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Common fixed point theorems for weakly biased mappings with an application in dynamic programming

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Abstract

The main objective of this paper is to investigate common fixed point theorems for weakly biased mappings satisfying property $(E.A)$ and a weak contraction condition involving cubic terms of distance functions. Our results generalize and improve the results by Kumar and Kumar [R. Kumar, S. Kumar, J. Math. Comput. Sci., **11** (2021), 1922–1954]. Results are supported with relevant application and example.

Keywords: Common fixed point, weak contraction, weakly biased mapping, property $(E.A)$, dynamic programming.

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1. Introduction and preliminaries

Fixed point theory is an essential part of the study of nonlinear functional analysis and it is useful for demonstrating the existence theorems for nonlinear differential and integral equations. The Banach contraction principle [3] is the simplest and one of the most versatile elementary results in fixed point theory, which is a very popular tool for solving existence problems in many branches of mathematical analysis. This principle states that if σ is a contraction mapping on a complete metric space $(\mathcal{M}, \Delta)$ into itself then σ has a unique fixed point in $\mathcal{M}$. Several authors explored some new type contraction and proved various fixed point theorems in order to generalize the Banach fixed point theorem (see [2, 6, 13–18]). In 1976, for generalization of Banach's fixed point theorem, Jungck [7] used the notion of commuting mappings to prove a common fixed point theorem. In 1982, Sessa [19] generalized the notion of commutativity to weak commutativity and proved some common fixed point theorems for weakly commuting mappings. In 1986, Jungck [8] extended the notion of weakly commuting mappings to a larger class of mappings known as compatible mappings.

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Definition 1.1 ([8]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to be compatible if $\lim_{n\to\infty} \Delta(\xi\zeta\omega_n, \zeta\xi\omega_n) = 0$, whenever $\{\omega_n\} \in \mathcal{M}$ is a sequence such that $\lim_{n\to\infty} \xi\omega_n = \lim_{n\to\infty} \zeta\omega_n = \varkappa$ for some $\varkappa \in \mathcal{M}$.

In 1996, Jungck [11] introduced the concept of weakly compatible mappings. In fact, weakly compatible mappings relax the condition of continuity of the mappings.

Definition 1.2 ([11]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to be weakly compatible if ξ and ζ commute at their coincidence points, i.e., if $\xi\varkappa = \zeta\varkappa$ for some $\varkappa \in \mathcal{M}$ implies $\xi\zeta\varkappa = \zeta\xi\varkappa$.

In the general setting, the notion of property $(E.A)$, which requires the closedness of the subspace, was introduced by Aamri and El-Moutawakil [1].

Definition 1.3 ([1]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to satisfy property $(E.A)$ if there exists a sequence $\{\omega_n\} \in \mathcal{M}$ such that $\lim_{n\to\infty} \xi\omega_n = \lim_{n\to\infty} \zeta\omega_n = \varkappa$ for some $\varkappa \in \mathcal{M}$.

Generalizing the concept of compatible mappings, Jungck and Pathak [10] introduced the concepts of biased and weakly biased mappings and established some fixed point theorems of Meir-Keeler type.

Definition 1.4 ([10]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to be ζ-biased if whenever $\{\omega_n\} \in \mathcal{M}$ is a sequence such that $\lim_{n\to\infty} \xi\omega_n = \lim_{n\to\infty} \zeta\omega_n = \varkappa$ for some $\varkappa \in \mathcal{M}$,

$$\alpha\Delta(\zeta\xi\omega_n, \zeta\omega_n) \leqslant \alpha\Delta(\xi\zeta\omega_n, \xi\omega_n),$$

where $\alpha = \limsup$ or $\liminf$.

By interchanging the role of ξ and ζ in Definition 1.4, we get the pair (ξ, ζ) to be ξ-biased. If the pair (ξ, ζ) is compatible, then it is both ζ and ξ-biased but the converse does not hold (see for details [10, Remark 1.1 and Example 1.2]).

Definition 1.5 ([10]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to be weakly ζ-biased if $\xi\varkappa = \zeta\varkappa$ implies $\Delta(\zeta\xi\varkappa, \zeta\varkappa) \leqslant \Delta(\xi\zeta\varkappa, \xi\varkappa)$.

In 1993, Jungck et al. [9] introduced the notion of compatible mappings of type (A).

Definition 1.6 ([9]). A pair of self mappings (ξ, ζ) on a metric space $(\mathcal{M}, \Delta)$ is said to be compatible of type (A) if $\Delta(\zeta\xi\omega_n, \xi\xi\omega_n) = 0$ and $\Delta(\xi\zeta\omega_n, \zeta\zeta\omega_n) = 0$, whenever $\{\omega_n\} \in \mathcal{M}$ is a sequence such that $\lim_{n\to\infty} \xi\omega_n = \lim_{n\to\infty} \zeta\omega_n = \varkappa$ for some $\varkappa \in \mathcal{M}$.

Remark 1.7 ([10]). If the pair (ξ, ζ) of self mappings defined on metric space $(\mathcal{M}, \Delta)$ is ζ-biased, then it is weakly ζ-biased.

Remark 1.8 ([20]). If the pair (ξ, ζ) of self mappings defined on metric space $(\mathcal{M}, \Delta)$ is weakly compatible, then it is both weakly ζ- and ξ-biased but the converse does not hold.

Example 1.9. Let $(\mathcal{M}, \Delta)$ be a usual metric space, where $\mathcal{M} = \mathbb{R}$. Let $\xi, \zeta : \mathcal{M} \to \mathcal{M}$ be defined by $\xi\varkappa = 2 - 4\varkappa$ and $\zeta\varkappa = 4\varkappa$. For $\varkappa = \frac{1}{4}$, we have $\xi\zeta\varkappa = -2$ and $\zeta\xi\varkappa = 4$. Thus $\Delta(\xi\zeta\varkappa, \xi\varkappa) = 3$ and $\Delta(\zeta\xi\varkappa, \zeta\varkappa) = 3$. Hence the pair (ξ, ζ) is weakly ξ- and ζ-biased. But the mappings ξ and ζ do not commute at its point of coincidence $\varkappa = \frac{1}{4}$. So the mappings are not weakly compatible. Also, the mappings are not compatible and not compatible of type (A).

In this paper, we generalize the results of Kumar et al. [12] for weakly biased mappings satisfying weak contraction condition.

2. Main results

In 2021, Kumar et al. [12] introduced a new weak contraction condition that contains cubic terms of distance functions and established common fixed point theorems for compatible mappings and its variants.

Theorem 2.1 ([12]). *Let ξ, ζ, θ and σ be four mappings of a complete metric space $(\mathcal{M}, \Delta)$ into itself satisfying the following assertions:*

(C1) $\xi(\mathcal{M}) \subset \sigma(\mathcal{M}), \zeta(\mathcal{M}) \subset \theta(\mathcal{M})$;

(C2)

$$\begin{aligned}\Delta^3(\xi\kappa, \zeta\nu) \leqslant& \rho \max \Big\{\frac{1}{2}[\Delta^2(\theta\kappa, \xi\kappa)\Delta(\sigma\nu, \zeta\nu) + \Delta(\theta\kappa, \xi\kappa)\Delta^2(\sigma\nu, \zeta\nu)],\\ &\Delta(\theta\kappa, \xi\kappa)\Delta(\theta\kappa, \zeta\nu)\Delta(\sigma\nu, \xi\kappa), \Delta(\theta\kappa, \zeta\nu)\Delta(\sigma\nu, \xi\kappa)\Delta(\sigma\nu, \zeta\nu)\Big\} - \phi(m(\theta\kappa, \sigma\nu)),\end{aligned}$$

for all $\kappa, \nu \in \mathcal{M}$, where

$$\begin{aligned}m(\theta\kappa, \sigma\nu) = \max \Big\{&\Delta^2(\theta\kappa, \sigma\nu), \Delta(\theta\kappa, \xi\kappa)\Delta(\sigma\nu, \zeta\nu), \Delta(\theta\kappa, \zeta\nu)\Delta(\sigma\nu, \xi\kappa),\\ &\frac{1}{2}[\Delta(\theta\kappa, \xi\kappa)\Delta(\theta\kappa, \zeta\nu) + \Delta(\sigma\nu, \xi\kappa)\Delta(\sigma\nu, \zeta\nu)]\Big\}\end{aligned}$$

and ρ is a real number satisfying $0 < \rho < 1$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$;

(C3) *one of the mappings $\xi, \zeta, \theta, \sigma$ is continuous.*

Assume that the pairs (ξ, θ) and (ζ, σ) are compatible or compatible of type (A) or compatible of type (B) or compatible of type (C) or compatible of type (P). Then ξ, ζ, θ, and σ possess a unique common fixed point in $\mathcal{M}$.

Now, we present our main result for weakly biased mappings enjoying property $(E.A)$ that generalizes the above mentioned result.

Theorem 2.2. *Let ξ, ζ, θ and σ be four mappings of a metric space $(\mathcal{M}, \Delta)$ into itself satisfying* (C1), (C2), *and the following conditions:*

(C4) *(ξ, θ) or (ζ, σ) satisfy the property $(E.A)$;*

(C5) *(ξ, θ) and (ζ, σ) are weakly θ- and σ-biased mappings, respectively.*

If one of the range spaces $\xi(\mathcal{M})$, $\zeta(\mathcal{M})$, $\theta(\mathcal{M})$, and $\sigma(\mathcal{M})$ is a closed subspace of $\mathcal{M}$, then ξ, ζ, θ and σ possess a unique common fixed point.

Proof. Assume that the pair (ζ, σ) satisfies the property $(E.A)$. Then there exists a sequence $\{\kappa_n\} \in \mathcal{M}$ such that $\lim_{n\to\infty} \zeta\kappa_n = \lim_{n\to\infty} \sigma\kappa_n = t$ for some $t \in \mathcal{M}$. Since $\zeta(\mathcal{M}) \subset \theta(\mathcal{M})$, there exists a sequence $\{\nu_n\} \in \mathcal{M}$ such that $\zeta\kappa_n = \theta\nu_n$ and hence $\lim_{n\to\infty} \theta\nu_n = t$.

Now, we show that $\xi\nu_n \to t$. Since $\Delta(\xi\nu_n, t) \leqslant \Delta(\xi\nu_n, \zeta\kappa_n) + \Delta(\zeta\kappa_n, t)$, it is sufficient to show that $\lim_{n\to\infty} \Delta(\xi\nu_n, \zeta\kappa_n) = 0$. On contrary, suppose that $\lim_{n\to\infty} \Delta(\xi\nu_n, \zeta\kappa_n) \neq 0$. Then there exist two subsequences $\{\nu_{n_k}\}$ of $\{\nu_n\}$ and $\{\kappa_{n_k}\}$ of $\{\kappa_n\}$ in $\mathcal{M}$ and a real number $\epsilon > 0$ such that for some positive integer $k \geqslant n$, $\lim_{k\to\infty} \Delta(\xi\nu_{n_k}, \zeta\kappa_{n_k}) \geqslant \epsilon$. Using (C2) with $\kappa = \nu_{n_k}$ and $\nu = \kappa_{n_k}$, we get

$$\begin{aligned}\Delta^3(\xi\nu_{n_k}, \zeta\kappa_{n_k}) \leqslant& \rho \max \Big\{\frac{1}{2}[\Delta^2(\theta\nu_{n_k}, \xi\nu_{n_k})\Delta(\sigma\kappa_{n_k}, \zeta\kappa_{n_k}) + \Delta(\theta\nu_{n_k}, \xi\nu_{n_k})\Delta^2(\sigma\kappa_{n_k}, \zeta\kappa_{n_k})],\\ &\Delta(\theta\nu_{n_k}, \xi\nu_{n_k})\Delta(\theta\nu_{n_k}, \zeta\kappa_{n_k})\Delta(\sigma\kappa_{n_k}, \xi\nu_{n_k}), \Delta(\theta\nu_{n_k}, \zeta\kappa_{n_k})\Delta(\sigma\kappa_{n_k}, \xi\nu_{n_k})\\ &.\Delta(\sigma\kappa_{n_k}, \zeta\kappa_{n_k})\Big\} - \phi(m(\theta\nu_{n_k}, \sigma\kappa_{n_k})),\end{aligned}$$

where

$$\begin{aligned}m(\theta\nu_{n_k},\sigma\kappa_{n_k}) =\max\Big\{&\Delta^2(\theta\nu_{n_k},\sigma\kappa_{n_k}),\Delta(\theta\nu_{n_k},\xi\nu_{n_k})\Delta(\sigma\kappa_{n_k},\zeta\kappa_{n_k}),\Delta(\theta\nu_{n_k},\zeta\kappa_{n_k})\Delta(\sigma\kappa_{n_k},\xi\nu_{n_k}),\\ &\frac{1}{2}[\Delta(\theta\nu_{n_k},\xi\nu_{n_k})\Delta(\theta\nu_{n_k},\zeta\kappa_{n_k})+\Delta(\sigma\kappa_{n_k},\xi\nu_{n_k})\Delta(\sigma\kappa_{n_k},\zeta\kappa_{n_k})]\Big\}.\end{aligned}$$

Taking the limit $k\to\infty$, we get $\epsilon^3\leqslant 0$, which is a contradiction. So $\lim_{n\to\infty}\Delta(\xi\nu_n,\zeta\kappa_n)=0$ and hence $\lim_{n\to\infty}\xi\nu_{n_k}=t$. Thus we obtain

$$\lim_{n\to\infty}\zeta\kappa_n=\lim_{n\to\infty}\sigma\kappa_n=\lim_{n\to\infty}\xi\nu_n=\lim_{n\to\infty}\theta\nu_n=t.$$

Suppose that $\theta(\mathcal{M})$ is a closed subspace of $\mathcal{M}$. Then there exists $\mathfrak{h}\in\mathcal{M}$ such that $t=\theta\mathfrak{h}$. We show that $\theta\mathfrak{h}=\xi\mathfrak{h}$. Using (C2) with $\kappa=\mathfrak{h}$ and $\nu=\kappa_n$, we obtain

$$\begin{aligned}\Delta^3(\xi\mathfrak{h},\zeta\kappa_n)\leqslant\rho\max\Big\{&\frac{1}{2}[\Delta^2(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\sigma\kappa_n,\zeta\kappa_n)+\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta^2(\sigma\kappa_n,\zeta\kappa_n)],\\ &\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\theta\mathfrak{h},\zeta\kappa_n)\Delta(\sigma\kappa_n,\xi\mathfrak{h}),\Delta(\theta\mathfrak{h},\zeta\kappa_n)\Delta(\sigma\kappa_n,\xi\mathfrak{h}).\Delta(\sigma\kappa_n,\zeta\kappa_n)\Big\}-\phi(m(\theta\mathfrak{h},\sigma\kappa_n)),\end{aligned}$$

where

$$\begin{aligned}m(\theta\mathfrak{h},\sigma\kappa_n)=\max\Big\{&\Delta^2(\theta\mathfrak{h},\sigma\kappa_n),\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\sigma\kappa_n,\zeta\kappa_n),\Delta(\theta\mathfrak{h},\zeta\kappa_n)\Delta(\sigma\kappa_n,\xi\mathfrak{h}),\\ &\frac{1}{2}[\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\theta\mathfrak{h},\zeta\kappa_n)+\Delta(\sigma\kappa_n,\xi\mathfrak{h})\Delta(\sigma\kappa_n,\zeta\kappa_n)]\Big\}.\end{aligned}$$

Taking the limit $n\to\infty$, we get $\Delta^3(\xi\mathfrak{h},\theta\mathfrak{h})\leqslant 0$, i.e., $\xi\mathfrak{h}=\theta\mathfrak{h}$. Thus ξ and θ have a coincidence point. Since $\xi(\mathcal{M})\subset\sigma(\mathcal{M})$ there exists a point $\mathfrak{p}\in\mathcal{M}$ such that $\xi\mathfrak{h}=\sigma\mathfrak{p}$. Let $\sigma\mathfrak{p}\neq\zeta\mathfrak{p}$. Letting $\kappa=\mathfrak{h}$ and $\nu=\mathfrak{p}$ in (C2), we obtain

$$\begin{aligned}\Delta^3(\xi\mathfrak{h},\zeta\mathfrak{p})\leqslant\rho\max\Big\{&\frac{1}{2}[\Delta^2(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})+\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta^2(\sigma\mathfrak{p},\zeta\mathfrak{p})],\\ &\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\theta\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\mathfrak{h}),\Delta(\theta\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\mathfrak{h}).\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})\Big\}-\phi(m(\theta\mathfrak{h},\sigma\mathfrak{p})),\end{aligned}$$

where

$$\begin{aligned}m(\theta\mathfrak{h},\sigma\mathfrak{p})=\max\Big\{&\Delta^2(\theta\mathfrak{h},\sigma\mathfrak{p}),\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p}),\Delta(\theta\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\mathfrak{h}),\\ &\frac{1}{2}[\Delta(\theta\mathfrak{h},\xi\mathfrak{h})\Delta(\theta\mathfrak{h},\zeta\mathfrak{p})+\Delta(\sigma\mathfrak{p},\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})]\Big\}.\end{aligned}$$

Thus we get $\Delta^3(\xi\mathfrak{h},\zeta\mathfrak{p})\leqslant 0$, which implies that $\xi\mathfrak{h}=\zeta\mathfrak{p}$ and hence $\sigma\mathfrak{p}=\theta\mathfrak{p}$. Therefore, σ and θ have a coincidence point. Since $\xi\mathfrak{h}=\theta\mathfrak{h}$ and $\sigma\mathfrak{p}=\zeta\mathfrak{p}$, 'weakly θ-biased of (ξ,θ)' implies $\Delta(\theta\xi\mathfrak{h},\theta\mathfrak{h})\leqslant\Delta(\xi\theta\mathfrak{h},\xi\mathfrak{h})$. Similarly, 'weakly σ-biased of the pair (ζ,σ)' implies that $\Delta(\sigma\zeta\mathfrak{p},\sigma\mathfrak{p})\leqslant\Delta(\zeta\sigma\mathfrak{p},\zeta\mathfrak{p})$. On the other hand, we obtain $\xi\mathfrak{h}=\theta\mathfrak{h}$ implies $\xi\theta\mathfrak{h}=\xi\xi\mathfrak{h}$, $\theta\theta\mathfrak{h}=\theta\xi\mathfrak{h}$ and $\sigma\mathfrak{p}=\zeta\mathfrak{p}$ implies $\zeta\theta\mathfrak{p}=\zeta\zeta\mathfrak{p}$.

Now, we show that $\xi\mathfrak{h}$ is a common fixed point of ξ, θ, σ, and ζ. Using (C2) with $\kappa=\xi\mathfrak{h}$ and $\nu=\mathfrak{p}$, we obtain

$$\begin{aligned}\Delta^3(\xi\xi\mathfrak{h},\xi\mathfrak{h})=\Delta^3(\xi\xi\mathfrak{h},\zeta\mathfrak{p})\leqslant\rho\max\Big\{&\frac{1}{2}[\Delta^2(\theta\xi\mathfrak{h},\xi\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})+\Delta(\theta\xi\mathfrak{h},\xi\xi\mathfrak{h})d^2(\sigma\mathfrak{p},\zeta\mathfrak{p})],\\ &\Delta(\theta\xi\mathfrak{h},\xi\xi\mathfrak{h})\Delta(\theta\xi\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\xi\mathfrak{h}),\Delta(\theta\xi\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\xi\mathfrak{h}).\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})\Big\}-\phi(m(\theta\xi\mathfrak{h},\sigma\mathfrak{p})),\end{aligned}$$

where

$$m(\theta\xi\mathfrak{h},\sigma\mathfrak{p})=\max\Big\{d^2(\theta\xi\mathfrak{h},\sigma\mathfrak{p}),\Delta(\theta\xi\mathfrak{h},\xi\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p}),\Delta(\theta\xi\mathfrak{h},\zeta\mathfrak{p})\Delta(\sigma\mathfrak{p},\xi\xi\mathfrak{h}),$$

$$\frac{1}{2}[\Delta(\theta\xi\mathfrak{h},\xi\xi\mathfrak{h})\Delta(\theta\xi\mathfrak{h},\zeta\mathfrak{p})+\Delta(\sigma\mathfrak{p},\xi\xi\mathfrak{h})\Delta(\sigma\mathfrak{p},\zeta\mathfrak{p})]\Big\}.$$

Thus we obtain $\Delta^3(\xi\xi\mathfrak{h},\xi\mathfrak{h}) \leqslant -\phi(\Delta^2(\xi\xi\mathfrak{h},\xi\mathfrak{h}))$, which is a contradiction and hence $\xi\xi\mathfrak{h} = \xi\mathfrak{h}$. Thus $\xi\mathfrak{h}$ is a fixed point of ξ and θ.

Similarly, one can show that $\zeta\mathfrak{p}$ is a common fixed point of ζ and σ. Since $\xi\mathfrak{h} = \zeta\mathfrak{p}$, we therefore conclude that $\xi\mathfrak{h}$ is a common fixed point of ξ, ζ, θ, and σ. The proof is similar when $\xi\mathcal{M}$, $\zeta\mathcal{M}$, or $\sigma\mathcal{M}$ is a closed subspace of $\mathcal{M}$. Using condition (C2), one can easily check the uniqueness of common fixed point of ξ, ζ, θ, and σ. This completes the proof. □

The following example supports our result.

Example 2.3. Consider $\mathcal{M} = [0, 20]$ with usual metric Δ on $\mathcal{M}$. Let ξ, ζ, θ and σ be four self mappings on $\mathcal{X}$ defined as

$$\xi(\mathfrak{h}) = \begin{cases} 10, & \mathfrak{h} \in [0,10], \\ 20-\mathfrak{h}, & \mathfrak{h} \in (10,20], \end{cases} \qquad \zeta(\mathfrak{h}) = \begin{cases} \frac{\mathfrak{h}}{2}+5, & \mathfrak{h} \in [0,10], \\ 20-\mathfrak{h}, & \mathfrak{h} \in (10,20], \end{cases}$$

$$\theta(\mathfrak{h}) = \begin{cases} \frac{-3\mathfrak{h}}{5}+16, & \mathfrak{h} \in [0,10], \\ 20-\mathfrak{h}, & \mathfrak{h} \in (10,20], \end{cases} \qquad \sigma(\mathfrak{h}) = 20-\mathfrak{h}, \mathfrak{h} \in \mathcal{M}.$$

Let $\phi : [0,\infty) \to [0,\infty)$ be a function defined by $\phi(t) = \frac{t}{30}$, for $t \geqslant 0$. Then one can easily check that all the conditions of Theorem 2.2 are satisfied for $\rho = 0.9$ and 10 is the unique common fixed point of ξ, ζ, θ, and σ.

Remark 2.4. Theorem 2.2 improves the results of Kumar et al. [12]. Of course, our results do not depend on the continuities of the mappings involved and it just requires weakly biased in place of compatible mappings or its variants and also one of the range spaces is closed in place of $\mathcal{M}$ being complete.

3. Application to functional equations in dynamic programming

In this section, we present the application of our result in finding a common solution of functional equations that arise in dynamic programming. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be Banach spaces. Let $\mathcal{S} \subset \mathcal{U}_1$ and $\mathcal{D} \subset \mathcal{U}_2$ be the state space and decision space respectively. Assume that $\mathcal{B}(\mathcal{S}) = \{\theta \mid \theta : \mathcal{S} \to \mathbb{R} \text{ is bounded}\}$.

Bellman and Lee [5] defined the fundamental form of the functional equation of dynamic programming as follows:

$$f(\mathfrak{h}) = \text{opt}_{\mathfrak{p}}\mathcal{H}(\mathfrak{h},\mathfrak{p},f(\mathcal{T}(\mathfrak{h},\mathfrak{p}))),$$

where $\mathfrak{h}$ and $\mathfrak{p}$ stand for state and decision vectors, respectively, $\mathcal{T}$ stands for the transformation of the process, $f(\mathfrak{h})$ stands for the optimal return given the initial state $\mathfrak{h}$, and the opt stands for max or min. The following functional equations that arise in dynamic programming ([4, 5]) will be discussed here:

$$f_i(\mathfrak{h}) = \sup_{\mathfrak{p}\in\mathcal{D}} \mathcal{H}_i(\mathfrak{h},\mathfrak{p},f_i(\mathcal{T}(\mathfrak{h},\mathfrak{p}))), \quad \mathfrak{h} \in \mathcal{S}, \tag{3.1}$$

$$g_i(\mathfrak{h}) = \sup_{\mathfrak{p}\in\mathcal{D}} \mathcal{F}_i(\mathfrak{h},\mathfrak{p},g_i(\mathcal{T}(\mathfrak{h},\mathfrak{p}))), \quad \mathfrak{h} \in \mathcal{S}, \tag{3.2}$$

where $\mathcal{T} : \mathcal{S} \times \mathcal{D} \to \mathcal{S}$ and $\mathcal{H}_i, \mathcal{F}_i : \mathcal{S} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$.

Theorem 3.1. *Assume that the following conditions hold.*

(i) *For $i \in \{1, 2\}$, $\mathcal{H}_i$ and $\mathcal{F}_i$ are bounded.*

(ii)

$$|\mathcal{H}_1(\mathfrak{h},\mathfrak{p},\theta(t)) - \mathcal{H}_2(\mathfrak{h},\mathfrak{p},\sigma(t))|^3$$

$$\leqslant \rho \max\Big\{\frac{1}{2}[|\mathcal{N}_1\theta(t)-\mathcal{P}_1\theta(t)|^2\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_2\sigma(t)|+|\mathcal{N}_1\theta(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_2\sigma(t)|^2],$$
$$|\mathcal{N}_1\theta(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_1\theta(t)-\mathcal{P}_2\sigma(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_1\theta(t)|,$$
$$|\mathcal{N}_1\theta(t)-\mathcal{P}_2\sigma(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_2\sigma(t)|\Big\}-\phi(m(\mathcal{N}_1\theta(t),\mathcal{N}_2\sigma(t)))$$

for all $(\mathfrak{h},\mathfrak{p})\in\mathcal{S}\times\mathcal{D}$, $\theta,\sigma\in\mathcal{B}(\mathcal{S})$ *and* $t\in\mathcal{S}$, *where*

$$m(\mathcal{N}_1\theta(t),\mathcal{N}_2\sigma(t))=\max\Big\{|\mathcal{N}_1\theta(t)-\mathcal{N}_2\sigma(t)|^2,|\mathcal{N}_1\theta(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_2\sigma(t)|,$$
$$|\mathcal{N}_1\theta(t)-\mathcal{P}_2\sigma(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_1\theta(t)|,\frac{1}{2}[|\mathcal{N}_1\theta(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_1\theta(t)-\mathcal{P}_2\sigma(t)|$$
$$+|\mathcal{N}_2\sigma(t)-\mathcal{P}_1\theta(t)|\cdot|\mathcal{N}_2\sigma(t)-\mathcal{P}_2\sigma(t)|]\Big\}.$$

Further, ρ *and* ϕ *are the same as in Theorem 2.2. Also, the mappings* $\mathcal{P}_i$ *and* $\mathcal{N}_i$ *are defined as follows:*

$$\mathcal{P}_i\theta(\mathfrak{h})=\sup_{\mathfrak{p}\in\mathcal{D}}\mathcal{H}_i(\mathfrak{h},\mathfrak{p},\theta(\mathcal{N}(\mathfrak{h},\mathfrak{p}))),\mathfrak{h}\in\mathcal{S},\theta\in\mathcal{B}(\mathcal{S}),\ i=1,2,$$

$$\mathcal{N}_i\sigma(\mathfrak{h})=\sup_{\mathfrak{p}\in\mathcal{D}}\mathcal{F}_i(\mathfrak{h},\mathfrak{p},\sigma(\mathcal{N}(\mathfrak{h},\mathfrak{p}))),\mathfrak{h}\in\mathcal{S},\sigma\in\mathcal{B}(\mathcal{S}),\ i=1,2.$$

(iii) *There exist a sequence* $\{\sigma_n\}\in\mathcal{B}(\mathcal{S})$ *and* $\sigma(\mathfrak{h})\in\mathcal{B}(\mathcal{S})$ *such that*

$$\lim_{n\to\infty}\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{P}_1\sigma_n(\mathfrak{h})-\sigma(\mathfrak{h})|=0 \text{ and } \lim_{n\to\infty}\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{N}_1\sigma_n(\mathfrak{h})-\sigma(\mathfrak{h})|=0$$

or

$$\lim_{n\to\infty}\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{P}_2\sigma_n(\mathfrak{h})-\sigma(\mathfrak{h})|=0 \text{ and } \lim_{n\to\infty}\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{N}_2\sigma_n(\mathfrak{h})-\sigma(\mathfrak{h})|=0.$$

(iv) *For any* $\theta\in\mathcal{B}(\mathcal{S})$, *there exist* $\sigma_1,\sigma_2\in\mathcal{B}(\mathcal{S})$ *such that*

$$\mathcal{P}_1\theta(\mathfrak{h})=\mathcal{N}_2\sigma_1(\mathfrak{h}),\qquad \mathcal{P}_2\theta(\mathfrak{h})=\mathcal{N}_1\sigma_2(\mathfrak{h}),\mathfrak{h}\in\mathcal{S}.$$

(v) *For some* $\theta\in\mathcal{B}(\mathcal{S})$, $\mathcal{P}_1\theta=\mathcal{N}_1\theta$ *implies*

$$\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{P}_1\mathcal{N}_1\theta(\mathfrak{h})-\mathcal{P}_1\theta(\mathfrak{h})|\leqslant\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{N}_1\mathcal{P}_1\theta(\mathfrak{h})-\mathcal{N}_1\theta(\mathfrak{h})|$$

and for some $\sigma\in\mathcal{B}(\mathcal{S})$, $\mathcal{P}_1\sigma=\mathcal{N}_1\sigma$ *implies*

$$\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{P}_2\mathcal{N}_2\sigma(\mathfrak{h})-\mathcal{P}_2\sigma(\mathfrak{h})|\leqslant\sup_{\mathfrak{h}\in\mathcal{S}}|\mathcal{N}_2\mathcal{P}_2\sigma(\mathfrak{h})-\mathcal{N}_2\sigma(\mathfrak{h})|.$$

Then the system of the functional equations (3.1) and (3.2) has a unique common solution in $\mathcal{B}(\mathcal{S})$.

Proof. Let $\Delta(\theta,\sigma)=\sup\{|\theta(\mathfrak{h})-\sigma(\mathfrak{h})|:\mathfrak{h}\in\mathcal{S}\}$ for any $\theta,\sigma\in\mathcal{B}(\mathcal{S})$. Then $(\mathcal{B}(\mathcal{S}),\Delta)$ is a complete metric space. From the conditions (i)-(v), $\mathcal{P}_i$ and $\mathcal{N}_i$ are self mappings of $B(\mathcal{S})$, $i=1,2$, $\mathcal{P}_1(\mathcal{B}(\mathcal{S}))\subset\mathcal{N}_2(\mathcal{B}(\mathcal{S}))$, $\mathcal{P}_2(\mathcal{B}(\mathcal{S}))\subset\mathcal{N}_1(\mathcal{B}(\mathcal{S}))$ and the pairs of mappings $(\mathcal{P}_i,\mathcal{N}_i)$ are weakly $\mathcal{N}_i$-mappings, $i=1,2$. Either $(\mathcal{P}_1,\mathcal{N}_1)$ or $(\mathcal{P}_2,\mathcal{N}_2)$ satisfies property $(E.A)$. Let $\theta_i(i=1,2)$ be any two points of $\mathcal{B}(\mathcal{S})$, $\mathfrak{h}\in\mathcal{S}$ and α be any positive number. Assume that there exist $\mathfrak{p}_i(i=1,2)$ in $\mathcal{D}$ such that

$$\mathcal{P}_i\theta_i(\mathfrak{h})<\mathcal{H}_i(\mathfrak{h},\mathfrak{p}_i,\theta_i(\mathfrak{h}_i))+\alpha, \tag{3.3}$$

where $\mathfrak{h}_i=\mathcal{N}(\mathfrak{h},\mathfrak{p}_i)$, $i=1,2$. Also, we have

$$\mathcal{P}_1\theta_1(\mathfrak{h})\geqslant\mathcal{H}_1(\mathfrak{h},\mathfrak{p}_2,\theta_1(\mathfrak{h}_2)), \tag{3.4}$$

$$\mathcal{P}_2\theta_2(\mathfrak{h}) \geqslant \mathcal{H}_2(\mathfrak{h},\mathfrak{p}_1,\theta_2(\mathfrak{h}_1)). \tag{3.5}$$

From (3.3), (3.5), and (ii), we have

$$\begin{aligned}(\mathcal{P}_1\theta_1(\mathfrak{h})-\mathcal{P}_2\theta_2(\mathfrak{h}))^3 &< (\mathcal{H}_1(\mathfrak{h},\mathfrak{p}_1,\theta_1(\mathfrak{h}_1))-\mathcal{H}_2(\mathfrak{h},\mathfrak{p}_1,\theta_2(\mathfrak{h}_1)))^3+\alpha\\ &\leqslant |\mathcal{H}_1(\mathfrak{h},\mathfrak{p}_1,\theta_1(\mathfrak{h}_1))-\mathcal{H}_2(\mathfrak{h},\mathfrak{p}_1,\theta_2(\mathfrak{h}_1))|^3+\alpha\\ &\leqslant \rho\max\Big\{\frac{1}{2}[|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|^2\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|\\ &\quad +|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|^2],\\ &\quad\quad |\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\cdot|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|,\\ &\quad\quad |\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|\Big\}\\ &\quad -\phi(m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1))+\alpha,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1)) = \max\Big\{&|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{N}_2\theta_2(\mathfrak{h}_1)|^2, |\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|,\\ &|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|, \frac{1}{2}[|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\\ &\cdot|\mathcal{N}_1\theta_1(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|+|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_1\theta_1(\mathfrak{h}_1)|\cdot|\mathcal{N}_2\theta_2(\mathfrak{h}_1)-\mathcal{P}_2\theta_2(\mathfrak{h}_1)|]\Big\}.\end{aligned}$$

From (3.6), we have

$$\begin{aligned}(\mathcal{P}_1\theta_1(\mathfrak{h})-\mathcal{P}_2\theta_2(\mathfrak{h}))^3 \leqslant \rho\max\Big\{&\frac{1}{2}[\Delta^2(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)+\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta^2(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)],\\ &\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1),\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1)\\ &\cdot\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)\Big\}-\phi(m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1))+\alpha,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1)) = \max\Big\{&\Delta^2(\mathcal{N}_1\theta_1,\mathcal{N}_2\theta_2),\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2),\\ &\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1),\frac{1}{2}[\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\\ &\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)+\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)]\Big\}.\end{aligned}$$

From (3.3), (3.4), and (ii), we have

$$\begin{aligned}(\mathcal{P}_1\theta_1(\mathfrak{h})-\mathcal{P}_2\theta_2(\mathfrak{h}))^3 \geqslant -\rho\max\Big\{&\frac{1}{2}[\Delta^2(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)+\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta^2(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)],\\ &\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1),\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1)\\ &\cdot\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)\Big\}+\phi(m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1))-\alpha,\end{aligned} \tag{3.8}$$

where $m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1))$ is same as in (3.7). Combination of (3.7) and (3.8) gives

$$\begin{aligned}|\mathcal{P}_1\theta_1(\mathfrak{h})-\mathcal{P}_2\theta_2(\mathfrak{h})|^3 \leqslant \rho\max\Big\{&\frac{1}{2}[\Delta^2(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)+\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1)\Delta^2(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)],\\ &\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_1\theta_1).\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1),\Delta(\mathcal{N}_1\theta_1,\mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_1\theta_1)\\ &\cdot\Delta(\mathcal{N}_2\theta_2,\mathcal{P}_2\theta_2)\Big\}-\phi(m(\mathcal{N}_1\theta_1(\mathfrak{h}_1),\mathcal{N}_2\theta_2(\mathfrak{h}_1))+\alpha.\end{aligned} \tag{3.9}$$

Since (3.9) holds for any $\mathfrak{h} \in \mathcal{S}$ and α is any positive number, on taking supremum over all $\mathfrak{h} \in \mathcal{S}$, we have

$$\begin{aligned}\Delta^3(\mathcal{P}_1\theta_1, \mathcal{P}_2\theta_2) \leqslant \rho \max \Big\{ &\frac{1}{2}[\Delta^2(\mathcal{N}_1\theta_1, \mathcal{P}_1\theta_1)\Delta(\mathcal{N}_2\theta_2, \mathcal{P}_2\theta_2) + \Delta(\mathcal{N}_1\theta_1, \mathcal{P}_1\theta_1)\Delta^2(\mathcal{N}_2\theta_2, \mathcal{P}_2\theta_2)], \\ &\Delta(\mathcal{N}_1\theta_1, \mathcal{P}_1\theta_1)\Delta(\mathcal{N}_1\theta_1, \mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2, \mathcal{P}_1\theta_1), \Delta(\mathcal{N}_1\theta_1, \mathcal{P}_2\theta_2)\Delta(\mathcal{N}_2\theta_2, \mathcal{P}_1\theta_1) \\ &\cdot \Delta(\mathcal{N}_2\theta_2, \mathcal{P}_2\theta_2) \Big\} - \phi(m(\mathcal{N}_1\theta_1(\mathfrak{h}_1), \mathcal{N}_2\theta_2(\mathfrak{h}_1))).\end{aligned}$$

Therefore, by Theorem 2.2, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{N}_1$, and $\mathcal{N}_2$ possess a unique common fixed point $\theta' \in \mathcal{B}(\mathcal{S})$. Thus $\theta'(\mathfrak{h})$ is a unique solution of the functional equations (3.1) and (3.2). □

4. Conclusion

We have investigated common fixed point theorems for weakly biased mappings satisfying property (E.A) and a weak contraction condition involving cubic terms of distance functions. Our results generalize and improve the results by Kumar and Kumar [12]. Results have been supported with relevant application and example.

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Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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