The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .2 No.4 (2011) 607-618

On Some Geometric Properties of the Sphere S^n

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Received: September 2010, Revised: December 2010
Online Publication: January 2011

Abstract

*It is known that the sphere S^n admits an almost complex structure only when $n=2$ or $n=6$. In this paper, we show that the sphere S^n is a space of constant sectional curvature and using the results of T. Sato in [4], we determine the scalar curvature and the *-scalar curvature of S^6. We shall also prove that S^6 is a non-Kähler nearly Kähler manifold using the Levi-Civita connection on S^6 defined by H. Hashimoto and K. Sekigawa [3]. In [2], A. Gray and L. Hervella defined sixteen classes of almost Hermitian manifolds. We shall define quasi-Hermitian, a class of almost Hermitian manifolds and partially characterize almost Hermitian manifolds that belong to this class. Finally, under certain conditions, we shall show the sphere S^6 is quasi-Hermitian.*

Keywords: Sphere, Kähler manifolds**,** Hermitian manifolds, quasi-Hermitian manifolds
AMS Subject Classification (MSC2010): 53B35, 53C55

1 Preliminaries

Let $M=(M,J,g)$ be a $2n$-dimensional almost Hermitian manifold with the almost complex structure J and Riemannian metric g. Let ∇ be the Levi-Civita connection on M and R the Riemannian curvature tensor defined by

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$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X)Z - \nabla_{[X,Y]}Z,$$

for X, Y and $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M.

If U is a unit normal vector to M and $V \in \mathfrak{X}(M)$, the *shape operator* of M in $\mathbb{R}^{n+1}$, denoted $S(V)$, is defined by

$$\begin{aligned} S(V) &= -\nabla_V^{\mathbb{R}^{n+1}} U \\ &= -\sum_{i=1}^{n+1} \mathbf{V}[U^i]e_i, \end{aligned}$$

where e_i is the standard i^{th} basis vector for $\mathbb{R}^{n+1}$ and $\mathbf{V}[\cdot]$ denotes the ordinary directional derivative $\mathbf{v}[f] = \nabla f \cdot \mathbf{V}$. For X, Y, $Z \in \mathfrak{X}(M)$,

$$R(X,Y)Z = g\left(S(X),Z\right)S(Y) - g\left(S(Y),Z\right)S(X).$$

The *Ricci tensor* ρ is a symmetric tensor of type $(0,2)$ defined by

$$\begin{aligned} \rho(X,Y) &= trace[Z \mapsto R(X,Z)Y] \\ &= \sum_{i=1}^{2n} R(e_i, X, e_i, Y), \end{aligned}$$

where $\{e_1,\dots,e_{2n}\}$ is an arbitrary orthonormal basis for $T_p(M)$, the tangent space to M at the point p. The *Ricci tensor transformation* Q of type (1,1) is given by

$$\rho(X,Y) = g(QX,Y),$$

and the trace of Q is called the *scalar curvature* τ of R. Furthermore, we denote by $\rho*$ and $\tau*$ the Ricci *-tensor and the *-scalar curvature on M, respectively. The tensor $\rho*$ is defined pointwise by

$$\begin{aligned} \rho*(X,Y) &= \mathrm{trace}(Z \mapsto R(JZ,X)JY) \\ &= -\sum_{i=1}^{2n} R(X,e_i,JY,Je_i) \\ &= -\frac{1}{2}\sum_{i=1}^{2n} R(X,JY,e_i,Je_i), \end{aligned}$$

where X , Y and Z $\in$ $T_p(M)$, $R(X,Y,Z,W)=g(R(X,Y)Z,W)$ and $\{e_i\}$ is an orthonormal basis of $T_p(M)$. We also define analogously the *Ricci *-operator*, denoted by $Q*$, by $\rho*(X,Y)=g(Q*X,Y)$ for X and $Y \in T_p(M)$. The trace of $Q*$ is called the **-scalar curvature* $\tau*$ on M . We note that $\rho*$ satisfies $\rho*(JX,JY)=\rho*(Y,X)$ but is not symmetric in general.

An almost complex structure J is said to be *integrable* if M admits a complex structure and the derived almost complex structure coincides with J. We also say that the almost complex manifold (M,J) is integrable if J is integrable.

The *Nijenhuis tensor* N of an almost complex structure J is a tensor field of type (1,2) defined by

$$N(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y],$$

for X , Y $\in \mathfrak{X}(M)$. In [5], B. Kruglikov showed that the Nijenhuis tensor N can be expressed in terms of any symmetric connection ∇ on M , i.e.,

$$N(X,Y)=(\nabla_X J)(JY)+(\nabla_{JX} J)Y-(\nabla_Y J)(JX)-(\nabla_{JY} J)X .$$

It is easy to verify that the Nijenhuis tensor satisfies the following properties:

$$N(X,Y)=-N(Y,X)$$
$$N(JX,Y)=-JN(X,Y)$$
$$N(X,JY)=-JN(X,Y).$$

A. Newlander and L. Nirenberg, in [1], established the following result on the integrability of an almost complex structure J.

Theorem 1. *An almost complex structure J is integrable if and only if the Nijenhuis tensor N vanishes on M .*

As a consequence of the Newlander and Nirenberg above, we have the following corollary.

Corollary 1. *Any 2-dimensional almost complex manifold (M,J) is integrable.*

Let X and Y $\in T_p(M)$ such that X and Y are linearly independent. The *sectional curvature* K_π of the 2-dimensional subspace π of $T_p(M)$ spanned by $\{X,Y\}$ is given by

$$K_\pi = K(X,Y) = \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y)-(g(X,Y))^2},$$

while the *holomorphic sectional curvature*, $H(X)$, is the sectional curvature of the subspace of $T_p(M)$ spanned by $\{X, JX\}$, i.e.,

$$H(X) = \frac{g(R(X,JX)X,JX)}{(g(X,X))^2-(g(X,JX))^2} = K(X,JX).$$

An almost complex manifold (M,J) equipped with a Riemannian metric g that satisfies

$$g\left(JX,JY\right) = g\left(X,Y\right),$$

for all X, $Y \in \mathfrak{X}(M)$, is called an *almost Hermitian manifold*, denoted by (M,J,g). In [2], sixteen classes of almost Hermitian manifolds were defined by Gray and Hervella. The list includes Kähler, Hermitian and nearly Kähler manifolds. An almost Hermitian manifold M is called a Kähler manifold if $\nabla J = 0$, for all X, $Y \in \mathfrak{X}(M)$. It is called *Hermitian* if $N = 0$. It is a *nearly Kähler manifold* if $(\nabla_X J)Y + (\nabla_Y J)X = 0$, or equivalently $(\nabla_X J)X = 0$, for all $X \in \mathfrak{X}(M)$.

2 Geometry of S^n

It is known that the standard sphere S^n admits an almost complex structure only when $n=2$ or $n=6$. To construct an almost complex structure on S^2, we use the span of the quaternions. Let $H = span_{\mathbb{R}}\{1, i, j, k\}$, where

$$i^2 = j^2 = k^2 = -1$$
$$ij = k,\ jk = i,\ ki = j$$
$$ji = -k,\ kj = -i,\ ik = -j.$$

Then $\mathbb{R}^3$ can be identified with the set $\mathrm{Im}H = span_{\mathbb{R}}\{i, j, k\}$. For $X = x^1\, i + x^2\, j + x^3\, k$ and $Y = y^1\, i + y^2\, j + y^3\, k \in \mathrm{Im}H$, the metric g in $\mathrm{Im}H$ and the exterior product are defined as

$$g(X,Y) = x^1y^1 + x^2y^2 + x^3y^3$$
$$X \times Y = (x^2y^3 - x^3y^2)\, i + (x^3y^1 - x^1y^3)\, j + (x^1y^2 - x^2y^1)\, k,$$

and the sphere S^2 is given by $S^2 = \{ p \in \mathrm{Im}H \mid g(p,p) = 1 \}$. With this definition, the tangent space to S^2 at a point $p \in S^2$ is $T_p S^2 = \{ X \in \mathrm{Im}H \mid g(X,p) = 0 \}$.

Let $p = p^1 i + p^2 j + p^3 k \in S^2$ and $X = x^1 i + x^2 j + x^3 k \in T_p S^2$. Define a tensor $J_p : T_p S^2 \to T_p(S^2)$ by

$$J_p X = X \times p = \left(x^2 p^3 - x^3 p^2 \right) i + \left(x^3 p^1 - x^1 p^3 \right) j + \left(x^1 p^2 - x^2 p^1 \right) k .$$

This J_p induces a tensor J on S^2 such that $J^2 = -I$, hence the following theorem.

Theorem 2. *The sphere S^2 is an almost complex manifold.*

Theorem 3. *The sphere S^2 is an almost Hermitian manifold.*

Proof. Let $X = x^1 i + x^2 j + x^3 k$ and $Y = y^1 i + y^2 j + y^3 k \in T_p(S^2)$. Then, for any $p \in S^2$, we have

$$\begin{aligned} g\left(J_p X, J_p Y \right) &= (x^2 p^3 - x^3 p^2)(y^2 p^3 - y^3 p^2) + (x^3 p^1 - x^1 p^3)(y^3 p^1 - y^1 p^3) \\ &\quad + (x^1 p^2 - x^2 p^1)(y^1 p^2 - y^2 p^1) \\ &= x^2 y^2 (p^3)^2 - x^3 p^3 y^2 p^2 - x^2 p^2 y^3 p^3 + x^3 y^3 (p^2)^2 \\ &\quad + x^3 y^3 (p^1)^2 - x^1 p^1 y^3 p^3 - x^3 p^3 y^1 p^1 + x^1 y^1 (p^3)^2 \\ &\quad + x^1 y^1 (p^2)^2 - x^2 p^2 y^1 p^1 - x^1 p^1 y^2 p^2 + x^2 y^2 (p^1)^2 . \end{aligned}$$

Regrouping terms, we get

$$\begin{aligned} g\left(J_p X, J_p Y \right) &= x^1 y^1 (p^1)^2 + x^1 y^1 (p^3)^2 + x^2 y^2 (p^1)^2 + x^2 y^2 (p^3)^2 \\ &\quad + x^3 y^3 (p^1)^2 + x^3 y^3 (p^2)^2 - x^1 p^1 y^2 p^2 - x^1 p^1 y^3 p^3 \\ &\quad - x^2 p^2 y^1 p^1 - x^2 p^2 y^3 p^3 - x^3 p^3 y^1 p^1 - x^3 p^3 y^2 p^2 . \end{aligned}$$

Adding and subtracting $x^1 y^1 (p^1)^2 + x^2 y^2 (p^2)^2 + x^3 y^3 (p^3)^2$ will yield

$$\begin{aligned} g\left(J_p X, J_p Y \right) &= x^1 y^1 (p^1)^2 + x^1 y^1 (p^2)^2 + x^1 y^1 (p^3)^2 + x^2 y^2 (p^1)^2 + x^2 y^2 (p^2)^2 + x^2 y^2 (p^3)^2 \\ &\quad + x^3 y^3 (p^1)^2 + x^3 y^3 (p^2)^2 + x^3 y^3 (p^3)^2 - x^1 p^1 y^1 p^1 - x^1 p^1 y^2 p^2 - x^1 p^1 y^3 p^3 \\ &\quad - x^2 p^2 y^1 p^1 - x^2 p^2 y^2 p^2 - x^2 p^2 y^3 p^3 - x^3 p^3 y^1 p^1 - x^3 p^3 y^2 p^2 - x^3 p^3 y^3 p^3 \\ &= x^1 y^1 g(p,p) + x^2 y^2 g(p,p) + x^3 y^3 g(p,p) - x^1 p^1 g(Y,p) \\ &\quad - x^2 p^2 g(Y,p) - x^3 p^3 g(Y,p) \\ &= x^1 y^1 + x^2 y^2 + x^3 y^3 \end{aligned}$$

$$= g(X,Y).$$

□

From Corollary 1 and Theorem 2, the Nijenhuis tensor on S^2 vanishes. Combining this result with Theorem 3 and the definition of a Hermitian manifold, we have the following.

Theorem 4. *The sphere S^2 is a Hermitian manifold.*

Before we can define the almost complex structure on the unit sphere S^6, we recall first the Cayley algebra.Let $\mathcal{C} = span_{\mathbb{R}}\{1, i, j, k, l, il, jl, kl\}$ such that

$$i^2 = j^2 = k^2 = l^2 = il^2 = jl^2 = kl^2 = -1,$$

and if $e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = l, e_5 = il, e_6 = jl$ and $e_7 = kl$, then Table 1 shows the multiplication of the basis elements of the Cayley algebra.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Table 1. Multiplication table of the basis elements of $\mathcal{C}$

Let $X, Y \in \mathcal{C}$. We define the metric g (inner product) and the exterior product, respectively, as

$$g(X,Y) = -(\text{real part of } XY)$$
$$X \times Y = \text{imaginary part of } XY$$

where XY is the product of X and Y in $\mathcal{C}$.

Remark. *For any X, Y, $Z \in \mathcal{C}$, the inner and exterior products satisfy the following:*

(i) $X \times Y = -(Y \times X)$

(ii) $g(X\times Y,Z)=g(X,Y\times Z)$
(iii) $X\times(Y\times Z)=g(X,Z)Y-g(X,Y)Z$

Notice that each element z of $\mathcal{C}$ can be expressed as $z=\sum_{i=0}^{7}a^i e_i$. Here, we call the number a^0 the real part of z and $\sum_{i=1}^{7}a^i e_i$ as its imaginary part. Denote the set of imaginary parts of elements of $\mathcal{C}$ by $\mathrm{Im}\mathcal{C}$. Let $p(x^1,\ldots,x^7)\in\Re^7$. If we denote by V_p the vector in $\mathbb{R}^7$ determined by the point p with the origin, then $V_p\in\mathrm{Im}\mathcal{C}$, i.e.,

$$V_p=x^1e_1+x^2e_2+x^3e_3+x^4e_4+x^5e_5+x^6e_6+x^7e_7.$$

This means that $\Re^7$ can be identified with $\mathrm{Im}\mathcal{C}$. Now observe that

$$\begin{aligned}g(V_p,V_p)=1 &\Leftrightarrow -\left\{\text{real part of }\left(V_pV_p\right)\right\}=1\\ &\Leftrightarrow -\left(-(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2-(x^5)^2-(x^6)^2-(x^7)^2\right)=1\\ &\Leftrightarrow (x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2+(x^5)^2+(x^6)^2+(x^7)^2=1\\ &\Leftrightarrow p\in S^6.\end{aligned}$$

Thus, we can define S^6 as

$$S^6=\left\{V_p\in\mathrm{Im}\mathcal{C}\mid g\left(V_p,V_p\right)=1\right\}.$$

Since the tangent space at a point $p\in S^6$ is the set of all vectors orthogonal to V_p, then

$$T_pS^6=\left\{X\in\mathrm{Im}\mathcal{C}\mid g(X,V_p)=0\right\}.$$

Let us define a tensor J_p from T_pS^6 to T_pS^6 by

$$J_pX=X\times V_p,$$

for $X\in T_pS^6$ and $p\in S^6$. Then

$$\begin{aligned}J_p^2X&=J_p(J_pX)\\ &=J_p(X\times V_p)\\ &=(X\times V_p)\times V_p\end{aligned}$$

$$
\begin{aligned}
&= -V_p \times (X \times V_p) \\
&= -[g(V_p, V_p)X - g(V_p, X)V_p] \\
&= -X .
\end{aligned}
$$

So, J_p induces a tensor J such that $J^2 = -I$, where I is the identity map. Also,

$$g(J_pX, J_pY) = g(X \times V_p, Y \times V_p) = g(X, V_p \times (Y \times V_p)).$$

Observe that

$$V_p \times (Y \times V_p) = g(V_p, V_p)Y - g(V_p, Y)V_p = Y .$$

Therefore,

$$g(J_pX, J_pY) = g(X, Y).$$

This proves the following theorem.

Theorem 5. *The sphere S^6 is an almost Hermitian manifold.*

H. Hashimoto and K. Sekigawa [3] derived the Levi-Civita connection on S^6 and obtained

$$(\nabla_X J)Y = -X \times Y + g(X \times Y, V_p)V_p ,$$

for any X , $Y \in T_pS^6$, $p \in S^6$. One can check that this linear connection is not always zero. But, for any $X \in T_pS^6$, we have

$$(\nabla_X J)X = -X \times X + g(X \times X, V_p)V_p = 0 .$$

Hence, we have the following theorem.

Theorem 6. *The sphere S^6 is a non-Kähler nearly Kähler manifold*.

We are now interested with the different curvature tensors on S^n. Let us determine first what the shape operator does with every tangent vector to S^n. In coordinates, the unit normal vector to S^n at a point $\left(x^1, \ldots, x^{n+1}\right)$ is given by $U = \left(x^1, \ldots, x^{n+1}\right)$. Let $V = \left(V^1, \ldots, V^{n+1}\right)$ be any tangent vector to S^n. The covariant derivative is the coordinate-wise directional derivative. So,

$$S(V) = -\nabla_V U = -\left(V[x^1], \ldots, V[x^{n+1}]\right).$$

But for $i=1,\dots,n+1$,

$$V[x^i]=\sum_{j=1}^{n+1}\frac{\partial x^i}{\partial x^j}V^j$$
$$=V^j\delta_i^j$$
$$=V^i.$$

Therefore,

$$S(V)=-\left(V^1,\dots,V^{n+1}\right)=-V.$$

Now, let X and Y be orthonormal tangent basis vectors at $p\in S^n$, i.e.,

$$g(X,X)=g(Y,Y)=1$$
$$g(X,Y)=g(Y,X)=0.$$

Then

$$R(X,Y)X=g(S(X),X)S(Y)-g(S(Y),X)S(X)$$
$$=g\left(-X,X\right)\left(-Y\right)-g\left(-Y,X\right)\left(-X\right)$$
$$=g(X,X)Y-g(Y,X)X$$
$$=Y.$$

Solving for the sectional curvature, we get

$$K(X,Y)=\frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y)-(g(X,Y))^2}$$

$$=\frac{g(Y,Y)}{g(X,X)g(Y,Y)-(g(X,Y))^2}$$

$$=1.$$

Theorem 7. *The sphere S^n is a space of constant sectional curvature with $K(X,Y)=1$ for X, $Y\in T_pS^n$ and for all $p\in S^n$.*

As stated earlier, S^2 and S^6 both admit an almost complex structure J. Hence, we have the following results.

Corollary 2. *The unit spheres S^2 and S^6 are spaces of constant holomorphic sectional curvature with $H(X)=1$, for any nonzero tangent vector X.*

Proof. For any nonzero tangent vector X,

$$H(X) = K(X, JX) = 1.$$

□

In [4], T. Sato showed that if M is a non-Kähler nearly Kähler manifold of dimension n with pointwise constant holomorphic sectional curvature $H(X) = c(p)$, then

$$\tau = \frac{5n(n+2)c(p)}{8}$$

and

$$\tau + 3\tau^* = n(n+2)c(p).$$

We now have the following result.

Theorem 8. *The sphere S^6 being a non-Kähler nearly Kähler manifold with sectional curvature 1, its scalar curvature τ and *-scalar curvature τ^* are*

$$\tau = 30 \quad and \quad \tau^* = 6.$$

Proof.

$$\tau = \frac{5(6)(6+2)}{8} = 30,$$

and

$$30 + 3\tau^* = 6(8) \Rightarrow \tau^* = \frac{6(8) - 30}{3} = 6.$$

□

It is interesting to note that in the 6-dimensional unit sphere S^6, the Ricci *-tensor ρ^* is a conformal of the Ricci tensor ρ, i.e., $\rho^* = \frac{1}{5}\rho$. T. Sato also proved the following theorems for the 6-dimensional case.

Theorem 9. *There does not exist any dimensional, except 6-dimensional, non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature.*

Theorem 10. *If M be a non-Kähler nearly Kähler manifold of pointwise constant holomorphic sectional curvature then M is locally isometric to a 6-dimensional sphere S^6.*

In [2], A. Gray and L. Hervella defined sixteen classes of almost Hermitian manifolds based on linear invariants. We now define a class of almost Hermitian manifolds. Our definition of this class is based on both linear invariants and the exterior product.

Definition 1. *An almost Hermitian manifold (M,J,g) is called quasi-Hermitian if it satisfies*

$$X \times J_p Y + J_p X \times Y = 0,$$

for all X , Y $\in T_p(M)$, $p \in M$.

With this definition, we have the following results.

Theorem 11. *Any 2-dimensional almost Hermitian manifold is quasi-Hermitian.*

Proof. Suppose M is a 2-dimensional almost-Hermitian manifold. Let $X \in T_p M$, such that X is nonzero. Then $\{X, J_p X\}$ is a local frame. Thus,

$$\begin{aligned} X \times J_p Y + J_p X \times Y &= X \times J_p(J_p X) + J_p X \times J_p X \\ &= X \times J_p^2 X \\ &= X \times (-X) \\ &= 0. \end{aligned}$$

□

Theorem 12. *The 6-dimensional sphere S^6 is Hermitian if and only if it is quasi-Hermitian.*

Proof. Let X , Y $\in T_p S^6$, $p \in S^6$. Since the Levi-Civita connection is torsion-free, we have

$$\begin{aligned} N(X,Y) &= (\nabla_X J)(JY) + (\nabla_{JX} J)Y - (\nabla_Y J)(JX) - (\nabla_{JY} J)X \\ &= [-X \times J_p Y + g(X \times J_p Y, V_p)V_p] + [-J_p X \times Y + g(J_p X \times Y, V_p)V_p] \\ &\quad -[-Y \times J_p X + g(Y \times J_p X, V_p)V_p] - [-J_p Y \times X + g(J_p Y \times X, V_p)V_p] \\ &= (-X \times J_p Y) - (J_p X \times Y) + (Y \times J_p X) + (J_p Y \times X) + g(X \times J_p Y, V_p)V_p \\ &\quad + g(J_p X \times Y, V_p)V_p - g(Y \times J_p X, V_p)V_p - g(J_p Y \times X, V_p)V_p \\ &= -(X \times J_p Y) - (J_p X \times Y) - (J_p X \times Y) - (X \times J_p Y) + g(X \times J_p Y, V_p)V_p \\ &\quad + g(J_p X \times Y, V_p)V_p - g(Y \times J_p X, V_p)V_p - g(J_p Y \times X, V_p)V_p \\ &= -2(X \times J_p Y + J_p X \times Y) + g(X \times J_p Y, V_p)V_p + g(J_p X \times Y, V_p)V_p \\ &\quad - g(Y \times J_p X, V_p)V_p - g(J_p Y \times X, V_p)V_p \\ &= -2(X \times J_p Y + J_p X \times Y) - g(X,Y)V_p \\ &\quad + g(X,Y)V_p + g(X,Y)V_p - g(X,Y)V_p \\ &= -2(X \times J_p Y + J_p X \times Y). \end{aligned}$$

Hence,

$$N(X,Y)=0 \text{ if and only if } X\times J_pY+J_pX\times Y=0.$$

□

Acknowledgment. We would like to thank the referees for their comments and suggestions.

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