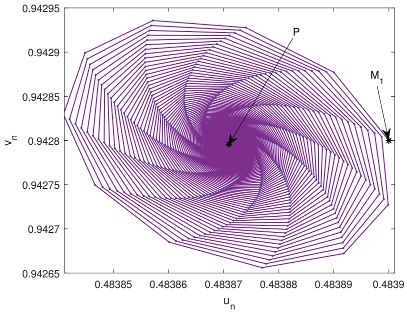
(b) Stability of u_n when $p = 0.12$.

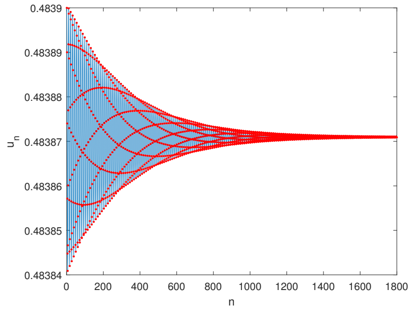


(c) Phase portrait of system (3.3) when $p = 0.22$.



(d) Stability of u_n when $p = 0.22$.

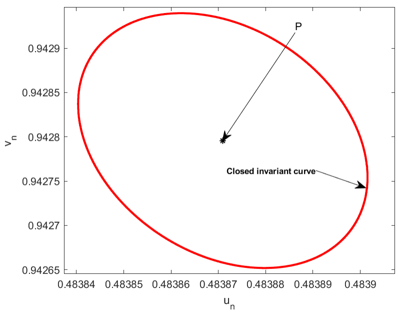


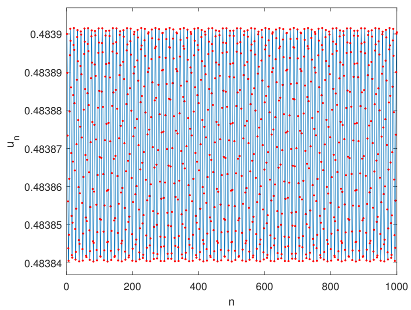


(e) Phase portrait of system (3.3) when $p = 0.238$.

(f) Stability of u_n when $p = 0.238$.

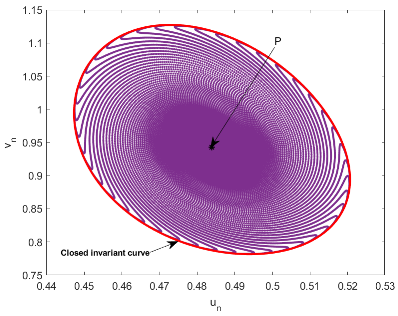
Figure 5: Diagrams (a), (c) and (e) present the phase plane of system (3.3) under various values of the parameter p while plots (b), (d) and (f) illustrate the stability of u_n under the values $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, $p \in [0, 0.2380]$, and initial condition M_1.

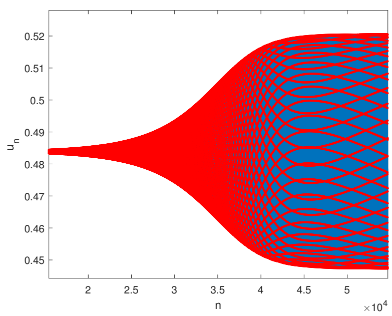




(a) Phase plane of system (3.3) when $p = 0.2443$.

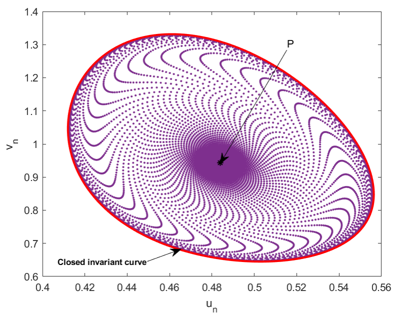
(b) Stability of u_n when $p = 0.2443$.

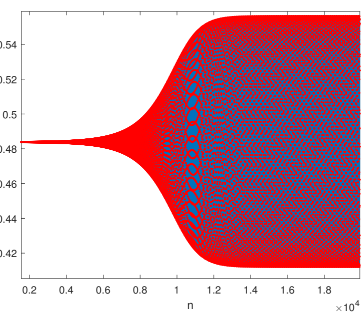




(c) Phase plane of model (3.3) when $p = 0.2447$.

(d) Stability of u_n when $p = 0.2447$.

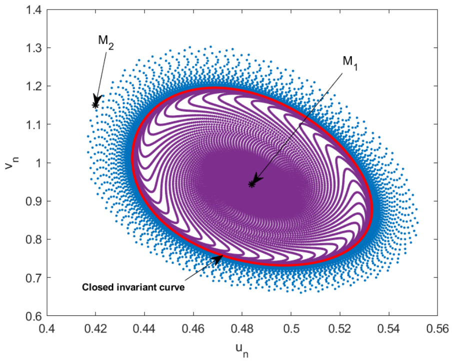




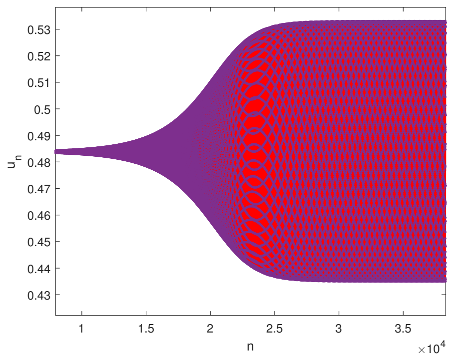
(e) Phase plane of model (3.3) when $p = 0.2458$.

(f) Stability of u_n when $p = 0.2458$.

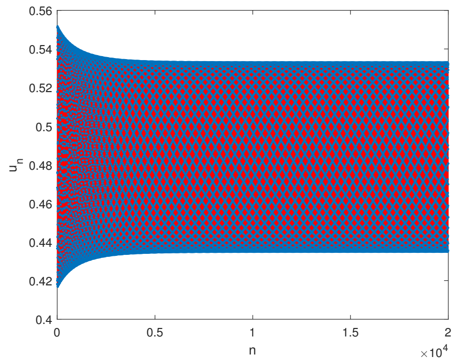
Figure 6: Figures (a), (c), and (e) depict the phase plane of system (3.3) when p varies whereas diagrams (b), (d), and (f) illustrate the stability of u_n under the values $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, $p \in [0.2380, 0.2458]$, and initial condition M_1.



(a) Phase plane of system (3.3) at $p = 0.245$.

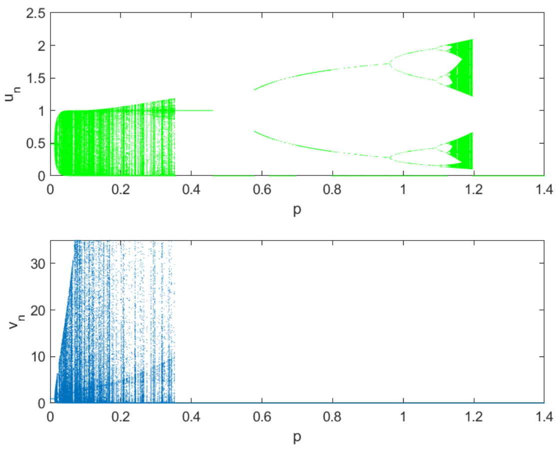


(b) Stability of u_n when $p = 0.245$ and initial condition M_1.

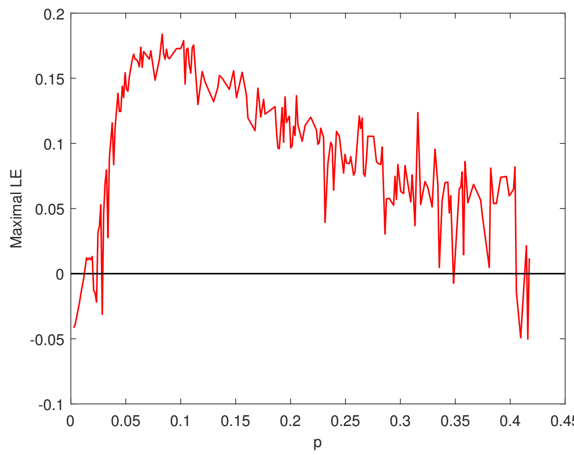


(c) Stability of u_n when $p = 0.245$ and initial condition M_2.

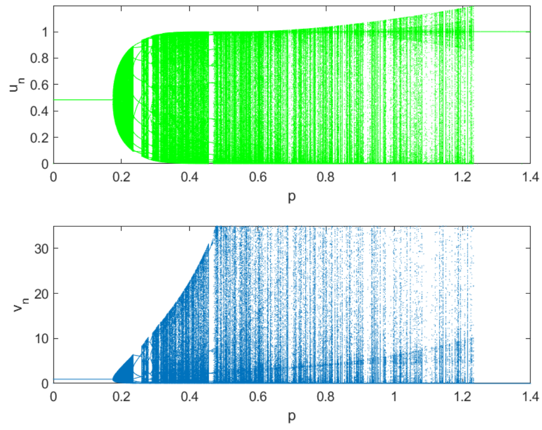
Figure 7: Phase portrait and stability of system (3.3) when the initial conditions $A_1 = (0.483, 0.942)$ and $M_2 = (0.42, 1.15)$.



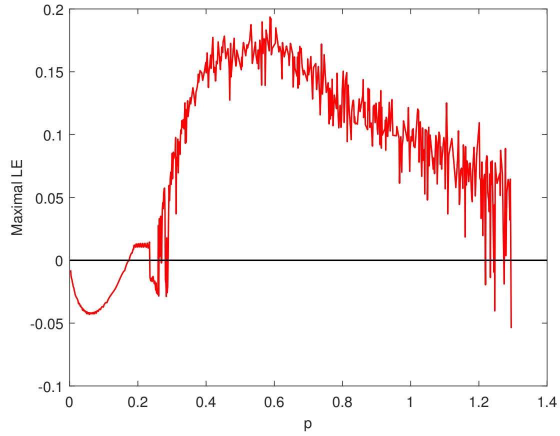
(a) For $\gamma = 0.36$.



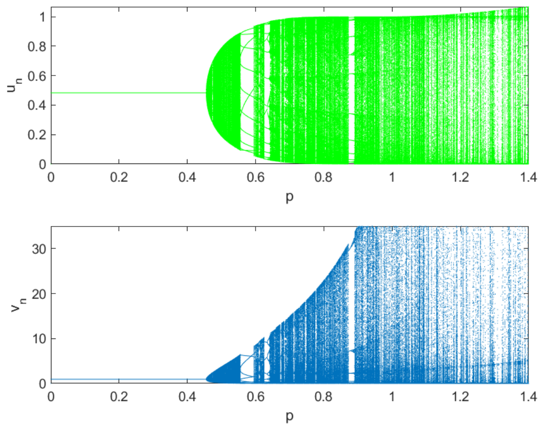
(b) For $\gamma = 0.36$.



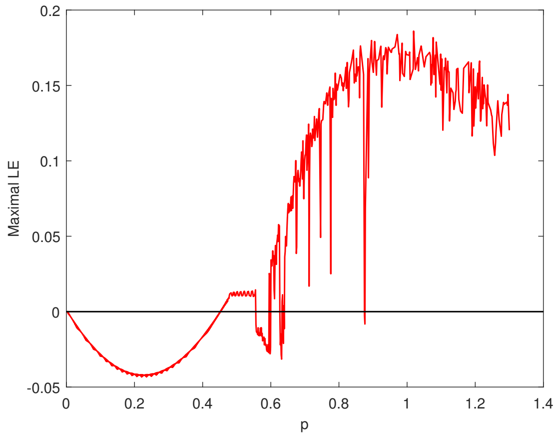
(c) For $\gamma = 0.67$.



(d) For $\gamma = 0.67$.

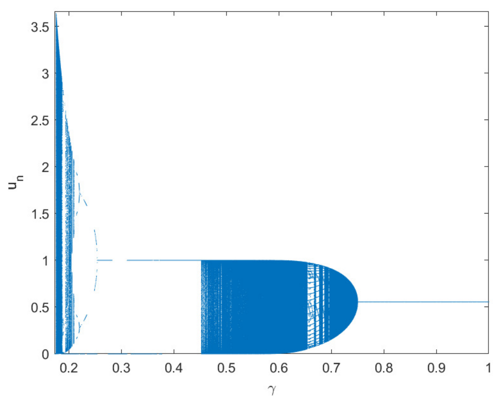


(e) For $\gamma = 0.99$.

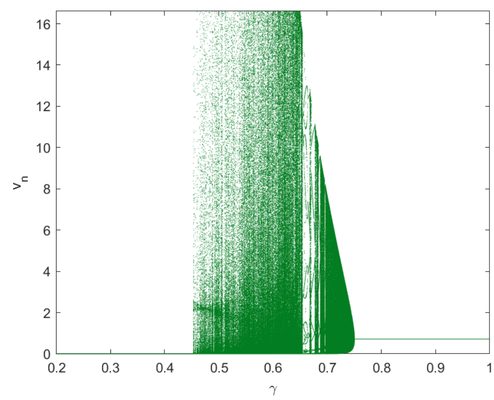


(f) For $\gamma = 0.99$.

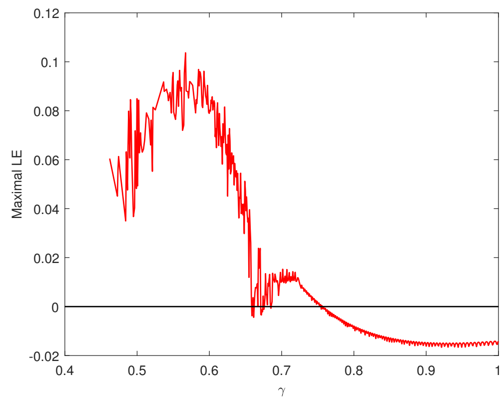
Figure 9: Graphs (a), (c), and (e) illustrate the Neimark-Sacker bifurcation of system (3.3) whereas figures (b), (d), and (f) show the maximum Lyapunov exponent for system (3.3) when $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, $p \in [0.2380, 0.2458]$, and γ take the values $0.36, 0, 67$, and 0.99, respectively.



(a) Bifurcation plot for x_n with $\gamma \in [0, 1]$.



(b) Bifurcation plot for v_n with $\gamma \in [0, 1]$.



(c) Maximum Lyapunov exponent (MLE) with $\gamma \in [0, 0.45]$.

Figure 11: Graphs (a) and (b) illustrate the Neimark-Sacker bifurcation of system (3.3) whereas figure (c) shows the maximum Lyapunov exponent for system (3.3) when $C = 0.34$, $M = 9$, $N = 16.2$, $p = 0.07$, and $\gamma \in [0, 1]$.

8. Conclusion

The dynamical behavior of system (1.1) has effectively examined in the present article. We have discretized this system using the piecewise-constant method and converted the system to difference equations. Three equilibrium points have been established for system (3.3) where one of them is positive if $N > M$. The stability of these points has been nicely presented under some certain conditions. The coexistence point P is found locally asymptotically stable, unstable, and non-hyperbolic under some specific conditions given in Theorem 4.5. In Theorem 5.3, we have proved that a period-doubling bifurcation occurs at the coexistence point P in system (3.3) if some conditions are satisfied. Similarly, this system faces a Neimark-Sacker bifurcation at the point P if some constraints are satisfied as illustrated in Theorem 5.4. The useful performance of the chaos control on the stability of systems (6.1) and (3.3) has been shown in Figure 1. In Figure 2a, we have illustrated the period doubling bifurcation of u_n shown in model (3.3) at $p = 1.2557$ while Figure 2b has presented the period doubling bifurcation of v_n shown in (3.3) at $p = 1.2557$. Figure 2c has depicted the maximum Lyapunov exponent for system (3.3) when $C = 2.3$, $M = 2.5$, $N = 2.59$, $\gamma = 0.5$, and $p \in [0, 3]$. In Figures 2a and 2b, we have plotted the Neimark-Sacker bifurcation of system (3.3) when $C = 0.4$, $M = 3$, $N = 6.2$, $\gamma = 0.75$, and $p \in [0, 1.42]$. If we increase the value of p, system (3.3) becomes unstable and limit cycles are formed as shown in Figures 5 and 6. Moreover, the closed invariant curve is clearly seen in Figure 7a in which we have assumed that the initial conditions are $M_1 = (0.483, 0.942)$ and $M_2 = (0.42, 1.15)$. In this figure, we note that the closed invariant curve is stable because all trajectories approach the cycle from inside and outside the closed invariant curve. Figures 10a and 10b show that the fractional parameter γ leads to good bifurcation figures when we use it as a bifurcation parameter. We can draw the conclusion that more nonlinear models can be discussed using the methodologies that were used.

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