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Introducing a novel family of Δ_h-Sheffer polynomials and their interconnected hybrid variants

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Abstract

The main objective of this work is to investigate a novel class of polynomials, called the Δ_h-Sheffer polynomials and to explore their various properties. The generating function, explicit representations, quasi-monomiality, and certain novel identities involving Δ_h-Sheffer polynomials are obtained. Also, the Δ_h-Sheffer polynomials are explored via determinant representation. Further, the Δ_h Gould-Hopper-Sheffer polynomials are introduced with the help of Δ_h-Sheffer and Δ_h Gould-Hopper polynomials. Certain fascinating results, such as the generating function, determinant form, multiplicative, and derivative operators and many more results for these hybrid form of the Δ_h-Sheffer polynomials are also obtained. Certain examples are considered as the special cases of Δ_h Gould-Hopper-Sheffer polynomials.

Keywords: Δ_h special polynomials, Δ_h Sheffer polynomials, monomiality principle, explicit forms, determinant form.

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1. Introduction

One significant category within polynomial sequences is the Sheffer polynomial sequences, as highlighted in [12]. The exploration of Sheffer polynomials has been extensive, not only because they emerge in various mathematical domains but also due to their relevance in applied sciences. Over the several decades, there has been increasing interest in these sequences and their diverse representations, as demonstrated in works like [4, 5]. Leveraging these findings, a determinant-based approach has been proposed in [15].

The Sheffer polynomials $s_n(q), n \in \mathbb{N}_0$ are defined through the generating function provided in [12] as follows:

$$A(\xi)\exp(q\mathcal{H}(\xi)) = \sum_{n=0}^{\infty} s_n(q)\frac{\xi^n}{n!}, \tag{1.1}$$

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where $A(\xi)$ and $\mathcal{H}(\xi)$ are power series, satisfying the conditions:

$$A(\xi)=\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}, \quad a_0 \neq 0, \qquad \text{and} \qquad \mathcal{H}(\xi)=\sum_{n=0}^{\infty} \hat{h}_n \frac{\xi^n}{n!}, \quad \hat{h}_0 \neq 0.$$

In recent times, various versions of special functions and polynomials and their properties have been explored by researchers using diverse techniques, as seen in works such as [7, 8, 10, 11, 16–20]. Presently, the Δ_h version of special polynomials has garnered considerable attention due to its distinctive properties and practical applicability in engineering and mathematics. Numerous authors have endeavored to present and delineate various characteristics of Δ_h special polynomials, as evidenced in works like [2, 9, 13, 21]. The significance of Δ_h-special polynomials extends beyond their combinatorial and mathematical features to their applications in various other areas of mathematics.

The forward difference operator Δ_h for any function $g : I \subset \mathbb{R} \to \mathbb{R}$ is defined in [6] as:

$$\Delta_h[g](q)=g(q+h)-g(q), \qquad h \in \mathbb{R}_+,$$

which for any given order $i \in \mathbb{N}$ can be expressed as follows:

$$\Delta_h^i[g](q)=\Delta_h\big(\Delta_h^{i-1}[g](q)\big)=\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l} g(q+lh).$$

Also, $\Delta_h^0 = I$ and $\Delta_h^1 = \Delta_h$, with I representing the identity operator.

A new form of the Appell polynomials, called Δ_h-Appell polynomials $\mathcal{Q}_n(q;h)$ is constructed by Costabile and Longo [2] and their numerous properties are proposed. These Δ_h-Appell polynomials $\mathcal{Q}_n(q;h)$ have been extensively studied for their significant applications not only in various mathematical domains but also in physics and statistics. For specific values of parameter q, Δ_h Appell sequences reduce to familiar sequences, as discussed in [2, 13]. These special cases prove valuable in applications requiring the use of well-known sequences and polynomials and the systematic derivation provided by Δ_h Appell sequences facilitates their acquisition. The Δ_h Appell polynomials $\mathcal{Q}_n(q;h)$ are provided in [2] by the following relation:

$$\sum_{n=0}^{\infty} \mathcal{Q}_n(q;h)\frac{\xi^n}{n!}=\delta(\xi)(1+h\xi)^{\frac{q}{h}}, \tag{1.2}$$

where

$$\delta(\xi)=\sum_{n=0}^{\infty} \delta_{n,h}\frac{\xi^n}{n!}, \qquad \delta_{0,h} \neq 0.$$

The Δ_h-Appell polynomials $\mathcal{Q}_n(q;h)$ exhibit the following relation, as stated in [2]:

$$\Delta_h[\mathcal{Q}_n](q;h)=nh\mathcal{Q}_{n-1}(q;h), \qquad n \in \mathbb{N}.$$

Remark 1.1. As $h \to 0$, expression (1.2) converges to the well-known Appell polynomials [1].

In 1941, Steffenson [14] introduced the notion of monomiality, which was refined by Dattoli in [3] and have now become an essential tool in exploring the properties of special polynomials. The study of special polynomials is significantly influenced by multiplicative operator $\hat{\mathcal{M}}$ and derivative operators $\hat{\mathcal{D}}$, which acts on the polynomial set $\{b_n(q)\}_{n\in\mathbb{N}}$ as:

$$b_{n+1}(q)=\hat{\mathcal{M}}\{b_n(q)\}, \tag{1.3}$$

$$nb_{n-1}(q)=\hat{\mathcal{D}}\{b_n(q)\}. \tag{1.4}$$

The collection of polynomials $\{b_n(q)\}_{n\in\mathbb{N}}$ with respect to the operators given in relations (1.3) and (1.4) is termed a quasi-monomial. These polynomials satisfy the following differential equation:

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_n(q)\}=nb_n(q), \tag{1.5}$$

which indicates that the quasi-monomials act as eigenfunctions of operator $\hat{\mathcal{M}}\hat{\mathcal{D}}$, with an eigenvalue of n.

This article aims to investigate Δ_h Sheffer polynomials and their related hybrid forms. The structure of remaining manuscript is outlined as follows. Section 2 introduces Δ_h Sheffer polynomials and explores some of their distinct features. In Section 3, the quasi-monomial characteristics and determinant form for Δ_h Sheffer polynomials are proposed. Section 4 is dedicated to the construction of Δ_h Gould-Hopper-Sheffer polynomials, where their properties are examined. Additionally, specific members of Δ_h Gould-Hopper-Sheffer polynomial family are scrutinized.

2. Δ_h Sheffer polynomials

This section starts by presenting generating function of the Δ_h Sheffer sequences. Inspired by the research conducted by Costabile and Longo [2], we introduce a novel set of polynomials termed Δ_h Sheffer polynomials, denoted as ${}_{\Delta_h}\mathcal{S}_n(q;h)$. The introduction of Δ_h Sheffer polynomials builds upon the groundwork laid by previous studies, and their generating expression is a key focus in this exploration. These polynomials exhibit a generating expression that is characterized by a specific form:

$$\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{S}_n(q;h)\frac{\xi^n}{n!}, \tag{2.1}$$

where

$$\delta(\xi) = \sum_{n=0}^{\infty} \delta_{n,h}\frac{\xi^n}{n!}, \qquad \mathcal{H}(\xi) = \sum_{n=1}^{\infty} \gamma_{n,h}\frac{\xi^n}{n!}, \qquad \delta_{0,h}, \quad \gamma_{1,h} \neq 0. \tag{2.2}$$

Taking $\delta(\xi) = 1$ in equation (2.1), Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$ give the Δ_h associated Sheffer polynomials ${}_{\Delta_h}\mathbb{H}_n(q;h)$, defined as

$$(1+h\mathcal{H}(\xi))^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n(q;h)\frac{\xi^n}{n!}. \tag{2.3}$$

For $q = 0$, equation (2.1) gives

$$\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{0}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{S}_n(0;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{S}_n(h)\frac{\xi^n}{n!}. \tag{2.4}$$

Remark 2.1. For $\mathcal{H}(\xi) = \xi$, expression (2.1) gives generating relation of the Δ_h Appell polynomials $\mathcal{Q}_n(q;h)$.

Remark 2.2. Taking the limit $h \longrightarrow 0$ in relation (2.1), we get the Sheffer polynomials $s_n(q)$, defined in equation (1.1).

Following this, the explicit series representation for Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$ is deduced. The outcome takes the form of a specific result that provides a detailed expression for these polynomials. Unraveling the explicit series representation is essential as it offers a clearer understanding of the mathematical structure and behavior of the Δ_h Sheffer polynomials. This result serves as a valuable tool for further analysis and applications involving these specific polynomials.

Theorem 2.3. *For the Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$, the following explicit representation holds:*

$${}_{\Delta_h}\mathcal{S}_n(q;h) = \sum_{s=0}^{n} \binom{n}{s} \delta_{s,h}\, {}_{\Delta_h}\mathbb{H}_{n-s}(q;h). \tag{2.5}$$

Proof. Inserting expressions (2.2) and (2.3) in the l.h.s. of relation (2.1), it follows that

$$\sum_{s=0}^{\infty} \delta_{s,h} \frac{\xi^s}{s!} \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n(q;h) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{S}_n(q;h) \frac{\xi^n}{n!},$$

which on applying the C.P. rule and on comparing coefficients of identical powers of ξ, yields assertion (2.5). □

Theorem 2.4. *For Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$, the following summation formula holds:*

$$ {}_{\Delta_h}\mathcal{S}_n(p+q;h) = \sum_{s=0}^{n} \binom{n}{s} {}_{\Delta_h}\mathcal{S}_s(q;h) {}_{\Delta_h}\mathbb{H}_{n-s}(p;h). \tag{2.6}$$

Proof. Replacing q by $p+q$ in equation (2.1), it follows that

$$\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{p+q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{S}_n(p+q;h) \frac{\xi^n}{n!},$$

which on using relations (2.1) and (2.3) in l.h.s and on comparing the coefficients of identical powers of ξ, assertion (2.6) is proved. □

Corollary 2.5. *The Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$ satisfy the following identity:*

$$ {}_{\Delta_h}\mathcal{S}_n(mq;h) = \sum_{s=0}^{n} \binom{n}{s} {}_{\Delta_h}\mathcal{S}_s(q;h) {}_{\Delta_h}\mathbb{H}_{n-s}((m-1)q;h),$$

which is obtained on letting $p = (m-1)q$ in relation (2.6).

Theorem 2.6. *For the Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$, the following identity holds:*

$$ {}_{\Delta_h}\mathcal{S}_n(q;h) = \sum_{s=0}^{n} \binom{n}{s} {}_{\Delta_h}\mathcal{S}_s(h) {}_{\Delta_h}\mathbb{H}_{n-s}(q;h). \tag{2.7}$$

Proof. Inserting expressions (2.4) and (2.3) in l.h.s. of equation (2.1), and after some simplifications and then on comparing the coefficients of similar powers of ξ, assertion (2.7) is proved. □

3. Monomiality principle and determinant form

In this context, we verify and establish the quasi-monomial properties inherent in the Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$. The subsequent results offer proof and confirmation of these quasi-monomial characteristics. Understanding and confirming these properties are crucial as they contribute to the broader comprehension of the algebraic and operational aspects of Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$. The validation of quasi-monomial properties adds depth to the mathematical analysis and enhances the applicability of these polynomials in various mathematical contexts.

Theorem 3.1. *For Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$, the following multiplicative and derivative operators are obtained:*

$$ {}_{\Delta_h}\mathcal{S}_{n+1}(q;h) = \hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h}\mathcal{S}_n(q;h)\} = \left(\frac{\delta'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)}{\delta\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)} + q \frac{\mathcal{H}'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)}{1+h\mathcal{H}\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)} \right) {}_{\Delta_h}\mathcal{S}_n(q;h) \tag{3.1}$$

and

$$ {}_{\Delta_h}\mathcal{S}_{n-1}(q;h) = \hat{\mathcal{D}}_{\Delta_h}\{{}_{\Delta_h}\mathcal{S}_n(q;h)\} = \mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\{{}_{\Delta_h}\mathcal{S}_n(q;h)\}, \tag{3.2}$$

respectively.

Proof. Using finite difference operator Δ_h in expression (2.1) yields

$$
\begin{aligned}
{}_q\Delta_h\Big[\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q}{h}}\Big] &= \Big[\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q+h}{h}} - \delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q}{h}}\Big],\\
{}_q\Delta_h\Big[\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q}{h}}\Big] &= h\mathcal{H}(\xi)\Big[\delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{q}{h}}\Big],
\end{aligned}
$$

or

$$
{}_q\Delta_h[{}_{\Delta_h}\mathcal{S}_n(q;h)] = h\mathcal{H}(\xi)[{}_{\Delta_h}\mathcal{S}_n(q;h)],
$$

or

$$
\frac{{}_q\Delta_h}{h}[{}_{\Delta_h}\mathcal{S}_n(q;h)] = \mathcal{H}(\xi)[{}_{\Delta_h}\mathcal{S}_n(q;h)]. \tag{3.3}
$$

Differentiating expression (2.1) w.r.t. ξ, it follows that

$$
{}_{\Delta_h}\mathcal{S}_{n+1}(q;h) = \hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h}\mathcal{S}_n(q;h)\} = \left(\frac{\delta'(\xi)}{\delta(\xi)} + q\frac{\mathcal{H}'(\xi)}{1+h\mathcal{H}(\xi)}\right)\{{}_{\Delta_h}\mathcal{S}_n(q;h)\},
$$

which on using identity (3.3) and in view of expressions (1.3) yields assertion (3.1). Further, in view of operator expression (1.4), expression (3.3) leads to the assertion (3.2). □

Corollary 3.2. *The Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$ satisfy the following differential equation:*

$$
\left(\frac{\delta'\Big(\mathcal{H}^{-1}\big(\frac{q\Delta_h}{h}\big)\Big)}{\delta\Big(\mathcal{H}^{-1}\big(\frac{q\Delta_h}{h}\big)\Big)}\mathcal{H}^{-1}\Big(\frac{q\Delta_h}{h}\Big) + q\frac{\mathcal{H}'\Big(\mathcal{H}^{-1}\big(\frac{q\Delta_h}{h}\big)\Big)}{1+h\mathcal{H}\Big(\mathcal{H}^{-1}\big(\frac{q\Delta_h}{h}\big)\Big)}\mathcal{H}^{-1}\Big(\frac{q\Delta_h}{h}\Big) - n\right){}_{\Delta_h}\mathcal{S}_n(q;h) = 0. \tag{3.4}
$$

Proof. Using relations (3.1) and (3.2) in equation (1.5), assertion (3.4) is proved. □

Theorem 3.3. *The Δ_h Sheffer polynomials ${}_{\Delta_h}\mathcal{S}_n(q;h)$ are defined by the following determinant form:*

$$
{}_{\Delta_h}\mathcal{S}_n(q;h) = \frac{(-1)^n}{(\zeta_{0,h})^{n+1}}\begin{vmatrix}
1 & {}_{\Delta_h}\mathbb{H}_1(q;h) & {}_{\Delta_h}\mathbb{H}_2(q;h) & \cdots & {}_{\Delta_h}\mathbb{H}_{n-1}(q;h) & {}_{\Delta_h}\mathbb{H}_n(q;h)\\
\zeta_{0,h} & \zeta_{1,h} & \zeta_{2,h} & \cdots & \zeta_{n-1,h} & \zeta_{n,h}\\
0 & \zeta_{0,h} & \binom{2}{1}\zeta_{1,h} & \cdots & \binom{n-1}{1}\zeta_{n-2,h} & \binom{n}{1}\zeta_{n-1,h}\\
0 & 0 & \zeta_{0,h} & \cdots & \binom{n-1}{2}\zeta_{n-3,h} & \binom{n}{2}\zeta_{n-2,h}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \zeta_{0,h} & \binom{n}{n-1}\zeta_{1,h}
\end{vmatrix},
$$

where the coefficients $\zeta_{n,h} \in \mathbb{C}$ are explicitly given by

$$
\zeta_{n,h} = -\frac{1}{\delta_{0,h}}\left(\sum_{k=1}^{n}\binom{n}{k}\delta_{k,h}\zeta_{n-k,h}\right), \qquad n = 1,2,\ldots. \tag{3.5}
$$

Proof. Multiplying both sides of equation (2.1) by $\hat{\delta}(\xi) = \frac{1}{\delta(\xi)} = \sum_{n=0}^{\infty}\zeta_{n,h}\frac{\xi^n}{n!}$, it follows that

$$
\sum_{n=0}^{\infty}{}_{\Delta_h}\mathbb{H}_n(q;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\zeta_{m,h}\frac{\xi^m}{m!}{}_{\Delta_h}\mathcal{S}_n(q;h)\frac{\xi^n}{n!},
$$

which in view of Cauchy product rule yields

$$ {}_{\Delta_h}\mathbb{H}_n(q;h) = \sum_{m=0}^{n} \binom{n}{m} \zeta_{m,h}\, {}_{\Delta_h}\mathbb{S}_{n-m}(q;h). $$

This equality gives rise to a system of m-equations, which involves unknowns ${}_{\Delta_h}\mathbb{S}_n(q;h)$ and n ranges from 0 to infinity. Solving this system is facilitated by employing Cramer's rule, leveraging the fact that the denominator corresponds to the determinant of a lower triangular matrix with a determinant of $(\delta_{0,h})^{n+1}$. By transposing the numerator and interchanging i^{th} row with the $(i+1)^{\text{th}}$ position for $i = 1, 2, \ldots, n-1$, we get the desired result. This process of solving the system of equations is instrumental in determining the values of ${}_{\Delta_h}\mathbb{S}_n(q;h)$ and is crucial for a comprehensive understanding of the behavior and properties of these specific polynomials. □

4. Δ_h Gould-Hopper-Sheffer polynomials

Incorporating Δ_h Sheffer polynomials ${}_{\Delta_h}\mathbb{S}_n(q;h)$ with Δ_h Gould-Hopper polynomials ${}_{\Delta_h}\mathbb{G}_n(p,q;h)$, Δ_h Gould-Hopper-Sheffer polynomials, denoted by ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ is constructed. The Δ_h Gould-Hopper polynomials ${}_{\Delta_h}\mathbb{G}_n(p,q;h)$, as recalled from [9], is expressed as follows:

$$ (1+h\xi)^{\frac{p}{h}}(1+h\xi^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{G}_n(p,q;h)\frac{\xi^n}{n!}. \tag{4.1} $$

With the help of relations (2.1) and (4.1), we get the following generating equation for Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$:

$$ \delta(\xi)(1+h\mathcal{H}(\xi))^{\frac{p}{h}}(1+h(\mathcal{H}(\xi))^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)\frac{\xi^n}{n!}, \tag{4.2} $$

where setting $q = 0$ reduces it to the Δ_h Sheffer polynomials ${}_{\Delta_h}\mathbb{S}_n(q;h)$, as given in equation (2.1).

Taking $\delta(\xi) = 1$ in relations (4.2), the following generating function for Δ_h Gould-Hopper-associated Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{H}_n(p,q;h)$ is obtained:

$$ (1+h\mathcal{H}(\xi))^{\frac{p}{h}}(1+h(\mathcal{H}(\xi))^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h\mathbb{G}}\mathbb{H}_n(p,q;h)\frac{\xi^n}{n!}, $$

which on taking $q = 0$ becomes the Δ_h associated Sheffer polynomials ${}_{\Delta_h}\mathbb{H}_n(q;h)$, given in equation (2.3).

Theorem 4.1. *The Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ are defined by the following determinant form:*

$$ {}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h) = \frac{(-1)^n}{(\zeta_{0,h})^{n+1}} \begin{vmatrix} 1 & {}_{\Delta_h\mathbb{G}}\mathbb{H}_1(p,q;h) & {}_{\Delta_h\mathbb{G}}\mathbb{H}_2(p,q;h) & \cdots & {}_{\Delta_h\mathbb{G}}\mathbb{H}_n(p,q;h) \\ \zeta_{0,h} & \zeta_{1,h} & \zeta_{2,h} & \cdots & \zeta_{n,h} \\ 0 & \zeta_{0,h} & \binom{2}{1}\zeta_{1,h} & \cdots & \binom{n}{1}\zeta_{n-1,h} \\ 0 & 0 & \zeta_{0,h} & \cdots & \binom{n}{2}\zeta_{n-2,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\zeta_{1,h} \end{vmatrix}, \tag{4.3} $$

where $n = 1, 2, \ldots$ and the coefficients $\zeta_{n,h} \in \mathbb{C}$ are same as defined in equation (3.5).

Remark 4.2. Inconsideration of $\mathcal{H}(\xi) = \xi$, equation (4.2) gives the generating equation and equation (4.3) gives the determinant expression for Δ_h Gould-Hopper-Appell polynomials $\mathcal{A}_n(p, q; h)$, given in [9].

Theorem 4.3. *The Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$ satisfy the following multiplicative and derivative operators:*

$$
\begin{aligned}
{}_{\Delta_h\mathbb{G}}\mathbb{S}_{n+1}(p, q; h) = \hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)\} = \Bigg(& \frac{\delta'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)}{\delta\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)} + p\frac{\mathcal{H}'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)}{1 + h\mathcal{H}\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)} \\
& + q\frac{2\mathcal{H}'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)\mathcal{H}'\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)}{1 + h\left(\mathcal{H}\left(\mathcal{H}^{-1}\left(\frac{q\Delta_h}{h}\right)\right)\right)^2}\Bigg){}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)
\end{aligned}
$$

and

$$
{}_{\Delta_h\mathbb{G}}\mathbb{S}_{n-1}(p, q; h) = \hat{\mathcal{D}}_{\Delta_h}\{{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)\} = \mathcal{H}^{-1}\left(\frac{p\Delta_h}{h}\right)\{{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)\},
$$

respectively.

Next, we give the certain identities satisfied by the Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$:

$$
\begin{aligned}
{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h) &= \sum_{s=0}^{n}\binom{n}{s}\delta_{s,h}\,{}_{\Delta_h\mathbb{G}}\mathbb{H}_{n-s}(p, q; h), \\
{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h) &= \sum_{s=0}^{n}\binom{n}{s}{}_{\Delta_h}\mathbb{H}_s(p; h)\,{}_{\Delta_h\mathbb{G}}\mathbb{H}_{n-s}(0, q; h), \\
{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p_1 + p_2, q_1; h) &= \sum_{s=0}^{n}\binom{n}{s}{}_{\Delta_h\mathbb{G}}\mathbb{S}_s(p_1, q_1; h)\,{}_{\Delta_h}\mathbb{H}_{n-s}(p_2; h), \\
{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p_1, q_1 + q_2; h) &= \sum_{s=0}^{n}\binom{n}{s}{}_{\Delta_h\mathbb{G}}\mathbb{S}_s(p_1, q_1; h)\,{}_{\Delta_h\mathbb{G}}\mathbb{H}_{n-s}(0, q_2; h), \\
{}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p_1 + p_2, q_1 + q_2; h) &= \sum_{s=0}^{n}\binom{n}{s}{}_{\Delta_h\mathbb{G}}\mathbb{S}_s(p_1, q_1; h)\,{}_{\Delta_h\mathbb{G}}\mathbb{H}_{n-s}(p_2, q_2; h).
\end{aligned}
$$

For certain choices of $\delta(\xi)$ and $\mathcal{H}(\xi)$, different members belonging to Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$ are obtained. Some members of Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$ are considered to give the application of the above results.

Taking $\delta(\xi) = (1 - h\xi^2)^{\frac{1}{h}}$ and $\mathcal{H}(\xi) = 2\xi$, Δ_h Sheffer polynomials reduce to Δ_h Hermite polynomials and therefore Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$ become the Δ_h Gould-Hopper-Hermite polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{R}_n(p, q; h)$ and given by the following generating relation:

$$
(1 - h\xi^2)^{\frac{1}{h}}(1 + 2h\xi)^{\frac{p}{h}}(1 + 4h\xi^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty}{}_{\Delta_h\mathbb{G}}\mathbb{R}_n(p, q; h)\frac{\xi^n}{n!}.
$$

Next, on taking $\delta(\xi) = (1 - h\xi^r)^{\frac{1}{h}}$ and $\mathcal{H}(\xi) = \nu\xi$, Δ_h Sheffer polynomials reduce to generalized Δ_h Hermite polynomials and therefore Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p, q; h)$ become the generalized Δ_h Gould-Hopper-Hermite polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{R}_{n,r,\nu}(p, q; h)$ and given by the following generating relation:

$$
(1 - h\xi^r)^{\frac{1}{h}}(1 + \nu h\xi)^{\frac{p}{h}}(1 + h\nu^2\xi^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty}{}_{\Delta_h\mathbb{G}}\mathbb{R}_{n,r,\nu}(p, q; h)\frac{\xi^n}{n!}.
$$

Also, for $\delta(\xi) = (1+h\xi)^{\frac{\beta}{h}}$ and $\mathcal{H}(\xi) = 1 - e^{\xi}$, Δ_h Sheffer polynomials reduce to Δ_h Acturial polynomials and therefore Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ become the Δ_h Gould-Hopper-Acturial polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{A}_n^{(\beta)}(p,q;h)$ and given by the following relation:

$$(1+h\xi)^{\frac{\beta}{h}}(1+h(1-e^{\xi}))^{\frac{p}{h}}(1+h(1-e^{\xi})^2)^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h\mathbb{G}}\mathbb{A}_n^{(\beta)}(p,q;h)\frac{\xi^n}{n!}.$$

Further, for $\delta(\xi) = (1-h\xi)^{\frac{1}{h}}$ and $\mathcal{H}(\xi) = \ln(1+\frac{\xi}{a})$, Δ_h Sheffer polynomials reduce to the Δ_h Poisson-Charlier polynomials and therefore Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ become the Δ_h Gould-Hopper-Acturial polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{C}_n(p,q;h;a)$ and given by the following generating relation:

$$(1-h\xi)^{\frac{1}{h}}\left(1+h\ln\left(1+\frac{\xi}{a}\right)\right)^{\frac{p}{h}}\left(1+2h\ln\left(1+\frac{\xi}{a}\right)\right)^{\frac{q}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h\mathbb{G}}\mathbb{C}_n(p,q;h;a)\frac{\xi^n}{n!}.$$

Posing a problem

Establishing the determinant form, quasi-monomial properties, explicit representation, and some other novel identities for the members of Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$, introduced in Section 4.

5. Conclusion

In this research work, we have investigated Δ_h Sheffer polynomials and delve into their distinctive attributes. Section 3 of the paper focuses on unveiling specific features of these polynomials, including the establishment of quasi-monomial characteristics and a determinant form. The significance of determinants transcends their traditional association with matrices, finding independent applications in diverse areas such as solving problems related to reflection-less transmission of electromagnetic waves through dielectrics. Additionally, Section 3 reveals several intriguing identities associated with the Δ_h Sheffer polynomials, contributing to a deeper understanding of their mathematical properties. A hybrid form, namely the Δ_h Gould-Hopper-Sheffer polynomials ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ is constructed and their properties are explored. Furthermore, a few members of Δ_h Gould-Hopper-Sheffer polynomials family ${}_{\Delta_h\mathbb{G}}\mathbb{S}_n(p,q;h)$ are observed and their generating functions are found. Further, integral representations, recurrence relations, numerical investigation of zeros, and other properties of Δ_h Sheffer polynomials can also be a problem for future investigations.

References

[1] P. Appell, *Sur une classe de polynômes*, Ann. Sci. École Norm. Sup. (2), **9** (1880), 119–144. 1.1

[2] F. A. Costabile, E. Longo, *Δ_h-Appell sequences and related interpolation problem*, Numer. Algorithms, **63** (2013), 165–186. 1, 1, 2

[3] G. Dattoli, *Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle*, In: Advanced special functions and applications (Melfi, 1999), Aracne, Rome, **1** (2000), 147–164. 1

[4] A. Di Bucchianico, D. Loeb, *A selected survey of umbral calculus*, Electron. J. Combin., **2** (1995), 28 pages. 1

[5] T.-X. He, L. C. Hsu, P. J.-S. Shiue, The Sheffer group and the Riordan group, Discrete Appl. Math., **155** (2007), 1895–1909. 1

[6] C. Jordan, *Calculus of finite differences*, Chelsea Publishing Co., New York, (1965). 1

[7] A. Muhyi, *A new class of Gould-Hopper-Eulerian-type polynomials*, Appl. Math. Sci. Eng., **30** (2022), 283–306. 1

[8] A. Muhyi, S. Araci, *A note on q-Fubini-Appell polynomials and related properties*, J. Funct. Spaces, **2021** (2021), 9 pages. 1

[9] M. A. Özarslan, B. Y. Yaşar *Δ_h-Gould-Hopper Appell polynomials*, Acta Math. Sci. Ser. B (Engl. Ed.), **41** (2021), 1196–1222. 1, 4, 4.2

[10] W. Ramírez, C. Cesarano, S. Díaz, *New Results for Degenerated Generalized Apostol–bernoulli, Apostol–euler and Apostol–genocchi Polynomials*, WSEAS Trans. Math., **21** (2022), 604–608. 1

[11] W. Ramírez, C. Cesarano, *Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Carpathian Math. Publ., **14** (2022), 354–363. 1

[12] I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke. Math. J. **5** (1939), 590–622. 1

[13] H. M. Srivastava, M. A. Özarslan, B. Y. Yaşar, *Difference equations for a class of twice-iterated Δ_h-Appell sequences of polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113** (2019), 1851–1871. 1

[14] J. F. Steffensen, *The poweriod, an extension of the mathematical notion of power*, Acta. Math., **73** (1941), 333–366. 1

[15] W. Wang, *A determinantal approach to Sheffer sequences*, Linear Algebra Appl., **463** (2014), 228–254. 1

[16] S. A. Wani, S. Khan, *Properties and applications of the Gould-Hopper-Frobenius-Euler polynomials*, Tbilisi Math. J., **12** (2019), 93–104. 1

[17] S. A. Wani, S. Khan, T. Nahid, *Gould-Hopper based Frobenius-Genocchi polynomials and their generalized form*, Afr. Mat., **31** (2020), 1397–1408.

[18] S. A. Wani, T. Nahid, *Finding determinant and integral forms of the 2-iterated 2 D Appell polynomials*, J. Anal., **30** (2022), 731–747.

[19] S. A. Wani, T. Nahid, K. Hussain, M. B. Jeelani, *A unified matrix approach to the Legendre-Sheffer and certain hybrid polynomial sequences*, J. Anal., **32** (2024), 843–867.

[20] G. Yasmin, A. Muhyi, *Extended forms of Legendre-Gould-Hopper-Appell polynomials*, Adv. Stud. Contemp. Math., **29** (2019), 489–504. 1

[21] M. Zayed, S. A. Wani, G. I. Oros, W. Ramírez, *Unraveling multivariable Hermite-Apostol-type Frobenius-Genocchi polynomials via fractional operators*, AIMS Math., **9** (2024), 17291–17304. 1