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# A study on differential equations associated with $(q,h)$-Frobenius-Euler polynomials

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## Abstract

In recent years, utilizing the generalized quantum exponential function (also known as the $(q,h)$-exponential function) that extends and unifies the q- and h-exponential functions into a single and convenient form, $(q,h)$-generalizations of the diverse polynomials and numbers, such as Euler and tangent polynomials and numbers, have been introduced and studied. Inspired by these studies, in this work, we focus on defining and analyzing extensions of Frobenius-Euler polynomials and numbers using the $(q,h)$-exponential function. Also, we show that the mentioned polynomials are solutions to some higher-order differential equations. Furthermore, we examine that $(q,h)$-Frobenius-Euler polynomials are solutions to higher-order differential equations combined with the q-Bernoulli, q-Euler, and q-Genocchi numbers and polynomials, respectively. Finally, we use a computer program to visualize the approximate roots of the mentioned polynomials.

**Keywords:** q-numbers, $(q,h)$-derivative, degenerate q-Frobenius-Euler polynomials, differential equation.

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## 1. Introduction

Scientists, mostly mathematicians, have recently established and built generating maps for new families of special polynomials, such as q-Genocchi, q-Euler, q-Bernoulli polynomials, and so forth, [10–13, 16, 17] and also see the references cited therein. Elementary properties such as recurrence relations, symmetric properties, explicit and implicit summation formulas, and varied applications, such as differential equations, number theory, functional analysis, quantum mechanics, mathematical analysis, and mathematical physics, have been worked and analyzed by these types of studies mentioned above.

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Inspired and motivated by the above, in this study, we consider $(q, h)$-extension of Frobenius-Euler polynomials, and then some of their formula and relations are analyzed and derived. Also, we provided that the mentioned polynomials are solutions to some higher-order differential equations. Moreover, we investigate that $(q, h)$-Frobenius-Euler polynomials are solutions to higher-order differential equations combined with the q-Bernoulli, q-Euler, and q-Genocchi numbers and polynomials, respectively. Furthermore, we utilize a computer program to give the structures and shapes of approximate roots of the mentioned polynomials.

Due to its applications in mathematics, physics, and engineering, the subject of q-calculus began to surface in the nineteenth century. The references [1–9, 14–19] contain the definitions and notations of q-calculus and $(q, h)$-calculus that we review here.

The q-shifted factorial is provided as follows

$$(\delta; q)_0 = 1, (\delta; q)_s = \prod_{m=0}^{s-1}(1 - q^m\delta), \quad s \in \mathbb{N}.$$

The q-number and q-factorial are provided as follows

$$[\delta]_q = \frac{1-q^\delta}{1-q}, \ q \in \mathbb{C} - \{1\}; \delta \in \mathbb{C}, \qquad [s]_q! = \prod_{m=1}^{s}[m]_q = [1]_q[2]_q \cdots [s]_q = \frac{(q;q)_s}{(1-q)^s}, \ q \neq 1; s \in \mathbb{N},$$

and

$$[0]_q! = 1, \quad q \in \mathbb{C}; 0 < q < 1.$$

The q-binomial coefficient is provided as follows

$$\binom{s}{r}_q = \frac{[s]_q!}{[r]_q![s-r]_q!} = \frac{(q;q)_s}{(q;q)_r(q;q)_{s-r}}, \quad r = 0, 1, \ldots, s.$$

The q-power basis is provided as follows

$$(\mu + \nu)_q^s = \sum_{r=0}^{s}\binom{s}{r}_q q^{r(r-1)/2}\mu^{s-r}\nu^r, \quad s \in \mathbb{N}_0.$$

The q-exponential function is provided as

$$e_q(\mu) = \sum_{s=0}^{\infty}\frac{\mu^s}{[s]_q!} = \frac{1}{((1-q)\mu; q)_\infty}, \quad 0 <| q |< 1; | \mu |<| 1-q |^{-1}, \tag{1.1}$$

The q-Bernoulli, q-Euler, and q-Genocchi polynomials are given, respectively, as follows (see [16, 17]):

$$\frac{l}{e_q(l)-1}e_q(\mu l) = \sum_{s=0}^{\infty}\mathbb{B}_{s,q}(\mu)\frac{l^s}{[s]_q!}, \quad (| l |) < 2\pi, \tag{1.2}$$

$$\frac{2}{e_q(l)+1}e_q(\mu l) = \sum_{s=0}^{\infty}\mathbb{E}_{s,q}(\mu)\frac{l^s}{[s]_q!}, \quad (| l |) < \pi, \tag{1.3}$$

$$\frac{2l}{e_q(l)+1}e_q(\mu l) = \sum_{s=0}^{\infty}\mathbb{G}_{s,q}(\mu)\frac{l^s}{[s]_q!}, \quad (| l |< \pi). \tag{1.4}$$

The corresponding numbers of these polynomials above are obtained by choosing $\mu = 0$, namely $\mathbb{B}_{s,q}(0) := \mathbb{B}_{s,q}, \mathbb{E}_{s,q}(0) := \mathbb{E}_{s,q}$ and $\mathbb{G}_{s,q}(0) := \mathbb{G}_{s,q}$, respectively.

The q-Frobenius-Euler polynomials are given by (cf. [15])

$$\frac{1-\omega}{e_q(l)-\omega}e_q(l\mu)=\sum_{s=0}^{\infty}\mathbb{H}_{s,q}(\mu;\omega)\frac{l^s}{[s]_q!},$$

with a suitable parameter $\omega$.

The corresponding numbers of q-Frobenius-Euler polynomials are obtained just by choosing $\mu=0$, namely $\mathbb{H}_{s,q}(0;\omega):=\mathbb{H}_{s,q}(\omega)$. The usual Frobenius-Euler polynomials and numbers are attained by taking $q\to 1$. The bivariate time scale $\mathbf{T}_{q,h}(\mu)$ is provided as follows (cf. [2, 3, 7, 18])

$$\mathbf{T}_{q,h}(\mu)=\{q^s\mu+[\mu]_q h\mid \mu\in\mathbb{R},\mu\in\mathbb{Z},h,q\in\mathbb{R}^+,q\neq 1\}\cup\{\frac{h}{1-q}\}.$$

The delta $(q,h)$-derivative of f is provided as follows (cf. [2, 4])

$$\mathbb{D}_{q,h}f(\mu)=\frac{f(q\mu+h)-f(\mu)}{(q-1)\mu+h}, \tag{1.5}$$

for $f:\mathbf{T}_{q,h}(\mu)\to\mathbb{R}$ being any function. By (1.5), some properties can be observed as follows.

(i) $f(\mu)$ is a constant if and only if $\mathbb{D}_{q,h}f(\mu)=0$, for $\mu\in\mathbf{T}_{q,h}(\mu)$.
(ii) $f(\mu)=g(\mu)+c$ with some constant c if and only if $\mathbb{D}_{q,h}f(\mu)=\mathbb{D}_{q,h}g(\mu)$ for all $\mu\in\mathbf{T}_{q,h}(\mu)$.
(iii) $f(\mu)=c_1\mu+c_2$, where $c_1$ and $c_2$ are constants if and only if $\mathbb{D}_{q,h}f(\mu)=c_1$, for $\mu\in\mathbf{T}_{q,h}(\mu)$.

We note from (1.5) that, when $q\to 1$, the delta $(q,h)$-derivative operator becomes q-derivative operator $\mathbb{D}_q(f)$ (cf. [1, 3, 6, 7]) and also when $h\to 0$, the delta $(q,h)$-derivative operator becomes the h-derivative operator $\mathbb{D}_h(f)$ (cf. [7]). Furthermore, the product rule and quotient rule for $\mathbb{D}_{q,h}f(\mu)$ are discovered as per the following.

(i) Product rule:

$$\mathbb{D}_{q,h}(f(\mu)g(\mu))=g(q\mu+h)\mathbb{D}_{q,h}f(\mu)+f(\mu)\mathbb{D}_{q,h}g(\mu)=f(q\mu+h)\mathbb{D}_{q,h}g(\mu)+g(\mu)\mathbb{D}_{q,h}f(\mu).$$

(ii) Quotient rule

$$\mathbb{D}_{q,h}\left(\frac{f(\mu)}{g(\mu)}\right)=\frac{g(\mu)\mathbb{D}_{q,h}f(\mu)-f(\mu)\mathbb{D}_{q,h}g(\mu)}{g(\mu)g(q\mu+h)}=\frac{g(q\mu+h)\mathbb{D}_{q,h}f(\mu)-f(q\mu+h)\mathbb{D}_{q,h}g(\mu)}{g(\mu)g(q\mu+h)}.$$

The $(q,h)$-power basis is provided as follows (cf. [19])

$$(\mu-\mu_0)^s_{q,h}=\begin{cases}1, & \text{if } s=0,\\ \prod_{i=1}^{s}(s-(q^{i-1}\mu_0+[i-1]_q h)), & \text{if } s\geqslant 0,\end{cases}$$

where $\mu_0\in\mathbb{R}$. We note from above that the $(q,h)$-power basis reduces to q-power basis (denoted by $(\mu-\mu_0)^s_q$) when $q\to 1$ and the $(q,h)$-power basis reduces to h-power basis (denoted by $(\mu-\mu_0)^s_h$) when $h\to 0$. In addition, it is not hard to observe that $\lim_{(q,h)\to(1,0)}(\mu-\mu_0)^s_{q,h}=(\mu-\mu_0)^s$.

For $\alpha$ being an arbitrary nonzero constant, the $(q,h)$-exponential function is provided as follows

$$\exp_{q,h}(\alpha\mu)=\sum_{i=0}^{\infty}\frac{\alpha^i(\mu-0)^i_{q,h}}{[i]!}. \tag{1.6}$$

It can be observed from (1.6) that the $(q,h)$-exponential function reduces to q-exponential function in (1.1) when $q\to 1$ with $\alpha=1$ and the $(q,h)$-exponential function reduces to h-exponential function (denoted by $e_h(\mu)=(1+h)^{\frac{\mu}{h}}$) when $h\to 0$ with $\alpha=1$. In addition, it is not hard to observe that $\lim_{(q,h)\to(1,0)}\exp_{q,h}(\alpha\mu)=e^{\alpha\mu}$.

Recently, Kang [8] considered the degenerate form of the $(q,h)$-exponential function as follows

$$\exp_{q,h}(\mu:l)=\sum_{s=0}^{\infty}(\mu)_{q,h}^{s}\frac{l^s}{[s]_q!}, \tag{1.7}$$

where $(\mu)_{q,h}^{s}=\prod_{r=1}^{s}(\mu-[r-1]_q h$ with $(\mu)_{q,h}^{0}:=1$. The $(q,h)$-tangent polynomials are considered as follows (cf. [8])

$$\sum_{s=0}^{\infty}T_{s,q}(\mu:h)\frac{l^s}{[s]_q!}=\frac{2}{e_{q,h}(2:l)}e_{q,h}(\mu:l), \tag{1.8}$$

where $|q|<1$ and $h$ being a non-negative integer. We readily attain from (1.8) that

$$T_{s,q}(\mu:h)=\sum_{r=0}^{s}\binom{s}{r}_q(\mu)_{q,h}^{s-r}T_{r,q}(h).$$

The corresponding numbers of the polynomials in (1.8) are obtained by choosing $\mu=0$, namely $T_{s,q}(0:h):=T_{s,q}(h)$. Many formulas and properties of the degenerate $(q,h)$-tangent polynomials have been derived in [8].

## 2. Differential equations of $(q,h)$-Frobenius-Euler polynomials

Here, by motivating the definition of the polynomials in (1.8), we consider $(q,h)$-analog Frobenius-Euler polynomials. Then, we investigate many properties and relations. We state our main definition.

**Definition 2.1.** The $(q,h)$-Frobenius-Euler polynomials are introduced as follows:

$$\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}=\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l), \tag{2.1}$$

with $|q|<1$, $\omega\in\mathbb{C}$ with $\omega\neq1$, and $h\in\mathbb{N}_0$. The $(q,h)$-Frobenius-Euler polynomials are abbreviated with qhFEP throughout the paper.

We now analyze some special cases of (2.1). We readily observe from (2.1) that when $\omega=-1$, qhFEPs become to the $(q,h)$-Euler polynomials $E_{s,q,h}(\mu)$ (cf. [9]) provided by

$$\sum_{s=0}^{\infty}E_{s,q,h}(\mu)\frac{l^s}{[s]_q!}=\frac{2}{e_{q,h}(1:l)+1}e_{q,h}(\mu:l),$$

when $\mu=0$, qhFEPs become to the $(q,h)$-Frobenius-Euler numbers (which are the corresponding numbers of the $(q,h)$-Frobenius-Euler polynomials) $\mathbb{H}_{s,q,h}(\omega)$ provided by

$$\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\omega)\frac{l^s}{[s]_q!}=\frac{1-\omega}{e_{q,h}(1:l)-\omega},$$

when $q\to1$, qhFEPs become the degenerate Frobenius-Euler polynomials $\mathbb{H}_{s,h}(\mu;\omega)$ (cf. [9, 14]) provided by

$$\sum_{s=0}^{\infty}\mathbb{H}_{s,h}(\mu;\omega)\frac{l^s}{s!}=\frac{1-\omega}{(1+hl)^{\frac{1}{h}}-\omega}(1+hl)^{\frac{\mu}{h}},$$

when $q\to1$ and $\mu=0$, qhFEP become to the degenerate Frobenius-Euler numbers $\mathbb{H}_{s,h}(\omega)$ (cf. [9, 14]), provided by

$$\sum_{s=0}^{\infty}\mathbb{H}_{s,h}(\omega)\frac{l^s}{s!}=\frac{1-\omega}{(1+hl)^{\frac{1}{h}}-\omega},$$

when $h \to 0$, qhFEPs become to the q-Frobenius-Euler polynomials $\mathbb{H}_{s,q}(\mu;\omega)$ (cf. [15]), provided by

$$\sum_{s=0}^{\infty} \mathbb{H}_{s,q}(\mu;\omega)\frac{l^s}{[s]_q!} = \frac{1-\omega}{e_q(1:l)-\omega} e_q(\mu l),$$

when $h \to 0$ and $\mu = 0$, qhFEPs become to the q-Frobenius-Euler numbers $\mathbb{H}_{s,q}(\omega)$ (cf. [15]), provided by

$$\sum_{s=0}^{\infty} \mathbb{H}_{s,q}(\omega)\frac{l^s}{[s]_q!} = \frac{1-\omega}{e_q(1:l)-\omega},$$

when $h \to 0$ and $q \to 1$, qhFEPs become to the Frobenius-Euler polynomials $\mathbb{H}_s(\mu;\omega)$ (cf. [9, 14, 15]), provided by

$$\sum_{s=0}^{\infty} \mathbb{H}_s(\mu;\omega)\frac{l^s}{s!} = \frac{1-\omega}{e^l-\omega} e^{\mu l},$$

and also when $h \to 0$, $q \to 1$, and $\mu = 0$, qhFEPs become to the Frobenius-Euler numbers $\mathbb{H}_s(\omega)$ (cf. [9, 14, 15]), provided by

$$\sum_{s=0}^{\infty} \mathbb{H}_s(\omega)\frac{l^s}{s!} = \frac{1-\omega}{e^l-\omega}.$$

We obtain by (2.1) that

$$\sum_{s=0}^{\infty} \mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!} = \sum_{s=0}^{\infty} \mathbb{H}_{s,q,h}(\omega)\frac{l^s}{[s]_q!} \sum_{s=0}^{\infty} (\mu)_{q,h}^{s}\frac{l^s}{[s]_q!} = \sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q \mathbb{H}_{r,q,h}(\omega)(\mu)_{q,h}^{s-r}\right)\frac{l^s}{[s]_q!}.$$

Therefore, it is derived that

$$\mathbb{H}_{s,q,h}(\mu;\omega) = \sum_{r=0}^{s}\binom{s}{r}_q \mathbb{H}_{r,q,h}(\omega)(\mu)_{q,h}^{s-r}. \tag{2.2}$$

Now, some differential properties of qhFEP are examined as follows.

**Theorem 2.2.** *The* $(q, h)$*-derivative property*

$$D_{q,\mu}^{(r)}\mathbb{H}_{s,q}(\mu;\omega) = \frac{[s]_q!}{[s-r]_q!}\mathbb{H}_{s-r,q}(\mu;\omega), \tag{2.3}$$

*holds for* $h, s, r \in \mathbb{N}_0$*, and* $|\, q \,| < 1$.

*Proof.* We observe from (1.5) and (2.1) that

$$D_{q,h,\mu}^{(1)}\mathbb{H}_{s,q,h}(\mu;\omega) = [s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega).$$

Then if we successively apply, we readily attain

$$D_{q,\mu}^{(r)}\mathbb{H}_{s,q}(\mu;\omega) = \frac{[s]_q!}{[s-r]_q!}\mathbb{H}_{s-r,q}(\mu;\omega),$$

which is the claimed property in (2.3). □

**Theorem 2.3.** *The solutions of the following* $(q, h)$*-differential equation*

$$\frac{(1)_{q,h}^{s}}{[s]_q!}D_{q,h,\mu}^{(s)}\mathbb{H}_{s,q,h}(\mu;\omega) + \frac{(1)_{q,h}^{s-1}}{[s-1]_q!}D_{q,h,\mu}^{(s-1)}\mathbb{H}_{s,q,h}(\mu;\omega) + \frac{(1)_{q,h}^{s-2}}{[s-2]_q!}D_{q,h,\mu}^{(s-2)}\mathbb{H}_{s,q,h}(\mu;\omega)$$
$$+\cdots+\frac{(1)_{q,h}^{2}}{[2]_q!}D_{q,h,\mu}^{(2)}\mathbb{H}_{s,q,h}(\mu;\omega) + (1)_{q,h}^{1}D_{q,h,\mu}^{(1)}(\mu;\omega) - \omega\mathbb{H}_{s,q,h}(\mu;\omega) - (1-\omega)(\mu)_{q,h}^{s} = 0$$

*are* qhFEP.

*Proof.* We observe by (2.1) and (2.3) that

$$\begin{aligned}(1-\omega)e_{q,h}(\mu:l) &= \sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}(e_{q,h}(1:l)-\omega)\\ &= \sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}\left(\sum_{r=0}^{\infty}(1)_{q,h}^{r}\frac{l^r}{[r]_q!}-\omega\right)\\ &= \sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q(1)_{q,h}^{r}\mathbb{H}_{s-r,q,h}(\mu;\omega)-\omega\mathbb{H}_{s,q,h}(\mu;\omega)\right)\frac{l^s}{[s]_q!}.\end{aligned} \tag{2.4}$$

Also, it can be written that

$$(1-\omega)e_{q,h}(\mu:l) = (1-\omega)\sum_{s=0}^{\infty}(\mu)_{q,h}^{s}\frac{l^s}{[s]_q!}. \tag{2.5}$$

Hence, it can be seen by (2.4) and (2.5) that

$$\sum_{r=0}^{s}\binom{s}{r}_q(1)_{q,h}^{r}\mathbb{H}_{s-r,q,h}(\mu;\omega)-\omega\mathbb{H}_{s,q,h}(\mu;\omega) = (1-\omega)(\mu)_{q,h}^{s}.$$

We obtain by (2.3) that

$$\sum_{r=0}^{s}\frac{(1)_{q,h}^{r}}{[r]_q!}D_{q,h,\mu}^{(r)}\mathbb{H}_{s,q,h}(\mu;\omega)-\omega\mathbb{H}_{s,q,h}(\mu;\omega)-(1-\omega)(\mu)_{q,h}^{s}=0,$$

which is the asserted equation in the theorem. □

**Theorem 2.4.** *The solutions of the following* $(q,h)$*-differential equation*

$$\begin{aligned}&\frac{\mathbb{H}_{s,q,h}(1;\omega)-\omega\mathbb{H}_{s,q,h}(\omega)}{[s]_q!}D_{q,h,\mu}^{(s)}\mathbb{H}_{s,q,h}(\mu;\omega)+\frac{\mathbb{H}_{s-1,q,h}(1;\omega)-\omega\mathbb{H}_{s-1,q,h}(\omega)}{[s-1]_q!}D_{q,h,\mu}^{(s-1)}\mathbb{H}_{s,q,h}(\mu;\omega)\\ &+\cdots+\frac{\mathbb{H}_{2,q,h}(1;\omega)-\omega\mathbb{H}_{2,q,h}(\omega)}{[2]_q!}D_{q,h,\mu}^{(2)}\mathbb{H}_{s,q,h}(\mu;\omega)+(\mathbb{H}_{1,q,h}(1;\omega)\\ &-\omega\mathbb{H}_{1,q,h}(\omega))D_{q,h,\mu}^{(1)}\mathbb{H}_{s,q,h}(\mu;\omega)+(\mathbb{H}_{0,q,h}(1;\omega)-\omega\mathbb{H}_{0,q,h}(\omega)-(1-\omega))\mathbb{H}_{s,q,h}(\mu;\omega)=0\end{aligned}$$

*are* qhFEP.

*Proof.* We see by (2.1) and (2.3) that

$$\begin{aligned}\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!} &= \frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\ &= \frac{1}{1-\omega}\left(\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(1:l)-\omega\frac{1-\omega}{e_{q,h}(1:l)-\omega}\right)\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\end{aligned}$$

and then

$$(1-\omega)\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}=\sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q\left(\mathbb{H}_{r,q,h}(1;\omega)-\omega\mathbb{H}_{r,q,h}(\omega)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega)\right)\frac{l^s}{[s]_q!},$$

which yields the following formula

$$\sum_{r=0}^{s}\binom{s}{r}_q\left(\mathbb{H}_{r,q,h}(1;\omega)-\omega\mathbb{H}_{r,q,h}(\omega)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega)-(1-\omega)\mathbb{H}_{s,q,h}(\mu;\omega)=0. \tag{2.6}$$

It is attained from (2.3) and (2.6) that

$$\sum_{r=0}^{s} \frac{\left(\mathbb{H}_{r,q,h}(1;\omega) - \omega\mathbb{H}_{r,q,h}(\omega)\right)}{[r]_q!} D_{q,h,\mu}^{(r)} \mathbb{H}_{s,q,h}(\mu;\omega) - (1-\omega)\mathbb{T}_{s,q,h}(\mu;\omega) = 0,$$

which is the assertion in the theorem. □

By (1.7), we develop the following identity (cf. [8]):

$$e_{q,h}(q\mu;l) = e_{q,q^{-1}h}(\mu : ql). \tag{2.7}$$

**Theorem 2.5.** *The solutions of the* $(q,h)$*-differential equation*

$$\begin{aligned}
&\frac{q^s\left(\mathbb{H}_{s,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{s,q}(q^{-1}h;\omega)\right)}{[s]_q!} D_{q,h,\mu}^{(s)} \mathbb{H}_{s,q,h}(q\mu;\omega) \\
&\quad + \frac{q^{s-1}\left(\mathbb{H}_{s-1,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{s-1,q}(q^{-1}h;\omega)\right)}{[s-1]_q!} D_{q,h,\mu}^{(s-1)} \mathbb{H}_{s,q,h}(q\mu;\omega) \\
&\quad + \cdots + \frac{q^2\left(\mathbb{H}_{2,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{2,q}(q^{-1}h;\omega)\right)}{[2]_q!} D_{q,h,\mu}^{(2)} \mathbb{H}_{s,q,h}(q\mu;\omega) \\
&\quad + q\left(\mathbb{H}_{1,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{1,q}(q^{-1}h;\omega)\right) D_{q,h,\mu}^{(1)} \mathbb{H}_{s,q,h}(q\mu;\omega) \\
&\quad + \left(\mathbb{H}_{0,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{0,q}(q^{-1}h;\omega) - (1-\omega)\right) \mathbb{H}_{s,q,h}(q\mu;\omega) = 0
\end{aligned}$$

*are* qhFEP.

*Proof.* We attain from (2.1) and (2.7) that

$$\begin{aligned}
&\sum_{s=0}^{\infty} \mathbb{H}_{s,q,h}(q\mu;\omega) \frac{l^s}{[s]_q!} \\
&\quad = \frac{1-\omega}{e_{q,h}(1:l) - \omega} e_{q,h}(q\mu:l) \\
&\quad = \frac{1}{1-\omega}\left(\frac{1-\omega}{e_{q,q^{-1}h}(1:ql) - \omega} e_{q,q^{-1}h}(1:ql) - \omega\frac{1-\omega}{e_{q,q^{-1}h}(1:l) - \omega}\right) \frac{1-\omega}{e_{q,h}(1:l) - \omega} e_{q,h}(q\mu:l)
\end{aligned}$$

and then

$$\begin{aligned}
&(1-\omega)\sum_{s=0}^{\infty} \mathbb{H}_{s,q,h}(q\mu;\omega) \frac{l^s}{[s]_q!} \\
&\quad = \sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q q^r\left(\mathbb{H}_{r,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{s,q}(q^{-1}h;\omega)\right)\mathbb{H}_{s-r,q,h}(q\mu;\omega)\right)\frac{l^s}{[s]_q!},
\end{aligned}$$

which means

$$\sum_{r=0}^{s}\binom{s}{r}_q q^r\left(\mathbb{H}_{r,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{s,q}(q^{-1}h;\omega)\right)\mathbb{H}_{s-r,q,h}(q\mu;\omega) - (1-\omega)\mathbb{H}_{s,q,h}(q\mu;\omega) = 0.$$

Changing $\mu$ by $q\mu$ in (2.3) gives the following equation

$$\mathbb{H}_{s-r,q,h}(q\mu;\omega) = \frac{[s-r]_q!}{[s]_q!} D_{q,h,\mu}^{(r)} \mathbb{H}_{s,q,h}(q\mu;\omega).$$

Thus, we investigate

$$\sum_{r=0}^{s} \frac{q^r\left(\mathbb{H}_{r,q}(1:q^{-1}h;\omega) - \omega\mathbb{H}_{s,q}(q^{-1}h;\omega)\right)}{[r]_q!} D_{q,h,\mu}^{(r)} \mathbb{H}_{s,q,h}(q\mu;\omega) - (1-\omega)\mathbb{H}_{s,q,h}(q\mu;\omega) = 0,$$

which is the assertion in the theorem. □

## 3. Some differential equations with the coefficients of other polynomials

Here are some differential equations for qhFEP in conjunction with the coefficients of q-Genocchi, q-Bernoulli, and q-Euler polynomials.

**Theorem 3.1.** *The solutions of the following* $(q,h)$*-differential equation*

$$\begin{aligned}&\frac{(E_{s,q}+E_{s,q}(1))}{[s]_q!}D^{(s)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)+\frac{(E_{s-1,q}+E_{s-1,q}(1))}{[s-1]_q!}D^{(s-1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\&+\cdots+\frac{(E_{2,q}+E_{2,q}(1))}{[2]_q!}D^{(2)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)+(E_{1,q}+E_{1,q}(1))\,D^{(1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\&+(E_{0,q}+E_{0,q}(1)-2)\,\mathbb{H}_{s,q,h}(\mu;\omega)=0\end{aligned}$$

*are* qhFEP *in conjunction with the coefficients of* q*-Euler polynomials.*

*Proof.* We achieve from (1.3) and (2.1) that

$$\begin{aligned}\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}&=\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\&=\frac{1}{2}\left(\frac{2}{e_q(l)+1}e_q(l)+\frac{2}{e_q(l)+1}\right)\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\&=\frac{1}{2}\sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q(E_{r,q}+E_{r,q}(1))\,\mathbb{H}_{s-r,q,h}(\mu;\omega)\right)\frac{l^s}{[s]_q!}i,\end{aligned}$$

which means that

$$2\mathbb{H}_{s,q,h}(\mu;\omega)=\sum_{r=0}^{s}\binom{s}{r}_q(E_{r,q}+E_{r,q}(1))\,\mathbb{H}_{s-r,q,h}(\mu;\omega).\tag{3.1}$$

By means of (2.3) and (3.1), it can be written that

$$\sum_{r=0}^{s}\frac{(E_{r,q}+E_{r,q}(1))}{[r]_q!}D^{(r)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)-2\mathbb{H}_{s,q,h}(\mu;\omega)=0,$$

which is the just desired differential equation in the theorem. □

**Theorem 3.2.** *The solutions of the following* $(q,h)$*-differential equation*

$$\begin{aligned}&\frac{(B_{s,q}-B_{s,q}(1))}{[s]_q!}D^{(s)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)+\frac{(B_{s-1,q}-B_{s-1,q}(1))}{[s-1]_q!}D^{(s-1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\&+\cdots+\frac{(B_{2,q}-B_{2,q}(1))}{[2]_q!}D^{(2)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)+(B_{1,q}-B_{1,q}(1))\,D^{(1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\&+(B_{0,q}-B_{0,q}(1))\,\mathbb{H}_{s,q,h}(\mu;\omega)-[s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega)=0\end{aligned}$$

*are* qhFEP *in conjunction with the coefficients of* q*-Bernoulli polynomials.*

*Proof.* We obtain from (1.2) and (2.1) that

$$\begin{aligned}\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}&=\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\&=\frac{1}{l}\left(\frac{l}{e_q(l)-1}e_q(l)-\frac{l}{e_q(l)+1}\right)\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\end{aligned}$$

$$= \frac{1}{l}\sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q \left(B_{r,q} - B_{r,q}(1)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega)\right)\frac{l^s}{[s]_q!},$$

which yields that

$$[s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega) = \sum_{r=0}^{s}\binom{s}{r}_q \left(B_{r,q} - E_{r,q}(1)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega). \tag{3.2}$$

By means of (2.3) and (3.2), it can be written that

$$\sum_{r=0}^{s}\frac{\left(B_{r,q} - B_{r,q}(1)\right)}{[r]_q!}D^{(r)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega) - [s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega) = 0,$$

which is the just asserted differential equation in the theorem. □

**Theorem 3.3.** *The solutions of the following* $(q,h)$*-differential equation*

$$\begin{aligned}
&\frac{\left(G_{s,q} + G_{s,q}(1)\right)}{[s]_q!}D^{(s)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega) + \frac{\left(G_{s-1,q} + G_{s-1,q}(1)\right)}{[s-1]_q!}D^{(s-1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\
&\quad + \cdots + \frac{\left(G_{2,q} + G_{2,q}(1)\right)}{[2]_q!}D^{(2)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega) + \left(G_{1,q} + G_{1,q}(1)\right)D^{(1)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega)\\
&\quad + \left(G_{0,q} + G_{0,q}(1)\right)\mathbb{H}_{s,q,h}(\mu;\omega) - 2[s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega) = 0
\end{aligned}$$

*are* qhFEP *in conjunction with the coefficients of* $q$*-Genocchi polynomials.*

*Proof.* We observe from (1.4) and (2.1) that

$$\begin{aligned}
\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!} &= \frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\
&= \frac{1}{2l}\left(\frac{2l}{e_q(l)+1}e_q(l) + \frac{2l}{e_q(l)+1}\right)\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)\\
&= \frac{1}{2l}\sum_{s=0}^{\infty}\left(\sum_{r=0}^{s}\binom{s}{r}_q \left(G_{r,q} + G_{r,q}(1)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega)\right)\frac{l^s}{[s]_q!},
\end{aligned}$$

which means that

$$2[s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega) = \sum_{r=0}^{s}\binom{s}{r}_q \left(G_{r,q} + G_{r,q}(1)\right)\mathbb{H}_{s-r,q,h}(\mu;\omega). \tag{3.3}$$

By means of (2.3) and (3.3), it can be written that

$$\sum_{r=0}^{s}\frac{\left(G_{r,q} + G_{r,q}(1)\right)}{[r]_q!}D^{(r)}_{q,h,\mu}\mathbb{H}_{s,q,h}(\mu;\omega) - 2[s]_q\mathbb{H}_{s-1,q,h}(\mu;\omega) = 0,$$

which is the claimed differential equation in the theorem. □

## 4. Zeros and graphical representations for qhFEP

In this part, certain numerical computations are completed to derive certain zeros of qhFEP and show some intriguing graphical representations. Remember the definition of qhFEP as follows:

$$\frac{1-\omega}{e_{q,h}(1:l)-\omega}e_{q,h}(\mu:l)=\sum_{s=0}^{\infty}\mathbb{H}_{s,q,h}(\mu;\omega)\frac{l^s}{[s]_q!}.$$

Certain members of qhFEP are investigated and given as:

$$\begin{aligned}
\mathbb{H}_{0,q,h}(\mu;\omega)&=1,\\
\mathbb{H}_{1,q,h}(\mu;\omega)&=\frac{1}{-1+\omega}-\frac{\mu}{-1+\omega}+\frac{\omega\mu}{-1+\omega},\\
\mathbb{H}_{2,q,h}(\mu;\omega)&=\frac{1}{(-1+\omega)^3}-\frac{h}{(-1+\omega)^3}-\frac{2\omega}{(-1+\omega)^3}+\frac{2h\omega}{(-1+\omega)^3}+\frac{\omega^2}{(-1+\omega)^3}-\frac{h\omega^2}{(-1+\omega)^3}-\frac{h\mu}{1-\omega}\\
&\quad+\frac{h\omega\mu}{1-\omega}+\frac{\mu^2}{1-\omega}-\frac{\omega\mu^2}{1-\omega}-\frac{[2]_q!}{(-1+\omega)^3}+\frac{\omega[2]_q!}{(-1+\omega)^3}-\frac{\mu[2]_q!}{(-1+\omega)^2}+\frac{\omega\mu[2]_q!}{(-1+\omega)^2},\\
\mathbb{H}_{3,q,h}(\mu;\omega)&=-\frac{1}{(1-\omega)^2}+\frac{2h}{(1-\omega)^2}-\frac{h^2}{(1-\omega)^2}+\frac{hq}{(1-\omega)^2}-\frac{h^2q}{(1-\omega)^2}+\frac{\omega}{(1-\omega)^2}-\frac{2h\omega}{(1-\omega)^2}\\
&\quad+\frac{h^2\omega}{(1-\omega)^2}-\frac{hq\omega}{(1-\omega)^2}+\frac{h^2q\omega}{(1-\omega)^2}+\frac{h^2\mu}{1-\omega}+\frac{h^2q\mu}{1-\omega}-\frac{h^2\omega\mu}{1-\omega}-\frac{h^2q\omega\mu}{1-\omega}-\frac{2h\mu^2}{1-\omega}\\
&\quad-\frac{hq\mu^2}{-\omega}+\frac{2h\omega\mu^2}{1-\omega}+\frac{hq\omega\mu^2}{1-\omega}+\frac{\mu^3}{1-\omega}-\frac{\omega\mu^3}{1-\omega}-\frac{[3]_q!}{(1-\omega)^4}+\frac{\omega[3]_q!}{(1-\omega)^4}-\frac{\mu[3]_q!}{(-1+\omega)^3}\\
&\quad+\frac{\omega\mu[3]_q!}{(-1+\omega)^3}+\frac{2[3]_q!}{(1-\omega)^3[2]_q!}-\frac{2h[3]_q!}{(1-\omega)^3[2]_q!}-\frac{2\omega[3]_q!}{(1-\omega)^3[2]_q!}+\frac{2h\omega[3]_q!}{(1-\omega)^3[2]_q!}\\
&\quad+\frac{\mu[3]_q!}{(-1+\omega)^3[2]_q!}-\frac{h\mu[3]_q!}{(-1+\omega)^3[2]_q!}+\frac{h\mu[3]_q!}{(-1+\omega)^2[2]_q!}\\
&\quad-\frac{2\omega\mu[3]_q!}{(-1+\omega)^3[2]_q!}+\frac{2h\omega\mu[3]_q!}{(-1+\omega)^3[2]_q!}-\frac{h\omega\mu[3]_q!}{(-1+\omega)^2[2]_q!}+\frac{\omega^2\mu[3]_q!}{(-1+\omega)^3[2]_q!}\\
&\quad-\frac{h\omega^2\mu[3]_q!}{(-1+\omega)^3[2]_q!}-\frac{\mu^2[3]_q!}{(-1+\omega)^2[2]_q!}+\frac{\omega\mu^2[3]_q!}{(-1+\omega)^2[2]_q!}.
\end{aligned}$$

The zeros of the equality $\mathbb{H}_{s,q,h}(\mu;\omega)=0$ for $s=30$ are plotted with 2D structure in Figure 1. Here we take $\omega=2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on top-left of Figure 1; $\omega=2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$ on top-right of Figure 1; $\omega=-2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on bottom-left of Figure 1; $\omega=-2, q=\frac{9}{10}$; and $h=\frac{1}{1000}$ on bottom-right of Figure 1.

We now present the 3D behavior of the stacks of zeros of the $\mathbb{H}_{s,q,h}(\mu;\omega)=0$ for $1\leqslant s\leqslant 30$ by Figure 2. Here we take $\omega=2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on top-left of Figure 2; $\omega=2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$ on top-right of Figure 2; $\omega=-2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on bottom-left of Figure 2; $\omega=-2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$ on bottom-right of Figure 2.

We now provide real zeros of the $\mathbb{H}_{s,q,h}(\mu;\omega)=0$ for $1\leqslant s\leqslant 30$ by Figure 3. Here we take $\omega=2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on top-left of Figure 3; $\omega=2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$ on top-right of Figure 3; $\omega=-2, q=\frac{1}{10}$, and $h=\frac{1}{100}$ on bottom-left of Figure 3; $\omega=-2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$ on bottom-right of Figure 3.

The approximate solutions of $\mathbb{H}_{s,q,h}(\mu;\omega)=0$ (choosing $\omega=-2, q=\frac{9}{10}$, and $h=\frac{1}{1000}$) are calculated and listed in Table 1.

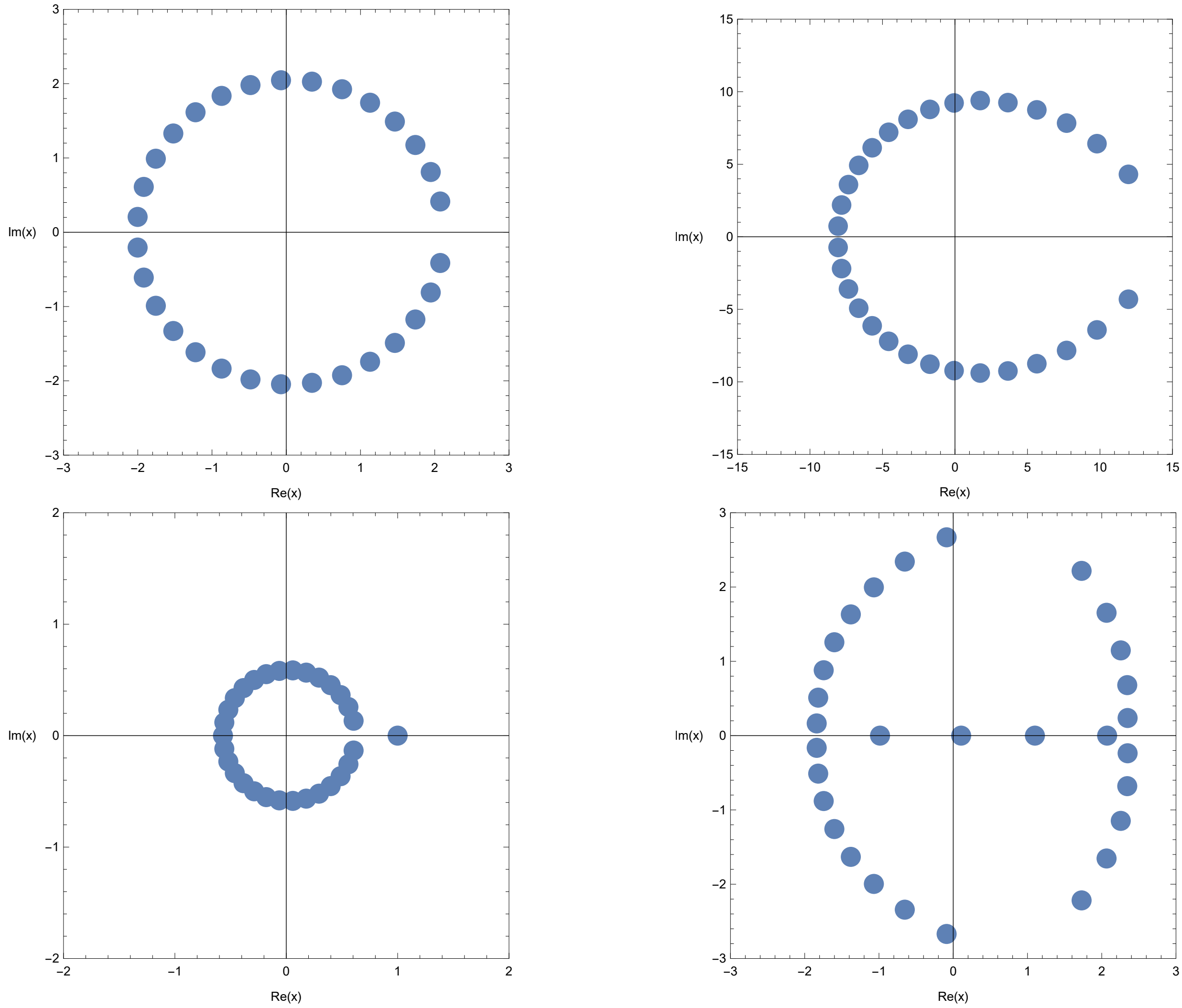


Figure 1: Zeros of $\mathbb{H}_{s,q,h}(\mu;\omega) = 0$.

Table 1: Approximate solutions of $\mathbb{H}_{s,q,h}(\mu;\omega) = 0$ for $u = -2$, $q = \frac{9}{10}$, and $h = \frac{1}{1000}$.

| Degree | $\mu$ |
|---|---|
| 1 | 0.33333 |
| 2 | -0.15451, 0.78885 |
| 3 | -0.40150, 0.17230, 1.1354 |
| 4 | -0.40666-0.17616i, -0.40666+0.17616i, 0.58499, 1.3803 |
| 5 | -0.55699-0.33972i, -0.55699+0.33972i, -0.018921, 0.98223, 1.5247 |
| 6 | -0.59217-0.51323i, -0.59217+0.51323i, -0.56122, 0.39438, 1.4631-0.0780i, 1.4631+0.0780i |
| 7 | -0.76008, -0.64110-0.68546i, -0.64110+0.68546i, -0.21928, 0.79615, 1.6111-0.2797i, 1.6111+0.2797i |
| 8 | -0.77171-0.18693i, -0.77171+0.18693i, -0.65913-0.85315i, -0.65913+0.85315i, 0.20197, 1.1901, 1.6955-0.3905i, 1.6955+0.3905i |
| 9 | -0.89195-0.30221i, -0.89195+0.30221i, -0.66964-1.01054i, -0.66964+1.01054i, -0.42380, 0.60852, 1.5396, 1.7348-0.5161i, 1.7348+0.5161i |
| 10 | -0.92811-0.46309i, -0.92811+0.46309i, -0.91432, -0.66815-1.16062i, -0.66815+1.16062i, 0.0061573, 1.0058, 1.7441, 1.7784-0.6511i, 1.7784+0.6511i |

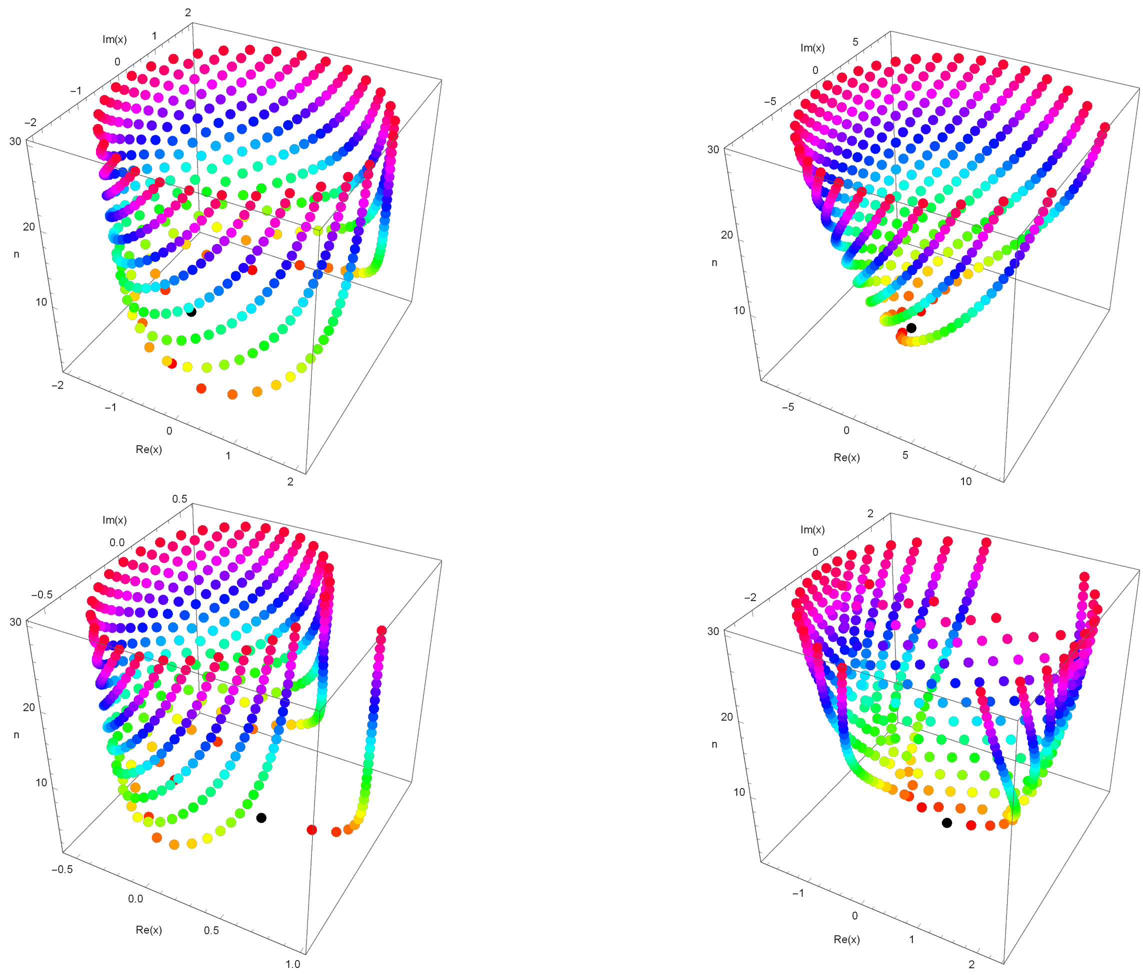

Figure 2: Zeros of $\mathbb{H}_{n,q,h}(x;u) = 0$.

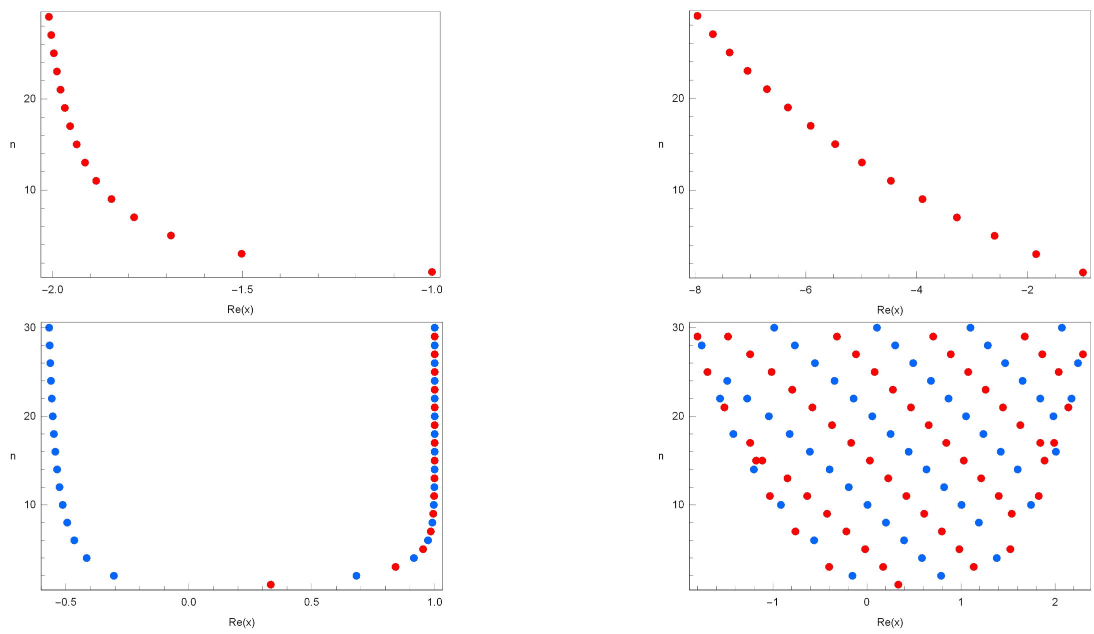

Figure 3: Real zeros of $\mathbb{H}_{s,q,h}(\mu;\omega) = 0$.

## 5. Conclusion

In recent years, by means of the generalized quantum exponential function (or, say $(q, h)$-exponential function) that unifies, extends $h$- and $q$-exponential functions in an efficient and convenient form, $(q, h)$-extensions of the several polynomials and numbers, such as Euler and tangent polynomials and numbers, have been studied and investigated. Motivated and inspired by the mentioned studies, in the presented work, we have introduced $(q, h)$-extensions of Frobenius-Euler polynomials and numbers, and we then have derived and analyzed some of their formulae and relations. Also, we have presented that these polynomials are solutions to some higher-order differential equations. Moreover, we have shown that $(q, h)$-Frobenius-Euler polynomials are solutions to higher-order differential equations combined with the $q$-Bernoulli, $q$-Euler, and $q$-Genocchi numbers and polynomials, respectively. In addition, we have utilized a computer program to show the structures and shapes of the approximate roots of the mentioned polynomials. For the subsequent plans, we will consider using the context of the monomiality principle and umbral calculus to analyze more deep results and properties for $(q, h)$-Frobenius-Euler polynomials.

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