

## The role of human shield in prey, crop-raiders and top predator species in southwestern Ethiopia's coffee forests: a modeling study



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### Abstract

In this work, we have developed a model that describes the relationships between top predators (such as tigers, hyenas, and others), crop raiders (such as baboons, warthogs, and deer), and prey (such as deer) in the coffee forests of southwest Ethiopia. Various potential equilibrium points are identified. Additionally, the model's stability in the vicinity of these equilibrium points is examined. An investigation of the model's Hopf bifurcation is conducted concerning several significant parameters. It is found that prey species may be extinct due to a lower growth rate and consumption by top predators in the absence of human interference in the carrying capacity of prey. It is observed that top predators may be extinct due to human interference in their carrying capacity and their smaller dependence on humans in terms of prey and crop raiders, respectively. It is also found that there is an increase in intra-species competition among the top predators, which may intensify the stability of the model. Again, it is also observed that the increase in the intrinsic growth rate of prey and top predators may improve the stability of the model. Lastly, some numerical simulation results have been shown to help visualize the model's dynamics.

**Keywords:** Human shield, prey-predator model, stability analysis, Hopf bifurcation, numerical simulation.

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### 1. Introduction

Due to its significance and widespread existence, the study of the dynamic relationship of the prey-predator system has long been. It will continue to be one of the main topics in both mathematical ecology and ecology. Mathematical modeling and comprehension of biological events have greatly benefited from

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the advances of recent decades. One major goal of mathematical models in biology is to mathematically represent the population dynamics of a prey-predator system, a goal that has drawn interest from numerous scholars [4, 6, 8, 14, 34, 36, 39, 40]. One of the prerequisites for comprehending how species interact with one another is knowledge of species co-occurrence patterns. Furthermore, it's critical to comprehend how human disturbance may impact these patterns, especially in settings with high wildlife-human crossing rates. Because resources and habitat are shared in these types of environments, human-wildlife coexistence can be common, and conflicts are likely to arise. This is true in many tropical forest landscapes in sub-Saharan Africa, where local livelihoods and forest animals coexist together, creating challenges for livelihood development and wildlife conservation as well as frequent conflicts [1, 17, 24, 28, 32, 33, 35]. Furthermore, a lot of these forest landscapes are being fragmented and cleared for development, two activities that might change the co-occurrence of species and have an effect on interacting species [18, 19]. Thus, gathering information on the spatiotemporal patterns of a species' activity can help shed light on how well-suited a species is to coexist with humans and help direct the creation of policies meant to co-manage animals and humans.

Here, we investigate the co-occurrence of prey, crop raiders, top predators, and people at a fine spatial scale in the smallholder landscapes of southwest Ethiopia, utilizing a multi-species occupancy model [27]. These landscapes exhibit a high degree of human-wildlife overlap, which makes this area very fascinating for studying human-wildlife interactions. In an ecological system, every population uses a different tactic to find food sources and protect itself, such as refuging or clustering. To create more realistic mathematical models, a wide range of ecological components and characteristics are used. Every prey-predator interaction in population dynamics must take into account the functional response, or the ratio of a predator's prey intake to the density of prey per unit of time [5, 37]. The functional response of Holling types I, II, III, and IV is generally employed for most arthropod predators [7, 9]. Afterward, Rosenzweig and MacArthur [29] looked into the Lotka-Volterra model, accounting for the predator's saturation with a Holling type II functional response and the prey's logistic growth rate. Majeed and Ghafel [16] are investigating how a prey-predator model, which takes into account intraspecific competitions and prey shelters, affects the fear that young animals feel due to the fear of more experienced predators. Panja et al. [26] have suggested using a three-species predator-prey model that includes nonlinear Lotka-Volterra rivalry between two rival species,  $x$  and  $y$ . SK et al. [30] are investigating the dynamics of a Lotka-Volterra model including three species and intraguild (IG) predation. Interactions between an omnivorous top/intraguild predator, intraguild prey, and base prey are included in the model. Mondal and el [20] gave a description of the qualitative behavior of a prey-predator model with prey refuge in discrete time. The prey fraction in the refuge is thought to be a self-limiting function that increases monotonically with the prey population. The authors in [31] focus on investigating a basic prey-predator model's global dynamics. The model accounts for the impact of harvesting on the population of predators as well as cooperative hunting behavior among predators. Mortoja and other authors [22] have developed an ecological model that permits the inverse feeding mechanism by utilizing the typical prey-predator interactions. A three-species food web structure with common prey, intermediate, and top predators makes up the mathematical model. On the other hand, there are several articles in this field see [3, 10–13, 15, 21, 25, 38].

According to the "humans as super-predators" hypothesis (Figure 1 (a)), the presence of people in the forest pushes crop-raiders and prey into the top predator space, displacing all other species groupings, including top predators and prey. According to this theory, all species groups will be more occupied in places without human presence than in areas with human presence. The "humans as shield" theory (Figure 1 (b)) states that the presence of people only displaces top predators, lowering the danger of predation for both crop raiders and prey. According to this theory, there will be more prey and crop raiders when humans are there than when they are not, and there will be more top predators when humans are not around.

The structure of the paper is as follows. A mathematical model is formulated in Section 2. The positiveness and boundedness are investigated in 3. Section 4 evaluates various potential equilibrium positions. The analysis of the model's local stability around these equilibrium locations is conducted in

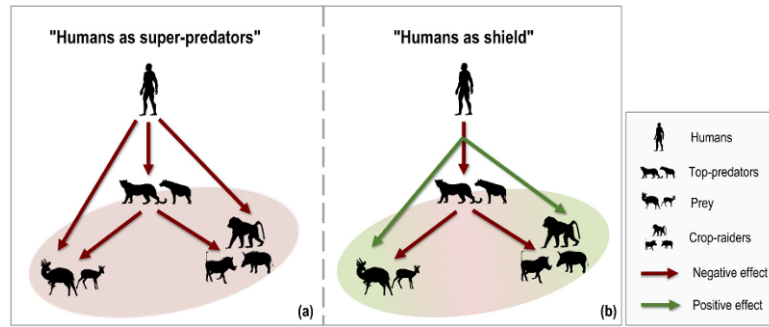


Figure 1: Conceptual representation of two study hypotheses: (a) the hypothesis that humans act as super-predators, and (b) the hypothesis that humans act as a shield.

Section 5. The existence of the Hopf bifurcation around the positive equilibrium point is shown in Section 6. Section 7 presents the findings of the numerical simulation. The conclusions of the primary results are presented in Section 8.

### 2. Model formulation

In this paper, we have constructed a model of the interactions of prey (i.e., Deer), crop-raiders (i.e., Baboon, Warthog, Bushpig, etc.), and top predator (i.e., Tiger, hyena, etc.) in the presence of the human. We have taken this idea from the coffee forests of southwestern Ethiopia. Here  $P(t)$ ,  $C_R(t)$ ,  $T(t)$ , and  $H(t)$  are the densities of prey, crop-raiders, top predators, and humans at time  $t$ , respectively. We have taken the following assumptions to construct the model.

- It is assumed that prey grows logistically in the absence of humans and top predators. It is also assumed that space (carrying capacity) for prey species increases due to the human shield.
- It is considered that the attack rate of prey by top predator is affected by the human shield. Holling type II functional response function is used for the consumption of prey by top predator.
- The logistic growth of crop-raiders has been taken into consideration in the absence of human, top predator and without natural death rate of crop-raiders. It is assumed that crop-raiders are also attacked by the top predator for food. Crowley-Martin type functional response is used for the consumption of crop-raiders by top predator.
- It is assumed that the space or carrying capacity for crop-raiders increase due to the human shield. It is also assumed that the consumption of crop-raiders by top predator may affected by human shield. The crop-raiders may decrease due to the natural death rate.
- It is considered that top predator species increases due to the consumption of prey and crop-raiders. It is also considered that top predator’s may decrease due to the intra-species competition and natural deaths.
- A constant rate of recruitment of human has been considered. The human species may decrease due to the natural deaths.

Keeping the above assumptions in mind, we have formulated the following model:

$$\begin{aligned}
 \frac{dP}{dt} &= r_1 P \left(1 - \frac{P}{k_1 + \xi_1 H}\right) - \frac{\alpha P T}{(1 + \beta P)(1 + b_1 H)}, \\
 \frac{dC_R}{dt} &= r_2 C_R \left(1 - \frac{C_R}{k_2 + \xi_2 H}\right) - \frac{\gamma C_R T}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R T)(1 + b_2 H)} - d_1 C_R, \\
 \frac{dT}{dt} &= \frac{\alpha_1 \alpha P T}{(1 + \beta P)(1 + b_1 H)} + \frac{\gamma_1 \gamma C_R T}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R T)(1 + b_2 H)} - m T^2 - d_2 T, \\
 \frac{dH}{dt} &= r_3 - d_3 H,
 \end{aligned}
 \tag{2.1}$$

with initial conditions  $P(0) = P_0 > 0$ ,  $C_R(0) = C_{R_0} > 0$ ,  $T(0) = T_0 > 0$ , and  $H(0) = H_0 > 0$ . The physical interpretations of the model parameters are given in the Table 1.

Parameter	Biological meaning
$r_1$	Intrinsic growth rate of prey
$k_1$	Environmental carrying capacity of prey
$\xi_1$	Increase rate of carrying capacity of prey due to human shield
$\alpha$	Consumption rate of prey by top predator
$\beta$	Inverse of half-saturation constant for the consumption of prey by top predator
$b_1$	Effects of human shield in the consumption of prey by top predator
$r_2$	Intrinsic growth rate of crop-raiders
$k_2$	Environmental carrying capacity of crop-raiders
$\xi_2$	Increase rate of carrying capacity of crop-raiders due to human shield
$\gamma$	Consumption rate of crop-raiders by top predator
$\beta_1$	Inverse of half-saturation constant for the consumption of crop-raiders by top predator
$b_2$	Effects of human shield in the consumption of crop-raiders by top predator
$m$	Intra-species competition among top predator
$r_3$	Intrinsic growth rate of human
$k_3$	Environmental carrying capacity for humans
$d_1$	Natural death rate of crop-raiders
$d_2$	Natural death rate of top predator
$d_3$	Natural death rate of human

Table 1: The physical meaning of all parameters of model.

### 3. Positiveness and boundedness

**Theorem 3.1.** All solutions of system (2.1) that start with positive initial conditions  $P_0 > 0$ ,  $C_{R_0} > 0$ ,  $T_0 > 0$ , and  $H_0 > 0$  are also positive.

*Proof.* By integrating system (2.1) for  $P(t), C_{R_0}(t), T(t) > 0$ , and  $H(t) > 0$ , we get

$$\begin{aligned}
 P(t) &= P(0) \exp\left\{\int_0^t \left[r_1\left(1 - \frac{P}{k_1 + \xi_1 H}\right) - \frac{\alpha T}{(1 + \beta P)(1 + b_1 H)}\right] ds\right\} > 0, \\
 C_R(t) &= C_R(0) \exp\left\{\int_0^t \left[r_2\left(1 - \frac{C_R}{k_2 + \xi_2 H}\right) - \frac{\gamma T}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R T)(1 + b_2 H)} - d_1\right] ds\right\} > 0, \\
 T(t) &= T(0) \exp\left\{\int_0^t \left[\frac{\alpha_1 \alpha P}{(1 + \beta P)(1 + b_1 H)} + \frac{\gamma_1 \gamma C_R}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R T)(1 + b_2 H)} - mT - d_2\right] ds\right\} > 0, \\
 H(t) &\geq H(0) \exp\left\{\int_0^t [-d_3] ds\right\}.
 \end{aligned}$$

Consequently, solutions that possess positive initial conditions will persist thereafter. □

**Theorem 3.2.** The set  $Z = \{(P, C_R, T, H) \in \mathbb{R}_+^4 : P + C_R + T \leq \frac{r_3}{\xi}, H \leq H_m\}$  attracts all the solutions  $P(t), C_R(t), T(t), H(t)$  initiating in  $\mathbb{R}_+^4$ .

*Proof.* By applying the Comparison lemma on the last equation of (2.1), we obtain

$$\limsup_{t \rightarrow \infty} [H(t)] \leq \frac{r_3}{d_4} = H_m.$$

Now, let  $Y(t) = \alpha_1 P(t) + \gamma_1 C_R(t) + T(t)$ , then  $\frac{dY}{dt} = \alpha_1 \frac{dP}{dt} + \gamma_1 \frac{dC_R}{dt} + \frac{dT}{dt}$ . Using (2.1) and the human above bound, we obtain

$$\frac{dY}{dt} + \xi \leq \nu,$$

where  $\xi = \min\{r_1\alpha_1, d_1\gamma_1, d_2\}$  and  $\nu = 2r_1\alpha_1(k_1 + \xi_1 H_m) + r_2\gamma_1(k_2 + \xi_2 H_m)$ . Then, using Gronwall's inequality, we get

$$0 \leq Y(t) \leq \frac{\nu}{\xi} (1 - e^{-\xi t}) + Y(0)e^{-\xi t}.$$

Hence,

$$0 \leq \limsup_{t \rightarrow \infty} Y(t) \leq \frac{\nu}{\xi}.$$

Thus, all solutions of system (2.1) that are initiated in  $\mathbb{R}_+^4$  are attracted to the region Z. □

### 4. Equilibrium points

The model (2.1) admits the following equilibria.

1. The top predator-free equilibrium point  $c_1 = (P_1, C_{R1}, 0, H_1) = (k_1 + \frac{\zeta_1 r_3}{d_3}, (1 - \frac{d_1}{r_2})(k_2 + \frac{\zeta_2 r_3}{d_3}), 0, \frac{r_3}{d_3})$ . Clearly,  $C_{R1} > 0$  if

$$\frac{d_1}{r_2} < 1 \tag{4.1}$$

2. The crop raiders-free equilibrium point  $c_2 = (P_2, 0, T_2, H_2)$ , where  $H_2 = \frac{r_3}{d_3}, T_2 = \frac{m\alpha\alpha_1 P_2}{(1+\beta P_2)(1+b_1 H_2)} - \frac{d_2}{m}$  and  $P_2$  is a root of the following polynomials:

$$w_0 P^3 + w_1 P^2 + w_2 P + w_3 = 0,$$

where  $w_0 = \frac{mr_1\beta^2(1+b_1 H_2)^2}{k_1+\zeta_1 H_2}, w_1 = w_0\beta - r_1 m\beta^2(1+b_1 H_2)^2, w_2 = \frac{w_0}{\beta} + \alpha^2\alpha_1 - r_1 m\beta(1+b_1 H_2)^2 - \alpha d_2\beta(1+b_1 H_2)$ , and  $w_3 = -(r_1 m(1+b_1 H_2)^2 + \alpha d_2(1+b_1 H_2))$ . So, according to Descartes' Rule of Signs, the above polynomial has a unique positive root say  $c_2$  if  $w_1 > 0$  or  $w_2 < 0$ . Clearly,  $T_2 > 0$  if

$$m^2\alpha\alpha_1 P_2 > d_2(1+\beta P_2)(1+b_1 H_2).$$

3. The prey-free equilibrium point  $c_3 = (0, C_{R3}, T_3, H_3)$ , where

$$H_3 = \frac{r_3}{d_3} = a^*, \quad C_{R3} = \frac{a^*(mT + d_2)}{\gamma\gamma_1 - a^*(m + m\beta_3 T^2 + d_2\beta_1 + d_2\beta_3)}.$$

Clearly,  $C_{R3} > 0$  if

$$\gamma\gamma_1 > a^*(m + m\beta_3 T^2 + d_2\beta_1 + d_2\beta_3).$$

$T_3$  is a root of the following fifth-degree polynomial

$$h_1 T^5 + h_2 T^4 + h_3 T^3 + h_4 T^2 + h_5 T + h_6 = 0,$$

where

$$h_1 = -\gamma,$$

$$h_2 = -a^* r_2 a_2^* \beta_3 a_3 m + a_2^* a_3^2 (r_2 - d_1),$$

$$h_3 = -2a^* a_3 (a^* a_2 - a_1) - a^{*2} a_3 m r_2 a_2^* \left( \beta_1 + \frac{d_2 \beta_3}{m} - \frac{1}{a_1^*} \right) - \frac{m^2 r_2 a_2^2 \beta_3}{a_1^*},$$

$$\begin{aligned}
 h_4 &= -2a^* a_2^* a_3 (r_2 - d_1) (a_1 - a^* a_2) - a^{*2} a_3 d_2 r_2 a_2^2 \left( \beta_1 - \frac{1}{a_1^*} \right) \\
 &\quad - \frac{r_2 a_2^2 m}{a_1^*} (\beta_1 m + 2d_2 \beta_3) - r_2 a_2^* \beta_3 a^* \beta_3 (a_2 a^* - a_1), \\
 h_5 &= -\gamma \left( a_1^* - 2a^* a_1 a_2 + a^{*2} a_2^* \right) - r_2 a^* (a^* a_2 - a_1) \left( a_2^* \beta_1 + a_2^2 \beta_3 - \frac{a_2^2}{a_1^*} \right) - \frac{r_2 a_2^2 d_2}{a_1^*} (2\beta_1 m + \beta_3 d_2), \\
 h_6 &= a_2^* (r_2 - d_1) \left( a_1^2 + a^{*2} a_2^2 - 2a^* a_1 a_2 \right) + r_2 a^* d_2 \left( \frac{a_2^*}{a_1^*} - a_2^* \beta_1 \right) (a_2 a^* - a_1) - \frac{r_2 a_2^* \beta_1 d_2^2}{a_1^*},
 \end{aligned}$$

and  $a_1 = \gamma\gamma_1$ ,  $a_2 = m + d_2\beta_1 + d_2\beta_3$ ,  $a_3 = m\beta_3$ ,  $a_1^* = k_1 + \xi_1 a^*$ ,  $a_2^* = 1 + b_2 a^*$ . So, according to Descartes' rule of signs, the above polynomial has a unique positive root, say  $T_3$ , if one of the following cases is hold:

- $h_i > 0, i = 3, 4, 5, 6;$
- $h_2 < 0, h_3 < 0, h_i > 0, i = 4, 5, 6;$
- $h_2 < 0, h_3 < 0, h_4 < 0, h_i > 0, i = 5, 6;$
- $h_i < 0, i = 2, 3, 4, 5, h_6 > 0.$

4. The PH-equilibrium point  $c_{PH} = (a_1^*, 0, 0, a^*)$ .

5. The  $C_R$ H-equilibrium point  $c_{C_RH} = \left( 0, \left( 1 - \frac{d_1}{r_2} \right) (k_2 + \xi_2 a^*), 0, a^* \right)$ , which exists under condition (4.1).

6. The positive equilibrium point  $c^* = (P^*, C_R^*, T^*, H^*)$ , where

$$H^* = a^*, T^*(P) = \frac{r_1}{\alpha} (1 + b_1 a^*) \left( 1 + \left( \beta - \frac{1}{k_1 + \xi_1 a^*} \right) P - \frac{\beta}{k_1 + \xi_1 a^*} P^2 \right)$$

and  $(P^*, C_R^*)$  is the positive intersection point of the following isoclines after substituting  $H^*, T^*$  in the second and third equations we obtained,

$$\begin{aligned}
 f_1(P, C_R) &= r_2 \left( 1 - \frac{C_R}{k_2 + \xi_2 a^*} \right) - \frac{\gamma T^*(P)}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R T^*(P))(1 + b_2 a^*)} - d_1 = 0, \\
 f_2(P, C_R) &= \frac{\alpha_1 \alpha P}{(1 + \beta P)(1 + b_1 a^*)} + \frac{\gamma_1 \gamma C_R T^*(P)}{(1 + \beta_1 C_R + \beta_2 P + \beta_3 C_R)(1 + b_2 a^*)} - m T^*(P) - d_2 = 0.
 \end{aligned}$$

As  $C_R \rightarrow 0$  we have

$$f_1(P, 0) = r_2 - \frac{\gamma T^*(P)}{(1 + \beta_2 P)(1 + b_2 a^*)} - d_1 = 0, \quad f_2(P, 0) = \frac{\alpha_1 \alpha P}{(1 + \beta P)(1 + b_1 a^*)} - m T^*(P) - d_2 = 0.$$

From  $f_1(P, 0)$  and  $f_2(P, 0)$  one can get  $P_{f_1}$  is a root of

$$\frac{\gamma r_1 \beta}{(k_1 + \xi_1 a^*)} P^2 + r_1 \left( \frac{1}{k_1 + \xi_1 a^*} + r_2 \beta_1 - \beta \right) P - r_1 + r_2 \alpha = 0,$$

which has a unique root provided that  $r_1 \left( \frac{1}{k_1 + \xi_1 a^*} + r_2 \beta_1 - \beta \right)$  and  $-r_1 + r_2 \alpha$  are both positive or both negative or  $r_1 \left( \frac{1}{k_1 + \xi_1 a^*} + r_2 \beta_1 - \beta \right)$  and  $-r_1 + r_2 \alpha$  is negative, and  $P_{f_2}$  is a root of

$$s_1 P^3 + s_2 P^2 + s_3 P + s_4 = 0, \tag{4.2}$$

where

$$s_1 = \frac{m r_1 (1 + b_1 a^*) \beta^2}{k_1 + \xi_1 a^*},$$

$$\begin{aligned}
 s_2 &= 2 \frac{mr_1(1 + b_1\alpha^*)\beta}{k_1 + \xi_1\alpha^*} - mr_1(1 + b_1\alpha^*)\beta^2, \\
 s_3 &= \frac{mr_1(1 + b_1\alpha^*)\beta}{k_1 + \xi_1\alpha^*} + \frac{\alpha_1\alpha^2}{(1 + b_1\alpha^*)} - 2mr_1(1 + b_1\alpha^*)\beta - \alpha d_2\beta, \\
 s_4 &= - (mr_1(1 + b_1\alpha^*) - \alpha d_2).
 \end{aligned}$$

It's clear that  $s_1$  is positive and  $s_4$  is negative. So, according to Descartes' rule of signs the polynomial (4.2) has a unique root provided that  $s_2$  and  $s_3$  are both positive or both negative. Now, we assume that

$$P_{f_1} < P_{f_2}. \tag{4.3}$$

Also, we assume that the following conditions are satisfied:

$$\frac{dP}{dC_R} = - \frac{\partial f_1}{\partial C_R} / \frac{\partial f_1}{\partial P} > 0, \quad \frac{dP}{dC_R} = - \frac{\partial f_2}{\partial C_R} / \frac{\partial f_2}{\partial P} < 0. \tag{4.4}$$

So, the slope of  $f_2(P, C_R)$  is negative. It is clear that the two isoclines  $f_1(P, C_R)$  and  $f_2(P, C_R)$  uniquely cross at a point  $(P^*, C_R^*)$  in the positive  $P$ - $C_R$  plane. Consequently, the interior equilibrium point is real and is defined as  $c^* = (P^*, C_R^*, T^*, H^*)$ . The following result is obtained.

**Lemma 4.1.** *Model (2.1) has a unique interior equilibrium point  $c^*(P^*, C_R^*, T^*, H^*)$  provided that the conditions (4.3) and (4.4) hold.*

### 5. Local stability

The Jacobian matrix of model (2.1) is

$$\begin{aligned}
 J(P, C_R, T, H) &= [a_{ij}]_{4 \times 4} = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}, \tag{5.1} \\
 a_{11} &= - \frac{\alpha T}{(b_1 H + 1)(\beta P + 1)^2} - r_1 \left( \frac{2P}{k_1 + \xi_1 H} - 1 \right), \\
 a_{13} &= - \frac{\alpha P}{(b_1 H + 1)(\beta P + 1)}, \\
 a_{14} &= \frac{r_1 \xi_1 P^2}{(k_1 + \xi_1 H)^2} + \frac{\alpha b_1 P T}{(b_1 H + 1)^2 (\beta P + 1)}, \\
 a_{21} &= \frac{\beta_2 \gamma C T}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2}, \\
 a_{22} &= - \frac{\gamma T (\beta_2 P + 1)}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2} - r_2 \left( \frac{2C}{k_2 + \xi_2 H} - 1 \right) - d_1, \\
 a_{23} &= - \frac{\gamma C (\beta_1 C + \beta_2 P + 1)}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2}, \\
 a_{24} &= \frac{r_2 \xi_2 C^2}{(k_2 + \xi_2 H)^2} + \frac{b_2 \gamma C T}{(b_2 H + 1)^2 (\beta_1 C + \beta_2 P + \beta_3 C T + 1)}, \\
 a_{31} &= \frac{\alpha \alpha_1 T}{(b_1 H + 1)(\beta P + 1)^2} - \frac{\beta_2 \gamma_1 \gamma C T}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2}, \\
 a_{32} &= \frac{\gamma_1 \gamma T (\beta_2 P + 1)}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2}, \\
 a_{33} &= \frac{\gamma_1 \gamma C (\beta_1 C + \beta_2 P + 1)}{(b_2 H + 1)(\beta_1 C + \beta_2 P + \beta_3 C T + 1)^2} - 2mT - d_2 + \frac{\alpha \alpha_1 P}{(b_1 H + 1)(\beta P + 1)},
 \end{aligned}$$

$$\alpha_{34} = -\frac{b_2\gamma_1\gamma CT}{(b_2H + 1)^2(\beta_1C + \beta_2P + \beta_3CT + 1)} - \frac{\alpha\alpha_1 b_1 PT}{(b_1H + 1)^2(\beta P + 1)},$$

$$\alpha_{44} = -d_3.$$

Now, to examine the local stability of each equilibrium, we should calculate the Jacobian matrix (5.1) around each of them. So, the Jacobian matrix around  $c_1$  is given as

$$J(c_1) = [u_{ij}]_{4 \times 4} \begin{bmatrix} -r_1 & 0 & u_{13} & r_1 \xi_1 \\ 0 & r_2(\frac{d_1}{r_2} - 1) & u_{23} & r_2 \xi_2(\frac{d_1}{r_2} - 1)^2 \\ 0 & 0 & u_{33} & 0 \\ 0 & 0 & 0 & -d_3 \end{bmatrix}, \tag{5.2}$$

$$u_{13} = -\frac{(\alpha k_1 + (\frac{r_3 \xi_1}{d_3}))}{((\frac{b_1 r_3}{d_3} + 1)(\beta(k_1 + \frac{r_3 \xi_1}{d_3}) + 1))},$$

$$u_{23} = \frac{\gamma(k_2 + (r_3 \xi_2)/d_3)(d_1/r_2 - 1)}{((b_2 r_3)/d_3 + 1)(\beta_2(k_1 + (r_3 \xi_1)/d_3) - \beta_1(k_2 + (r_3 \xi_2)/d_3)(d_1/r_2 - 1) + 1)},$$

$$u_{33} = \frac{\alpha\alpha_1(k_1 + (r_3 \xi_1)/d_3)}{((b_1 r_3)/d_3 + 1)(\beta(k_1 + (r_3 \xi_1)/d_3) + 1)} - d_2 - \frac{\gamma_1\gamma(k_2 + (r_3 \xi_2)/d_3)(d_1/r_2 - 1)}{((b_2 r_3)/d_3 + 1)(\beta_2(k_1 + (r_3 \xi_1)/d_3) - \beta_1(k_2 + (r_3 \xi_2)/d_3)(d_1/r_2 - 1) + 1)}.$$

The characteristic equation of (5.2) is

$$(-r_1 - \lambda)(r_2(\frac{d_1}{r_2} - 1) - \lambda)(u_{33} - \lambda)(-d_3 - \lambda) = 0.$$

So, the roots of above equation are  $\lambda_1 = -r_1$ ,  $\lambda_2 = r_2(\frac{d_1}{r_2} - 1)$ ,  $\lambda_3 = u_{33}$ , and  $\lambda_4 = -d_3$ . It's easy to investigate the negativity of all eigenvalues provided that condition (4.1) is satisfied along with

$$u_{33} < 0. \tag{5.3}$$

Therefore, according to the stability criterion, the equilibrium point  $c_1$  is locally asymptotically stable. The following theorem is given.

**Theorem 5.1.** *Provided that the conditions (4.1) and (5.3) hold, the equilibrium point  $c_1$  is locally asymptotically stable.*

We can calculate the jacobian matrix (5.1) around  $c_2$  as previous way by

$$J(c_2) = [e_{ij}]_{4 \times 4} = \begin{bmatrix} e_{11} & 0 & e_{13} & e_{14} \\ 0 & e_{22} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} \\ 0 & 0 & 0 & -d_3 \end{bmatrix}, \tag{5.4}$$

$$e_{11} = -\frac{\alpha T_2}{(b_1 \alpha^* + 1)(\beta P_2 + 1)^2} - r_1 \left( \frac{2P_2}{k_1 + \xi_1 \alpha^*} - 1 \right),$$

$$e_{31} = \frac{\alpha\alpha_1 T_2}{((b_1 r_3)/d_3 + 1)(\beta P_2 + 1)},$$

$$e_{22} = r_2 - d_1 - \frac{\gamma T_2}{((b_2 r_3)/d_3 + 1)(\beta_2 P_2 + 1)},$$

$$e_{32} = \frac{\gamma_1 \gamma T_2}{((b_2 r_3)/d_3 + 1)(\beta_2 P_2 + 1)},$$

$$e_{13} = -\frac{\alpha P_2}{((b_1 r_3)/d_3 + 1)(\beta P + 1)},$$



$$\begin{aligned}
 e_{33} &= \frac{\alpha\alpha_1P}{((b_1r_3)/d_3 + 1)(\beta P_2 + 1)} - 2mT_2 - d_2, \\
 e_{14} &= \frac{r_1\xi_1P^2}{(k_1 + (r_3\xi_1)/d_3)^2} + \frac{\alpha b_1P_2T_2}{(b_1r_3)/d_3 + 1)^2(\beta P_2 + 1)}, \\
 e_{34} &= -\frac{\alpha\alpha_1b_1P_2T_2}{((b_1r_3)/d_3 + 1)^2(\beta P + 1)}.
 \end{aligned}$$

The characteristic equation of (5.4) is

$$(-d_3 - \lambda)(e_{22} - \lambda)[\lambda^2 - (e_{11} + e_{33})\lambda + e_{11}e_{33} - e_{13}e_{31}] = 0. \tag{5.5}$$

The first two roots of (5.5) are  $\lambda_1 = -d_3$  and  $\lambda_2 = e_{22}$  and the other two roots are

$$\lambda_{3,4} = \frac{(e_{11} + e_{33}) \pm \sqrt{(e_{11} + e_{33})^2 - 4(e_{11}e_{33} - e_{13}e_{31})}}{2}.$$

So, due to the Routh-Hurwitz stability criterion the equilibrium point  $c_2$  is locally asymptotically stable if all roots of the characteristic equation (5.5) have negative real parts which are satisfied if the following conditions hold

$$e_{22} < 0, \quad e_{11} + e_{33} < 0, \quad e_{11}e_{33} > e_{13}e_{31}. \tag{5.6}$$

Thus, the following theorem is obtained.

**Theorem 5.2.** *Provided that the conditions (5.6) hold, the equilibrium point  $c_2$  is locally asymptotically stable.*

For equilibrium point  $c_3$ , the Jacobian matrix (5.1) around  $c_3$  can be calculated as

$$\begin{aligned}
 J(c_3) &= [d_{ij}]_{4 \times 4} \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ 0 & 0 & 0 & d_{44} \end{bmatrix}, \\
 d_{11} &= -\frac{\alpha T_3}{(b_1 a^* + 1)} + r_1, \\
 d_{21} &= \frac{\beta_2 \gamma C T_3}{(b_2 H_3 + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2}, \\
 d_{22} &= -\frac{\gamma T_3}{(b_2 a^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2} - r_2 \left( \frac{2C_3}{k_2 + \xi_2 a^*} - 1 \right) - d_1, \\
 d_{23} &= -\frac{\gamma C_3 (\beta_1 C_3 + 1)}{(b_2 a^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2}, \\
 d_{24} &= \frac{r_2 \xi_2 C_3^2}{(k_2 + \xi_2 a^*)^2} + \frac{b_2 \gamma C_3 T_3}{(b_2 a^* + 1)^2 (\beta_1 C_3 + \beta_3 C_3 T_3 + 1)}, \\
 d_{31} &= \frac{\alpha \alpha_1 T_3}{(b_1 a^* + 1)} - \frac{\beta_2 \gamma_1 \gamma C_3 T_3}{(b_2 a^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2}, \\
 d_{32} &= \frac{\gamma_1 \gamma T_3}{(b_2 a^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2}, \\
 d_{33} &= \frac{\gamma_1 \gamma C (\beta_1 C_3 + 1)}{(b_2 a^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2} - 2mT_3 - d_2, \\
 d_{34} &= -\frac{b_2 \gamma_1 \gamma C_3 T_3}{(b_2 a^* + 1)^2 (\beta_1 C_3 + \beta_3 C_3 T_3 + 1)}, \\
 d_{44} &= -d_3.
 \end{aligned}$$

The characteristic equation of  $J(c_3)$  is

$$(-d_3 - \lambda)(d_{11} - \lambda)[\lambda^2 - (d_{22} + d_{33})\lambda + d_{22}d_{33} - d_{23}d_{32}] = 0. \tag{5.7}$$

The first root of (5.7) is  $\lambda_1 = -d_3$ , where the second root of it is  $\lambda_2 = -\frac{\alpha T_3}{(b_1 a^* + 1)} + r_1$ , which is negative with

$$\frac{\alpha T_3}{(b_1 a^* + 1)} > r_1 \tag{5.8}$$

and the other two roots are  $\lambda_{3,4} = \frac{(d_{11} + d_{33}) \pm \sqrt{(d_{11} + d_{33})^2 - 4(d_{11}d_{33} - d_{13}d_{31})}}{2}$ , which are negative if

$$d_{11} + d_{33} < 0, \tag{5.9}$$

$$d_{11}d_{33} > d_{13}d_{31}. \tag{5.10}$$

So, according to Routh-Hurwitz stability criterion  $c_3$  is locally asymptotically stable if (5.8) and (5.9) are holds. Thus, the following theorem is given.

**Theorem 5.3.** *Provided that the conditions (5.8), (5.9), and (5.10) hold, the equilibrium point  $c_3$  is locally asymptotically stable.*

In the following, we find the Jacobian matrix (5.1) around  $c_{PH}$  as

$$J(c_{PH}) = [f_{ij}]_{4 \times 4} = \begin{bmatrix} f_{11} & 0 & f_{13} & f_{14} \\ 0 & f_{22} & 0 & 0 \\ 0 & 0 & f_{33} & 0 \\ 0 & 0 & 0 & f_{44} \end{bmatrix}, \quad \begin{aligned} f_{11} &= -r_1 \left( \frac{2a_1^*}{k_1 + \xi_1 a^*} - 1 \right), \\ f_{13} &= -\frac{\alpha a_1^*}{(b_1 a^* + 1)(\beta a_1^* + 1)}, & f_{14} &= \frac{r_1 \xi_1 a_1^{*2}}{(k_1 + \xi_1 a^*)^2}, \\ f_{22} &= r_2 - d_1, & f_{33} &= -d_2 + \frac{\alpha \alpha_1 a_1^*}{(b_1 a^* + 1)(\beta a_1^* + 1)}, & f_{44} &= -d_3. \end{aligned}$$

The characteristic equation of  $J(c_{PH})$  is

$$(\lambda_1 - f_{11})(\lambda_2 - f_{22})(\lambda - f_{33})(\lambda_4 - f_{44}) = 0,$$

where  $\lambda_1 = -r_1 \left( \frac{2a_1^*}{k_1 + \xi_1 a^*} - 1 \right)$ ,  $\lambda_2 = r_2 - d_1$ ,  $\lambda_3 = -d_2 + \frac{\alpha \alpha_1 a_1^*}{(b_1 a^* + 1)(\beta a_1^* + 1)}$ , and  $\lambda_4 = -d_3$ . So, all those roots are negative provided that

$$\frac{2a_1^*}{k_1 + \xi_1 a^*} > 1, \quad d_1 > r_2, \quad d_2 > \frac{\alpha \alpha_1 a_1^*}{(b_1 a^* + 1)(\beta a_1^* + 1)}. \tag{5.11}$$

Thus, we can write the following theorem.

**Theorem 5.4.** *Provided that the conditions (5.11) hold, the equilibrium point  $c_{PH}$  is locally asymptotically stable.*

We calculate the Jacobian matrix (5.1) around  $c_{CRH}$  as

$$J(c_{CRH}) = \begin{bmatrix} h_{11} & 0 & 0 & 0 \\ 0 & h_{22} & h_{23} & h_{24} \\ 0 & 0 & h_{33} & 0 \\ 0 & 0 & 0 & h_{44} \end{bmatrix}, \quad \begin{aligned} h_{11} &= r_1, & h_{22} &= -r_2 + d_1, \end{aligned}$$

$$h_{23} = -\frac{\gamma}{(b_2 a^* + 1)}, \quad h_{24} = r_2 \xi_2 \left(1 - \frac{d_1}{r_2}\right)^2, \quad h_{33} = \frac{\gamma_1 \gamma}{\beta_1 (b_2 a^* + 1)} - d_2, \quad h_{44} = -d_3.$$

Unfortunately, we get one root of it characteristic equation  $h_{11} = r_1$  which is positive. So, this point can not be stable. In the following, we discuss the local stability of positive equilibrium point  $c^*$ . The Jacobian matrix around  $c^*$  is given as

$$J(P^*, C_R^*, T^*, H^*) = [m_{ij}]_{4 \times 4} = \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & m_{44} \end{bmatrix}.$$

So, the eigenvalues of  $J(c^*)$  are the roots of the following equation

$$(-d_3 - \lambda)(\lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3) = 0, \tag{5.12}$$

where

$$\begin{aligned} M_1 &= -(m_{11} + m_{22} + m_{33}), \\ M_2 &= m_{11}(m_{22} + m_{33}) + m_{22}m_{33} - m_{13}m_{31} - m_{23}m_{32}, \\ M_3 &= m_{11}(m_{23}m_{32} - m_{22}m_{33}) + m_{13}(m_{22}m_{31} - m_{21}m_{32}), \end{aligned}$$

and

$$\begin{aligned} m_{11} &= -\frac{\alpha T^*}{(b_1 H^* + 1)(\beta P^* + 1)^2} - r_1 \left( \frac{2P^*}{k_1 + \xi_1 H^*} - 1 \right), \\ m_{13} &= -\frac{\alpha P^*}{(b_1 H^* + 1)(\beta P^* + 1)}, \\ m_{14} &= \frac{r_1 \xi_1 P^{*2}}{(k_1 + \xi_1 H^*)^2} + \frac{\alpha b_1 P^* T^*}{(b_1 H^* + 1)^2 (\beta P^* + 1)}, \\ m_{21} &= \frac{\beta_2 \gamma C_R^* T^*}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2}, \\ m_{22} &= -\frac{\gamma T^* (\beta_2 P^* + 1)}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2} - r_2 \left( \frac{2C_R^*}{k_2 + \xi_2 H^*} - 1 \right) - d_1, \\ m_{23} &= -\frac{\gamma C_R^* (\beta_1 C_R^* + \beta_2 P^* + 1)}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2}, \\ m_{24} &= \frac{r_2 \xi_2 C_R^{*2}}{(k_2 + \xi_2 H^*)^2} + \frac{b_2 \gamma C_R^* T^*}{(b_2 H^* + 1)^2 (\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)}, \\ m_{31} &= \frac{\alpha \alpha_1 T^*}{(b_1 H^* + 1)(\beta P^* + 1)^2} - \frac{\beta_2 \gamma_1 \gamma C_R^* T^*}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2}, \\ m_{32} &= \frac{\gamma_1 \gamma T^* (\beta_2 P^* + 1)}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2}, \\ m_{33} &= \frac{\gamma_1 \gamma C_R^* (\beta_1 C_R^* + \beta_2 P^* + 1)}{(b_2 H^* + 1)(\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)^2} - 2m T^* - d_2 + \frac{\alpha \alpha_1 P^*}{(b_1 H^* + 1)(\beta P^* + 1)}, \\ m_{34} &= -\frac{b_2 \gamma_1 \gamma C_R^* T^*}{(b_2 H^* + 1)^2 (\beta_1 C_R^* + \beta_2 P^* + \beta_3 C_R^* T^* + 1)} - \frac{\alpha \alpha_1 b_1 P T}{(b_1 H + 1)^2 (\beta P + 1)}, \\ m_{44} &= -d_3. \end{aligned}$$

Keep above in mind, based on the Routh-Hurwitz rule,  $c^*$  will exhibit asymptotic stability if and only if  $M_1 > 0$ ,  $M_3 > 0$ , and  $M_1 M_2 > M_3$ . Therefore,  $M_1 > 0$  if

$$m_{11} + m_{22} + m_{33} < 0, \tag{5.13}$$

and  $M_3 > 0$  if

$$m_{11}m_{23}m_{32} + m_{13}m_{22}m_{31} > m_{13}m_{21}m_{32} + m_{11}m_{22}m_{33}. \tag{5.14}$$

On other hand,

$$\begin{aligned} M_1M_2 - M_3 &= (m_{11} + m_{22} + m_{33})[m_{13}m_{31} - m_{22}m_{33} + m_{23}m_{32} - m_{11}(m_{22} + m_{33})] \\ &\quad + m_{11}m_{22}m_{33} - m_{11}m_{23}m_{32} - m_{13}m_{21}m_{32} - m_{13}m_{22}m_{31} \\ &= (m_{11} + m_{33})(m_{13}m_{31} - m_{22}^2) + m_{33}^2(m_{11} + m_{22}) \\ &\quad + (m_{22} + m_{33})(m_{23}m_{32} - m_{11}^2) - 2m_{11}m_{22}m_{33} - m_{13}m_{21}m_{32}. \end{aligned}$$

This condition  $M_1M_2 > M_3$  is satisfied provided that the following condition holds:

$$\frac{(m_{11} + m_{33})(m_{13}m_{31} - m_{22}^2) + m_{33}^2(m_{11} + m_{22}) + (m_{22} + m_{33})(m_{23}m_{32} - m_{11}^2)}{2m_{11}m_{22}m_{33} + m_{13}m_{21}m_{32}} > 1. \tag{5.15}$$

Hence, the following theorem is obtained.

**Theorem 5.5.** *Assume that the conditions (5.13), (5.14), and (5.15) are satisfied, then the positive equilibrium point  $c^*$  is locally asymptotically stable.*

### 6. Hopf bifurcation

By using [2, 23], the steady state  $c^*$  changes as the parameter  $\gamma_1$  crosses the threshold value  $\gamma_1^*$ , which implies that  $c^*$  may become unstable due to Hopf bifurcation when forced to operate within particular restrictions on its parameters. In the case where we use  $\gamma_1^*$  as the bifurcation parameter, the Hopf bifurcation threshold and its conditions are clearly clarified in the following theorem.

**Theorem 6.1.** *Assuming the following conditions are satisfied*

$$\gamma_1^* > 0, \tag{6.1}$$

$$M_i > 0, \quad i = 1, 2, \tag{6.2}$$

where  $M_i, i = 1, 2$  are the coefficients of the characteristic equation given in equation (5.12) with  $\gamma_1 = \gamma_1^*$  and the formula for  $\gamma_1^*$  is designated in the following proof, then, there exists a Hopf bifurcation for  $A_3$  at  $\gamma_1 = \gamma_1^*$ .

*Proof.* The value of the bifurcation parameter can be determined by setting  $M_1(\gamma_1^*)M_2(\gamma_1^*) - M_3(\gamma_1^*)=0$ , which gives:

$$\gamma_1^* = \frac{(b_2a^* + 1)(\beta_1C_3 + \beta_3C_3T_3 + 1)^2\Gamma}{\gamma T_3 m_{13} m_{21}}.$$

Clearly,  $\gamma_1^* > 0$  if condition (6.1) holds, where  $\Gamma = (m_{11} + m_{33})(m_{13}m_{31} - m_{22}^2) + m_{33}^2(m_{11} + m_{22}) + (m_{22} + m_{33})(m_{23}m_{32} - m_{11}^2) - 2m_{11}m_{22}m_{33}$ . Clearly,  $\gamma_1^* > 0$  if condition (5.15) holds. Now, at  $\gamma_1 = \gamma_1^*$  equation (5.12) can be written as

$$(\lambda + M_1)(\lambda^2 + M_2) = 0. \tag{6.3}$$

According to condition (6.2), the above equation has three roots: a negative root  $\lambda_1 = -M_1$  and two purely imaginary roots  $\lambda_{2,3} = \pm i\sqrt{M_2}$ . In a neighborhood of  $\gamma_1^*$ , the roots have the following forms:  $\lambda_1 = -M_1$ ,  $\lambda_{2,3} = \sigma_1(\lambda_1) \pm i\sigma_2(\gamma_1)$ . Clearly,  $\text{Re}(\lambda_{2,3})|_{\gamma_1=\gamma_1^*} = \sigma_1(\gamma_1^*) = 0$  indicates that the first condition for Hopf bifurcation has been met at  $\gamma_1 = \gamma_1^*$ . Now, to confirm the transversality condition, we substitute  $\sigma_1(\gamma_1) \pm$

$i\sigma_2(\gamma_1)$  into equation (6.3) and then compute its derivative with respect to  $\gamma_1^*, \theta(\gamma_1^*)\psi(\gamma_1^*) + \Sigma(\gamma_1^*)\phi(\gamma_1^*)$ , where the form of  $\theta(\gamma_1), \psi(\gamma_1), \Sigma(\gamma_1),$  and  $\phi(\gamma_1)$  are

$$\begin{aligned} \psi(\gamma_1) &= 3\sigma_1^2(\gamma_1) + 2M_1(\gamma_1)\sigma_1(\gamma_1) + M_2(\gamma_1) - 3\sigma_2^2(\gamma_1), \\ \phi(\lambda_1) &= 6\sigma_1(\lambda_1)\sigma_2(\lambda_1) + 2M_1(\lambda_1)\sigma_2(\lambda_1), \\ \theta(\lambda_1) &= \sigma_1^2(\lambda_1)M_1'(\lambda_1) + M_2'(\lambda_1)\sigma_1(\lambda_1) + M_3'(\lambda_1) - M_1'(\lambda_1)\sigma_2^2(\lambda_1), \\ \Sigma(\lambda_1) &= 2\sigma_1(\lambda_1)\sigma_2(\lambda_1)M_1'(\lambda_1) + M_2'(\lambda_1)\sigma_2(\lambda_1). \end{aligned}$$

Now at  $\lambda_1 = \lambda_1^*$ , substituting  $\sigma_1(\lambda_1^*) = 0$  and  $\sigma_2(\lambda_1^*) = \sqrt{M_2(\lambda_1^*)}$  into equation (6.3), the following is obtained:

$$\begin{aligned} \psi(\lambda_1^*) &= -2M_2(\lambda_1^*), & \phi(\lambda_1^*) &= 2M_1(\lambda_1^*)\sqrt{M_2(\lambda_1^*)}, \\ \theta(\lambda_1^*) &= M_3'(\lambda_1^*) - M_1'(\lambda_1^*)M_2(\lambda_1^*), & \Sigma(\lambda_1^*) &= M_2'(\lambda_1^*)\sqrt{M_2(\lambda_1^*)}, \end{aligned}$$

where

$$M_1'(\lambda_1^*) = \frac{\gamma C(\beta_1 C_3 + 1)}{(b_2 \alpha^* + 1)(\beta_1 C_3 + \beta_3 C_3 T_3 + 1)^2}, \quad M_2'(\lambda_1^*) = 0, \quad M_3'(\lambda_1^*) = 0.$$

Hence, condition (6.2) gives

$$\theta(\lambda_1^*)\psi(\lambda_1^*) + \Sigma(\lambda_1^*)\phi(\lambda_1^*) = 2M_1'(\lambda_1^*)M_2^2(\lambda_1^*) + 2M_1(\lambda_1^*)\sqrt{M_2(\lambda_1^*)} \neq 0.$$

That means the Hop bifurcation has occurred at  $\lambda_1^*$ . □

### 7. Numerical results

The dynamical behaviour of the proposed model has been checked with the help of the ode45 package of MATLAB 2015. The existence of a prey-free equilibrium point has been depicted in Figure 2. This is happening due to the small growth rate of prey and the increase in prey consumption by top predators in the coffee forests in southwestern Ethiopia.

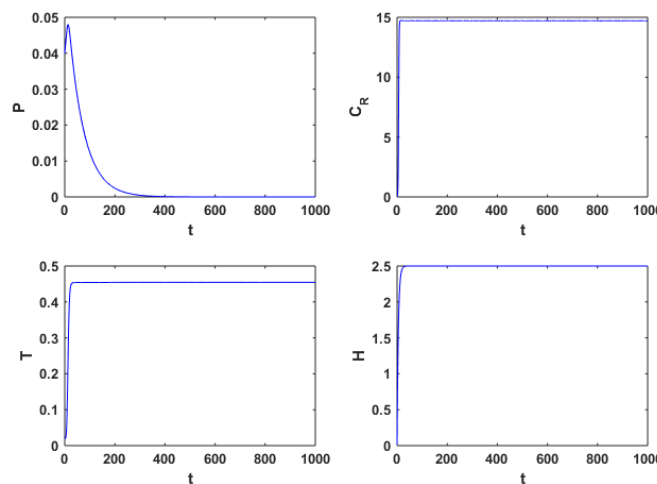


Figure 2: Stability of prey-free equilibrium point of model (2.1) for the parametric values  $r_1 = 0.02, k_1 = 10, \xi_1 = 0.0, \alpha = 0.1, \beta = 0.1, b_1 = 0.1, r_2 = 1.0, k_2 = 15, \xi_2 = 0.0, \gamma = 0.2, \beta_1 = 0.2, \beta_2 = 0.3, \beta_3 = 0.4, b_2 = 0.2, d_1 = 0.01, \alpha_1 = 0.02, \gamma_1 = 0.8, m = 0.5, d_2 = 0.01, r_3 = 0.5, d_3 = 0.2.$

Figure 3 represents the stability of the top predator-free equilibrium point for the proposed model. Due to the major influence of the human shield, the top predators are surviving to eat the prey, which can lead the top predator species towards extinction in the coffee forests of southwestern Ethiopia. Also, the stability of the existence of the prey, crop raiders, and top predator in the presence of a human shield has been seen in Figure 4. By adding humans to the list of species in coffee forest ecosystems, we can learn more about how human activity affects the occurrence of different species.

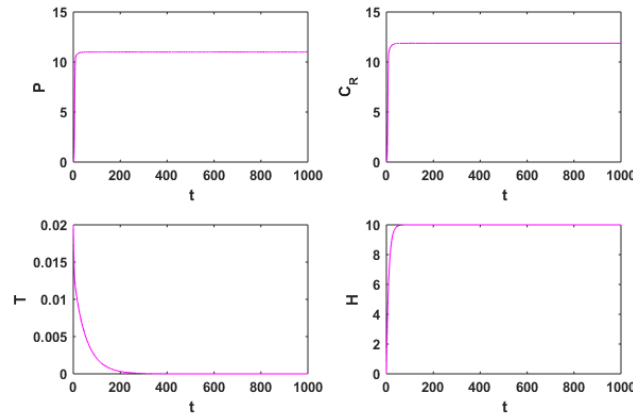


Figure 3: Stability of the top predator-free equilibrium point of model (2.1) for the parameters  $r_1 = 1.0, k_1 = 10, \xi_1 = 0.1, \alpha = 0.1, \beta = 0.1, b_1 = 0.00001, r_2 = 1.0, k_2 = 10, \xi_2 = 0.2, \gamma = 0.2, \beta_1 = 0.2, \beta_2 = 0.3, \beta_3 = 0.4, b_2 = 0.00002, d_1 = 0.01, \alpha_1 = 0.02, \gamma_1 = 0.2, m = 0.01, d_2 = 0.1, r_3 = 1.0, d_3 = 0.1$ .

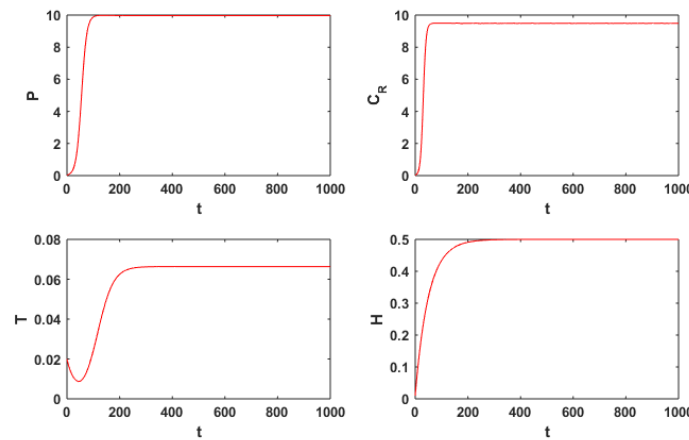


Figure 4: Stability of the interior or coexistence equilibrium point of model (2.1) for the set of parameters  $r_1 = 0.1, k_1 = 10, \xi_1 = 0.0, \alpha = 0.01, \beta = 0.1, b_1 = 0.01, r_2 = 0.2, k_2 = 10, \xi_2 = 0.0, \gamma = 0.02, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 1, \gamma_1 = 0.08, m = 0.5, d_2 = 0.02, r_3 = 0.01, d_3 = 0.02$ .

The bifurcation diagram of model (2.1) with respect to  $m$  has been presented in Figure 5. From this figure, it is observed that the model may undergo Hopf bifurcation due to the change of value of  $m$  from 0 to 0.02. It is seen that prey, crop-raiders, and top predator species are experienced with oscillatory or unstable solutions for  $0 \leq m < 0.0007$ . But the three species may maintain stability for  $0.0007 < m \leq 0.02$ . No change in the stability of the human species has been found for  $0 \leq m \leq 0.02$ . So, it can be concluded that the increased rate of intra-species competition among the top predators may increase the stability of the coffee forests in the presence of humans. Again, the bifurcation diagram of model (2.1) with respect to  $\beta$  has been depicted in Figure 6. This figure shows that the model may go through Hopf bifurcation for the change of  $\beta$  from 0.4 to 2.0. It is seen that prey, crop raiders, and top predator species may show stable steady state behaviour for  $0.4 \leq \beta < 0.528$  and  $1.344 < \beta \leq 2$ . But all these three species may

display unstable behaviour for  $0.528 < \beta < 1.344$ . It has been found that the human species may remain unaffected by the change in  $\beta$ . So, it can be concluded that the increase in the half-saturation constant for the top predator may change the system dynamics.

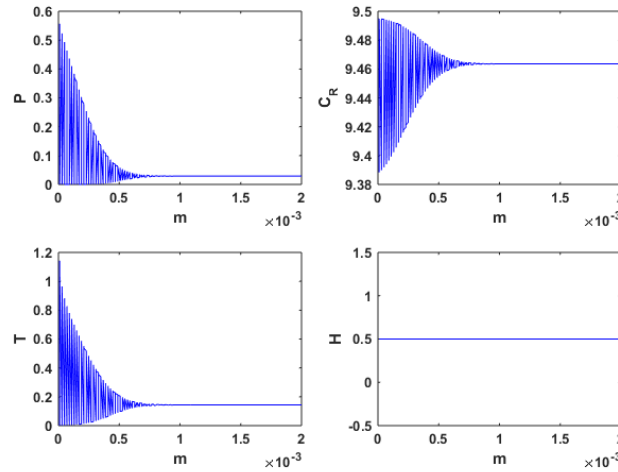


Figure 5: Bifurcation diagram of model (2.1) with respect to  $m$  for the set of parameters  $r_1 = 0.1, k_1 = 10, \xi_1 = 0.0, \alpha = 0.7, \beta = 0.1, b_1 = 0.01, r_2 = 0.2, k_2 = 10, \xi_2 = 0.0, \gamma = 0.02, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 1.0, \gamma_1 = 0.001, d_2 = 0.02, r_3 = 0.01, d_3 = 0.02$ .

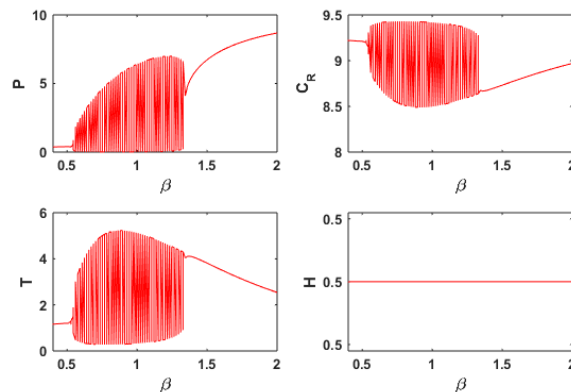


Figure 6: Bifurcation diagram of model (2.1) with respect to  $\beta$  for  $r_1 = 0.1, k_1 = 10, \xi_1 = 1.0, \alpha = 0.1, b_1 = 0.1, r_2 = 0.2, k_2 = 10, \xi_2 = 1.0, \gamma = 0.02, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 1.0, \gamma_1 = 0.01, m = 0.01, d_2 = 0.02, r_3 = 0.01, d_3 = 0.02$ .

The bifurcation diagram of model (2.1) with respect to  $b_1$  has been presented in Figure 7. From this figure, it is observed that the model may experience Hopf bifurcation due to the variation of  $b_1$  from 0.1 to 1.2. It is seen that prey may be extinct for  $0.188 \leq b_1 < 0.419$ , but the species may show stable steady state behaviour for  $0.419 < b_1 < 0.562$  and  $1.112 < b_1 \leq 1.2$ . The oscillatory behaviour of prey species has been observed for  $0.562 < b_1 < 1.112$ . Also, it has been found that crop raider species may show stable steady-state behaviour for  $0.1 \leq b_1 < 0.54$  and  $1.112 < b_1 \leq 1.2$ . But the crop raider species may show oscillatory behaviour for  $0.54 < b_1 < 1.112$ . The top predator species may show stable steady-state behaviour for  $0.1 \leq b_1 < 0.562$  and  $1.112 < b_1 \leq 1.2$ . But the top predator species may continue unstable behaviour for  $0.562 < b_1 < 1.112$ . So it can be concluded that an increase in human interference with prey species may be responsible for the stable dynamics of the model. Hence, human interference can be beneficial for the stability of the coffee forest ecosystem in southwestern Ethiopia. Also, the bifurcation diagram of model (2.1) with respect to  $\gamma_1$  has been depicted in Figure 8. This figure shows that the model may undergo

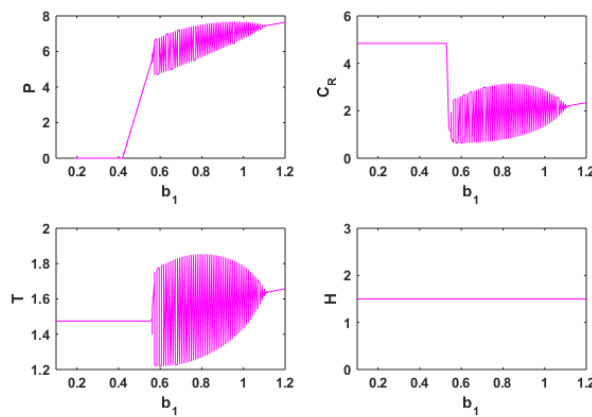


Figure 7: Bifurcation diagram of model (2.1) with respect to  $b_1$  for the set of parameters  $r_1 = 0.1, k_1 = 10, \xi_1 = 1.0, \beta = 0.1, \alpha = 0.1, r_2 = 0.2, k_2 = 10, \xi_2 = 1.0, \gamma = 0.2, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 0.1, \gamma_1 = 0.1, m = 0.01, d_2 = 0.02, r_3 = 0.03, d_3 = 0.02$ .

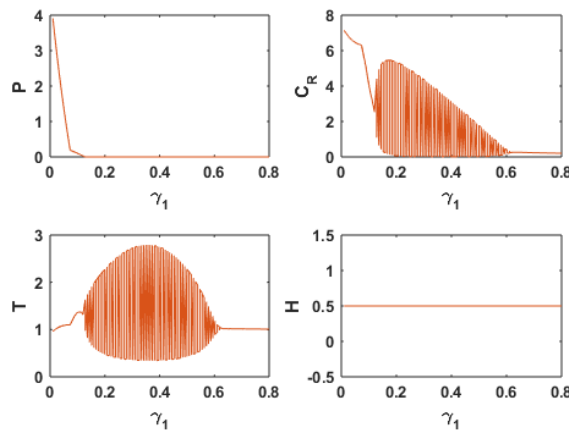


Figure 8: Bifurcation diagram of model (2.1) with respect to  $\gamma_1$  for the set of parametric values  $r_1 = 0.1, k_1 = 10, \xi_1 = 1.0, \beta = 0.1, b_1 = 0.2, \alpha = 0.1, r_2 = 0.2, k_2 = 10, \xi_2 = 1.0, \gamma = 0.2, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 0.1, m = 0.01, d_2 = 0.02, r_3 = 0.01, d_3 = 0.02$ .

Hopf bifurcation due to the change of  $\gamma_1$  from 0.01 to 0.8. It is observed that stable dynamics exist for prey species for  $0.01 < \gamma_1 < 0.1285$  and the prey may go extinct for  $0.1285 < \gamma_1 \leq 0.8$ . Crop raider and top predator species may show stable steady state behaviour for  $0.01 \leq \gamma_1 < 0.1206$  and  $0.6341 < \gamma_1 \leq 0.8$ . But crop raiders and top predators may show unstable behaviour for  $0.1206 < \gamma_1 < 0.6341$ . So, it can be concluded that an increase in the conversion rate of crop raiders to top predators may be responsible for the extinction of prey. Thus, the conditions outlined in Theorem 6.1 are fulfilled, and model (2.1) encounters a Hopf bifurcation for  $0.1206 < \gamma_1 < 0.6341$ .

The bifurcation diagram of model (2.1) with respect to  $r_1$  has been depicted in Figure 9. From this figure, it is seen that the model may go through Hopf bifurcation for the change in value of  $r_1$  from 0.001 to 0.01. It is observed that prey, crop raiders, and top predator species may show oscillatory behaviour for  $0.001 \leq r_1 < 0.002377$ , but they continue stable steady state behaviour for  $0.002377 < r_1 \leq 0.01$ . Hence, it can be concluded that the increase in the intrinsic growth rate of prey may be beneficial for the stability of the ecosystem, which consists of prey, crop raiders, and top predators. Finally, the bifurcation diagram of model (2.1) with respect to  $r_3$  has been presented in Figure 10. The model may possess Hopf bifurcation due to the variation of  $r_3$  from 0.08 to 0.11. It was found that prey, crop raiders, and top predators may show some oscillatory dynamics for  $0.08 \leq r_3 < 0.1072$ , but they may continue stable steady state



behaviour for  $0.1072 < r_3 \leq 0.11$ . It has been found that the increase of humans in coffee forests may increase the stability of prey, crop raiders, and top predator species.

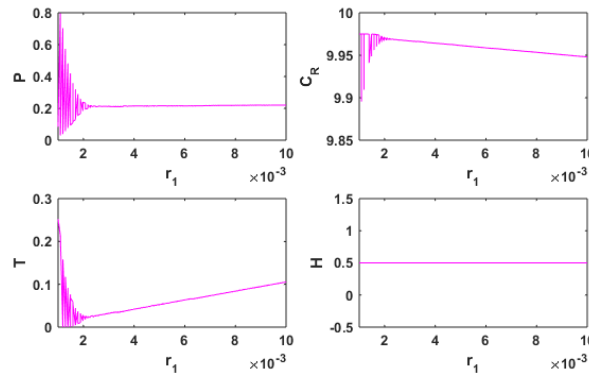


Figure 9: Bifurcation diagram of model (2.1) with respect to  $r_1$  for the set of parametric values  $k_1 = 10, \xi_1 = 1.0, \alpha = 0.1, \beta = 0.1, b_1 = 0.1, r_2 = 0.2, k_2 = 10, \xi_2 = 1.0, \gamma = 0.02, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 1.0, \gamma_1 = 0.01, m = 0.01, d_2 = 0.02, r_3 = 0.01, d_3 = 0.02$ .

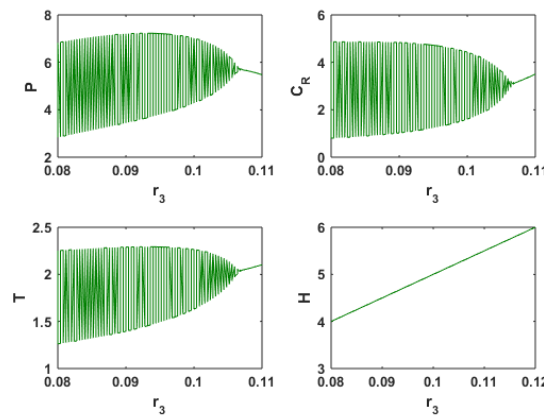


Figure 10: The bifurcation diagram of model (2.1)  $r_1 = 0.1, k_1 = 10, \xi_1 = 1.0, \beta = 0.1, b_1 = 0.2, \alpha = 0.1, r_2 = 0.2, k_2 = 10, \xi_2 = 1.0, \gamma = 0.2, \beta_1 = 0.3, \beta_2 = 0.04, \beta_3 = 0.03, b_2 = 0.03, d_1 = 0.01, \alpha_1 = 0.1, \gamma_1 = 0.1, m = 0.01, d_2 = 0.02, d_3 = 0.02$ .

### 8. Conclusion

In this paper, we have formulated a model among the interactions of prey (e.g., deer’s), crop raiders (e.g., baboons, warthogs, etc), and top predators (e.g., tigers, hyenas, etc) in the coffee forests of south-western Ethiopia. It is considered that prey grow logistically in the absence of humans and top predators. It is also considered that the environmental carrying of prey may be increased due to the human shield. It is considered that crop raiders grow logistically in the absence of humans and top predators. It is also considered that top predators consume prey, as well as crop raiders, through Holling type II functional form, depending on human intelligence. The intra-species competition among the top predators has been taken into consideration. It is assumed that humans are increasing in the coffee forests at a constant rate and decreasing due to the natural constant decay rate. From the analysis of the model, it is found that prey species may be extinct due to a lower growth rate and consumption by top predators in the absence of human interference in the carrying capacity of prey. It has been observed that top predators may be extinct due to the human interference in the carrying capacity and the smaller dependence on humans in the consumption terms of prey and crop raiders, respectively. It is also found that there is an increase

in intra-species competition among the top predators, which may intensify the stability of the model. This happens due to the absence of human interference in the carrying capacity of prey and crop raiders and the smaller conversion rate of crop raiders among top predators. It may create a food crisis for top predators, which creates intra-species competition among top predators. It is found that the stability of the model may increase with an increase in the half saturation constant of the top predator. It is seen that the increase in human dependency on the consumption of prey may improve the stability of the model. It is also reported that there has been an increase in the stability of the model with the increase in the conversion rate of crop raiders. Again, it is observed that the increase in the intrinsic growth rate of prey and top predators may improve the stability of the model.

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