J. Math. Computer Sci., 36 (2025), 432–443

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Boundary value problem of fractional fuzzy Volterra-Fredholm systems

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Abstract

This paper investigates the suitable conditions for the uniqueness and existence results for a class of fuzzy fractional Caputo Volterra-Fredholm integro differential equations (FFCV-FIDEs) with boundary conditions. The findings are based on Banach contraction principle and Schaefer's fixed point theorem. Additionally, the solution to the given problem is found using the Adomian decomposition technique (ADT). We support the concept with various instances. The relationship between the lower and upper reduce approximations of the fuzzy solutions has been demonstrated numerically and graphically via MATLAB.

Keywords: Volterra-Fredholm equation, Caputo fractional derivative, fixed point technique, ADT.

2020 MSC: 34A12, 26A33, 49M27.

1. Introduction

The idea of fractional calculus (FC) dates back to the days of Leibniz and Newton. Since then, a number of mathematicians have made contributions to the theoretical development of FC. Its application to real-world issues has garnered a lot of attention in recent years. The mathematical modelling of processes and systems in the domains of porous media, aerodynamics, electromagnetic, physics, viscoelasticity, control theory, electro-chemistry, signal processing, chemistry, and so on gives rise to fractional DEs in many engineering and scientific disciplines (see [7, 12, 16, 28, 30, 33, 34, 37]). Recently, fractional DEs have seen a substantial theoretical development (see [4, 9, 15, 26, 27] and the references therein). Some mathematicians are interested in solving problems involving IDEs. The uniqueness and existence of solutions to fractional IDEs were examined in a number of studies [6, 18, 24, 25, 35, 36]. However, the majority of works deal with numerical analysis of fractional IDEs, or fractional DEs. The number of approaches for locating these approximations has increased recently. A few of these techniques are the wavelet method [37], Homotopy analysis method [17, 23], variational iteration method [17, 32], ADT [19–22], fractional differential transform method [7], collocation method [29], and reproducing kernel method [26, 31], etc.

Many academics have developed the concept of FIDEs in recent years. Zadeh was the first to identify the relationship between arithmetic operations and fuzzy numbers. Additionally, they extension the idea

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doi: 10.22436/jmcs.036.04.06

Received: 2024-06-02 *Revised:* 2024-07-08 *Accepted:* 2024-07-20

of function fuzzy integration. The fuzzy mapping function was also suggested by Cheng and Zadeh [13], Dubois and Prade [15], and others. Furthermore, [16] provided a basic fuzzy calculus based on the extension idea. Numerous techniques have been developed recently for solving fuzzy IDEs. Abbasbandy and Hashemi (2011) worked on FVIDEs, which they formulated and solved using the homotopy analysis approach and variational iteration method [1]. Techniques: residual power, two-dimensional Legendre wavelet, flitted reproduction kernel Hilbert space, transform of fuzzy Laplace, and Abu Araub's discussion of the kernel technique for replicating results in fuzzy Fredholm-Volterra integral equation [8], Alaroud, Al-smadi et al.'s work on residual power series method under the generalised H-differentiability [5], and residual power The FFVIDE analysis that Naveed Ahmad and colleagues [4] examined. The fractional derivative (FD) in the Atangana-Baleanu meaning of FFDEs was examined by Arqub et al. [10]. FFVIDEs in the context of FD Caputo-Atangana-Baleanu have been worked on by writers in [9]. They made advantage of kernel function adoption. The ADT was developed by G. Adomian, and it has been successfully used to solve a large number of nonlinear DEs using approximations that converge fast to the desired result in [2].

Motivated by the above articles, in this paper, we examine a new class of FFCV-FIDEs and demonstrate the existence and uniqueness of solutions inside their specified domain. Additionally, we will study the the approximation of solution of the following model by using ADT:

$$D^{\eta}\,\tilde{\chi}(t,\theta) = \tilde{\varphi}(t,\theta) + B(t)\Im(\tilde{\chi}(t,\theta)) + \int_0^t \vartheta(t,\lambda)\Im(\tilde{\chi}(\lambda,\theta))d\lambda + \int_0^T \vartheta_1(t,\lambda)\Im(\tilde{\chi}(\lambda,\theta))d\lambda, \tag{1.1}$$

$$\beta_1\tilde{\chi}(0,\theta) + \beta_2\tilde{\chi}(T,\theta) = \tilde{\beta_3}, \tag{1.2}$$

where $t \in \Psi := (0,T]$, $\Im(\tilde{\chi}(t,\theta))$ is a nonlinear function and kernels $\vartheta(t,\lambda)$, $\vartheta_1(t,\lambda)$, and $\tilde{\varphi}(t,\theta)$ are sufficiently smooth functions on Ψ and furthermore, $B(t) \neq 0$ on Ψ. $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 + \beta_2 \neq 0$, and $\tilde{\beta_3} \in \mathbb{R}_\Omega$ represents $(\theta - 1, 1 - \theta)$.

An outline of the paper's structure is provided below. In Section 2, the fundamental concepts, notations, lemmas, and theorems of fuzzy and fuzzy FC were reviewed. In Section 3, we examine the existence and uniqueness of a solution of the given model (1.1)-(1.2). In Section 4, the solution approximation of the proposed model was tested using an ADT. Moreover, the convergence analysis is shown. Numerical experiments and a concrete computing technique are presented in Section 5.

2. Auxiliary results

The basic ideas of fuzzy calculus are defined in this part, and these definitions will be applied to the problems and approximate solutions that are put forth.

Definition 2.1 ([4]). $\tilde{\chi} : \mathbb{R} \to [0,1]$ is mapped to is shown as a fuzzy number that satisfies the following criteria: for example, $\tilde{\chi}$ is upper semi continuous on $\mathbb{R}$; $cl(\operatorname{supp}\tilde{\chi})$ is compact; and $\tilde{\chi}$ is normal, closure, and fuzzy convex set. The set of all fuzzy numbers is represented by the symbol $\mathbb{R}_\Omega$. For any $x, \nu \in \mathbb{R}_O$ and $k \in \mathbb{R}$, also, we define $(x \oplus \nu)_\theta = x_\theta \oplus \nu_\theta, (k \odot x)_\theta = \left[k\underline{x_\theta}, k\bar{x_\theta}\right]$.

Definition 2.2 ([5]). Let $\tilde{\chi}$ is a Fuzzy number in a parametric form given by $\tilde{\chi} = (\underline{\chi}(\theta), \bar{\chi}(\theta))$, it fulfills the following characteristics.

1. $\underline{\chi}(\theta)$ be a non decreasing, bounded, and right continuous function over $\theta \in [0,1]$.
2. $\bar{\chi}(\theta)$ be a non increasing, bounded, and left continuous function over $\theta \in [0,1]$.

$\underline{\chi}(\theta) \leqslant \bar{\chi}(\theta)$ for $\theta \in [0,1]$.

Definition 2.3 ([4]). The θ-level set of a fuzzy number $\tilde{\chi} \in \mathbb{R}_\Omega$ defined by $[\ \tilde{\chi}]_\theta$ is identified by

$$[\tilde{\chi}]_\theta = \begin{cases} r \in \mathbb{R}/\tilde{\chi}(r) \geqslant \theta, & \text{if } 0 < \theta \leqslant 1, \\ cl(\operatorname{supp}\tilde{\chi}), & \text{if } \theta = 0. \end{cases}$$

The fuzzy number appears in θ level set is a bounded and closed interval $[\underline{\chi}(\theta), \bar{\chi}(\theta)]$, where $\underline{\chi}(\theta)$ is the left side end point and $\bar{\chi}(\theta)$ right side end point.

Definition 2.4 ([11])**.** The extended Hukuhara derivative (eH-derivative) of fuzzy-valued function $\tilde{\Omega} : [x, y] \to \mathbb{R}_\Omega$ at c_0 is expressed by

$$\tilde{\Omega}'_{eH}(c_0) = \lim_{h \to 0} \frac{\Omega(c_0 + h) \ominus_{eH} \Omega(c_0)}{h}$$

if $\left(\tilde{\Omega}'\right)_{eH}(c_0) \in \mathbb{R}_\Omega$, we say that $\tilde{\Omega}$ is extended Hukuhara differentiable (eH-differentiable) at c_0. Furthermore, we say that $\tilde{\Omega}$ is [(i)-eH]-differentiable at c_0 if

$$\left(\tilde{\Omega}'_{eH}\right)_\theta(c_0) = \left[\left(\underline{\Omega_\theta}\right)'(c_0), \left(\overline{\Omega_\theta}\right)'(c_0)\right], \quad 0 \leqslant \theta \leqslant 1,$$

and that $\tilde{\Omega}$ is [(ii)-eH]-differentiable at c_0 if

$$\left(\tilde{\Omega}'_{eH}\right)_\theta(c_0) = \left[\left(\overline{\Omega_\theta}\right)'(c_0), \left(\underline{\Omega_\theta}\right)'(c_0)\right], \quad 0 \leqslant \theta \leqslant 1.$$

Definition 2.5 ([11])**.** For a function $\Omega(t)$, the fractional integral (FI) of order $\eta > 0$ in the Riemann-Liouville sense is defined as follows:

$$I^\eta \Omega(u) = \frac{1}{\Gamma(\eta)} \int_0^u (u-t)^{\eta-1} \Omega(t) dt, \ u > 0, \ \eta \in \mathbb{R}, \qquad I^0 \Omega(u) = \Omega(u).$$

Definition 2.6 ([11])**.** Based on its θ-level examples, the R-L FI of order η of the fuzzy function $\tilde{\Omega}(u, \eta)$ may be expressed as follows: $\left[I^\eta \tilde{\Omega}(u;\theta)\right] = \left[I^\eta \underline{\Omega}(u;\theta), I^\eta \bar{\Omega}(u;\theta)\right]$, where

$$I^\eta \underline{\Omega}(u, \theta) = \frac{1}{\Gamma(\eta)} \int_0^u (u-t)^{\eta-1} \underline{\Omega}(t, \theta) dt, u > 0, \eta \in \mathbb{R},$$
$$I^\eta \bar{\Omega}(u, \theta) = \frac{1}{\Gamma(\eta)} \int_0^u (u-t)^{\eta-1} \bar{\Omega}(t, \theta) dt, u > 0, \eta \in \mathbb{R}.$$

Definition 2.7 ([14])**.** The order η Caputo FD is expressed in the following equation.

$$D^\eta \Omega(u) = \begin{cases} \frac{1}{\Gamma(q-\eta)} \int_0^u (u-v)^{q-\eta-1} \Omega^q(v) dv, & q-1 \leqslant \eta < q, \\ \frac{d^q}{dv^q} \Omega(u), & \eta = q, \ q \in \mathbb{N}. \end{cases}$$

The characteristics are as

- $I^{\eta_1} I^{\eta_2} \Omega(u) = I^{\eta_1+\eta_2} \Omega(u), \ \eta_1, \eta_2 > 0;$
- $I^{\eta_1}(u^{\eta_2}) = \left\{ \frac{\Gamma(\eta_2+1) u_1 + \eta_2}{\Gamma(\eta_2+\eta_1+1)}, \ \eta_2 > 0, \eta_1 > -1, u > 0. \right.$

Definition 2.8 ([11])**.** Based on its θ-level examples, the Caputo FD of order η of the fuzzy function $\tilde{\Omega}(u, \theta)$ may be expressed as follows: $\left[D^\eta \tilde{\Omega}(u;\theta)\right] = \left[D^\eta \underline{\Omega}(u;\theta), D^\eta \bar{\Omega}(u;\theta)\right]$, where

$$D^\eta \underline{\Omega}(u, \theta) = \begin{cases} \frac{1}{\Gamma(m-\eta)} \int_0^u (u-t)^{m-\eta-1} \underline{\Omega}^m(u, \theta) dt, & m-1 \leqslant \eta < m, \\ \frac{d^m}{dt^m} \underline{\Omega}(u, \theta), & \eta = m, m \in \mathbb{N}, \end{cases}$$

$$D^\eta [\bar{\Omega}(u, \theta)] = \begin{cases} \frac{1}{\Gamma(m-\eta)} \int_0^u (u-t)^{m-\eta-1} \bar{\Omega}^m(u, \theta) dt, & m-1 \leqslant \eta < m, \\ \frac{d^m}{dt^m} \bar{\Omega}(u, \theta), & \eta = m, m \in \mathbb{N}. \end{cases}$$

Note that $\tilde{\Omega}_\theta(t)$ is shown as $\tilde{\Omega}(t, \theta)$.

3. Existence and uniqueness results

We will talk about the uniqueness and existence results for FFCV-FIDE (1.1) in this part. To make things easier, we've included a list of the theories we'll be using to deepen our discussion.

(H1) The function $\Im(\tilde{\chi}(t,\theta))$ satisfies the Lipschitz condition with respect to $\tilde{\chi}(t,\theta)$, with $L(>0)$ being Lipschitz constant, and $\Im(0)=0$, $\forall t \in \Psi$.

(H2) The kernels $\vartheta(t,\lambda), \vartheta_1(t,\lambda)$ are bounded and continuous by $\Theta_1 > 0$ and $\Theta_1^* > 0$ on $\Psi \times \Psi$.

(H3) The functions $B(t)$ and $\tilde{\varphi}(t,\theta)$ are bounded by $\Theta_2(>0)$ and $\Theta_3(>0)$, respectively.

Theorem 3.1. *Let $\tilde{\varphi}(t,\theta), B(t), \vartheta(t,\lambda)$, and $\vartheta_1(t,\lambda)$ are smooth functions on $[0,T]$. Then, the FFCV-FIDE (1.1)-(1.2) is comparable to the form below*

$$\begin{aligned}\tilde{\chi}(t,\theta) = \tilde{h}(t,\theta) &- \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) \\ &+ \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda \\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda,\end{aligned} \tag{3.1}$$

where

$$\tilde{h}(t,\theta) = \frac{\tilde{\beta}_3}{\beta_1+\beta_2} - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\tilde{\varphi}(\lambda,\theta)d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\tilde{\varphi}(\lambda,\theta)d\lambda, \tag{3.2}$$

here, $\tilde{\beta}_3$ is fuzzy value.

Proof. Applying I^η to both sides of (1.1) yields

$$\begin{aligned}\tilde{\chi}(t,\theta) = \tilde{\chi}(0,\theta) &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[\tilde{\varphi}(\lambda,\theta) + B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) \\ &+ \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda.\end{aligned} \tag{3.3}$$

Based on the aforementioned equation at $t=T$, we can readily obtain

$$\begin{aligned}\tilde{\chi}(T,\theta) = \tilde{\chi}(0,\theta) &+ \frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[\tilde{\varphi}(\lambda,\theta) + B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) \\ &+ \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda.\end{aligned}$$

The condition (1.2) provides us with the following identity:

$$\begin{aligned}\tilde{\chi}(0,\theta) = \frac{\tilde{\beta}_3}{\beta_1+\beta_2} &- \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[\tilde{\varphi}(\lambda,\theta) + B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) \\ &+ \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda.\end{aligned}$$

Therefore, by adding the aforementioned value of $\tilde{\chi}(0,\theta)$ to (3.3), we may obtain the required equivalent form. □

Theorem 3.2. *Let (H1)-(H3) hold. Furthermore, we consider that $|\Im(\tilde{\chi}(t,\theta))| \leqslant \Theta^*$, $\forall t \in [0,T]$ and $\tilde{\chi}(t,\theta) \in \mathbb{R}_\Omega$. The FFCV-FIDE (1.1)-(1.2) has at least one solution in Ψ.*

Proof. Let $\Phi : C([0,T],\mathbb{R}_\Omega) \to C([0,T],\mathbb{R}_\Omega)$ be an operator, which is described as

$$\begin{aligned}\Phi(\tilde{\chi}(t,\theta)) &= \tilde{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) \\ &\quad + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda,\end{aligned}$$

where the $\tilde{h}(t,\theta)$ is given in (3.2). Here, we demonstrate the fixed point of the operator Φ by demonstrating that Φ is continuous on $C([0,T],\mathbb{R}_\Omega)$ and compact on each bounded subset of $C([0,T],\mathbb{R}_\Omega)$. This implies that the statement part one in Schaefer's theorem is false, which implies that the part two in Schaefer's theorem must be true, as will be demonstrated in a series of steps.

(i) Initially, we establish the continuity of the operator Φ. Let $\tilde{\chi}_n$ be a sequence convergence to $\tilde{\chi}$ in $C([0,T],\mathbb{R}_\Omega)$ as $n\to\infty$. $\forall\, \tilde{\chi}_n, \tilde{\chi} \in C([0,T],\mathbb{R}_\Omega)$, for any $t\in[0,T]$, we get

$$\begin{aligned}|\Phi(\tilde{\chi}_n(t,\theta)) - \Phi(\tilde{\chi}(t,\theta))| &\leqslant \left|\frac{\beta_2}{\beta_1+\beta_2}\right|\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)|\,|\Im(\tilde{\chi}_n(\lambda,\theta)) - \Im(\tilde{\chi}(\lambda,\theta))| \\ &\quad + \int_0^\lambda |\vartheta(\lambda,s)|\,|\Im(\tilde{\chi}_n(s,\theta)) - \Im(\tilde{\chi}(s,\theta))|\,ds + \int_0^T |\vartheta_1(\lambda,s)|\Big|\Im(\tilde{\chi}_n(s,\theta)) \\ &\quad - \Im(\tilde{\chi}(s,\theta))\Big|ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[|B(\lambda)|\,|\Im(\tilde{\chi}_n(\lambda,\theta)) - \Im(\tilde{\chi}(\lambda,\theta))| \\ &\quad + \int_0^\lambda |\vartheta(\lambda,s)|\,|\Im(\tilde{\chi}_n(s,\theta)) - \Im(\chi(s,\theta))|\,ds \\ &\quad + \int_0^T |\vartheta_1(\lambda,s)|\,|\Im(\tilde{\chi}_n(s,\theta)) - \Im(\chi(s,\theta))|\,ds\Big]d\lambda \\ &\leqslant \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2 LT^\eta}{\Gamma(\eta+1)} + \frac{\Theta_1 LT^{\eta+1}}{\Gamma(\eta+2)} + \frac{\Theta_1^* LT^{\eta+1}}{\Gamma(\eta+2)}\right)\|\tilde{\chi}_n - \tilde{\chi}\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Φ is implied to be continuous by this.

(ii) Next, we will demonstrate how $C([0,T],\mathbb{R}_\Omega)$ translates bounded set to itself using the operator Φ, i.e., $\forall\kappa>0, \exists$ a $n>0$ and $\forall\tilde{\chi}\in E_\kappa$, we get $\|\Phi(\tilde{\chi})\|\leqslant n$, and E_κ is defined by $E_\kappa = \{\tilde{\chi}\in C([0,T],\mathbb{R}_\Omega) : \|\tilde{\chi}\|\leqslant\kappa\}$. $\forall t\in[0,T]$,

$$\begin{aligned}|\Phi(\tilde{\chi}(t,\theta))| &\leqslant \left|\frac{\beta_2}{\beta_1+\beta_2}\right|\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\chi}(\lambda,\theta))| \\ &\quad + \int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds + \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\chi}(\lambda,\theta))| \\ &\quad + \int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds + \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds\Big]d\lambda \\ &\leqslant \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2(\eta+1) + (\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)LT^\eta\|\tilde{\chi}\| \\ &\leqslant \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2(\eta+1) + (\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)\kappa LT^\eta.\end{aligned}$$

By choosing $n = \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)\kappa LT^\eta$, we have $\|\Phi(\tilde{\chi}(t,\theta))\|\leqslant n$. We get $\|\Phi\tilde{\chi}\|\leqslant n$, the set E_k is implied to be confined by this.

(iii) Φ operators bounded set into equi-continuous sets of $C([0,T],\mathbb{R}_\Omega)$, let $t_1, t_2 \in (0,T]$, and $t_1 < t_2$. $\forall \tilde{\chi} \in E_\kappa$, we get

$$\begin{aligned}
&|\,\Phi(\tilde{\chi}(t_2,\theta) - \Phi(\tilde{\chi}(t_1,\theta))\,| \\
&= \Big|\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-\lambda)^{\eta-1}\left[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\right]d\lambda \\
&\quad - \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-\lambda)^{\eta-1}\left[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\right]d\lambda\Big| \\
&= \Big|\frac{1}{\Gamma(\eta)}\int_0^{t_1}\left((t_2-\lambda)^{\eta-1} - (t_1-\lambda)^{\eta-1}\right)\Big[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds \\
&\quad + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-\lambda)^{\eta-1} \\
&\quad \times\left[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\right]d\lambda\Big| \\
&\leqslant \frac{\Theta_2 L\|\tilde{\chi}\|}{\Gamma(\eta+1)}\left|2(t_2-t_1)^\eta + (t_1^\eta - t_2^\eta)\right| + \frac{\Theta_1 L\|\tilde{\chi}\|}{\Gamma(\eta+2)}\left|2(t_2-t_1)^{\eta+1} + \left(t_1^{\eta+1} - t_2^{\eta+1}\right)\right| \\
&\quad + \frac{\Theta_1^* L\|\tilde{\chi}\|}{\Gamma(\eta+2)}\left|2(t_2-t_1)^{\eta+1} + \left(t_1^{\eta+1} - t_2^{\eta+1}\right)\right| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.
\end{aligned}$$

$|\Phi(\tilde{\chi}(t_2,\theta)) - \Phi(\tilde{\chi}(t_1,\theta))| \to 0$ as $t_1 \to t_2$. This illustrates how the operator converts $C([0,T],\mathbb{R}_\Omega)$ into an equicontinuous set from a bounded set. Consequently, the Arzela-Ascoli theorem states that operator Φ is compact. For the final stage, consider the set ι, which is given by

$$\iota = \{\tilde{\chi} \in C([0,T],\mathbb{R}_\Omega) : \tilde{\chi} = \sigma\Phi(\tilde{\chi}) \text{ for } 0 < \sigma < 1\}.$$

Now that the prior set is bounded, we may demonstrate it. Assume that $\tilde{\chi} \in \iota$. $\forall t \in [0,T]$, we get

$$\begin{aligned}
\tilde{\chi}(t,\theta) = \sigma\Bigg(&\tilde{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) \\
&+ \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \\
&\times\left[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\right]d\lambda\Bigg),
\end{aligned}$$

$\tilde{h}(t,\theta)$ is given in Theorem 3.1. Also, $|\tilde{\varphi}(t,\theta)| \leqslant \Theta_3$ from assumption (H3), we get

$$|\tilde{h}(t,\theta)| \leqslant \frac{|\beta_2|}{|\beta_1+\beta_2|} + \left(1 + \frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\frac{\Theta_3}{\Gamma(\eta+1)} := \Theta_4.$$

Then, $\forall t \in [0,T]$, using $|\mathfrak{I}(\tilde{\chi}(t,\theta))| \leqslant \Theta^*$, we have

$$\begin{aligned}
|\tilde{\chi}(t,\theta)| = \Bigg|\sigma\Bigg(&\tilde{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) \\
&+ \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \\
&\times\left[B(\lambda)\mathfrak{I}(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\mathfrak{I}(\tilde{\chi}(s,\theta))ds\right]d\lambda\Bigg)\Bigg|
\end{aligned}$$

$$\leqslant |\tilde{h}(t,\theta)| + \frac{|\beta_2|}{|\beta_1+\beta_2|}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\chi}(\lambda,\theta))|$$
$$+\int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds + \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\chi}(s,\theta))|ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}$$
$$\times\left[B(\lambda)||\Im(\tilde{\chi}(\lambda,\theta))| + \int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\chi}(s,\theta)) \mid ds + \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\chi}(s,\theta)) \mid ds\right] d\lambda$$
$$\leqslant \Theta_4 + \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\Theta^* T^\eta\right) = n^*,$$

where $n^* = \Theta_4 + \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)\left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)\Theta^* T^\eta$. This establishes the boundedness of every $\tilde{\chi} \in \iota$. The set ι is hence limited.

Additionally, Schaefer's theorem establishes the existence of a fixed point for the Φ. This indicates that there is at least one solution for the FFCV-FIDE (1.1)-(1.2), $\forall t \in [0,T]$, $\tilde{\chi}(t)$. Moreover, we can demonstrate that FFCV-FIDE (1.1)-(1.2) has a unique continuous solution on $[0,T]$ by employing the assumptions in (H1)-(H3),

$$\Upsilon = \frac{(\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T)}{\Gamma(\eta+2)} L T^\eta \left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right) < 1.$$

□

4. Methodology of ADT

This study demonstrates how to approximate the FFCV-FIDE solutions (1.1) ([2, 3]) using an ADT. Examine the FFCV-FIDE (1.1) that follows. The result of applying the I^η operator to both sides of the FFCV-FIDE (1.1) is

$$\tilde{\chi}(t,\theta) = \tilde{\chi}_0 + I^\eta(\tilde{\varphi}(\lambda,\theta)) + I^\eta(B(t)\Im\tilde{\chi}(t,\theta))) + I^\eta\left[\int_0^t \vartheta(t,\lambda)\Im(\tilde{\chi}(\lambda,\theta))d\lambda + \int_0^T \vartheta_1(t,\lambda)\Im(\tilde{\chi}(\lambda,\theta))d\lambda\right].$$

ADT describes the solution $\tilde{\chi}(t,\theta)$ as a series

$$\tilde{\chi}(t,\theta) = \sum_{i=0}^{\infty}\tilde{\chi}_i(t,\theta) \tag{4.1}$$

and M_1, the nonlinear term, is broken down as $M_1 = \sum_{i=0}^{\infty} P_i$, in which the Adomian polynomials P_i are supplied by

$$P_i = \frac{1}{i!}\frac{d^i}{dv^i}\left[M_1\left(\sum_{l=0}^{\infty} v^i\tilde{\chi}_l\right)\right]_{v=0}.$$

Hence,

$$P_0 = M_1(\tilde{\chi}_0), \quad P_1 = \tilde{\chi}_1 M_1'(\tilde{\chi}_0), \quad P_2 = \tilde{\chi}_2 M_1'(\tilde{\chi}_0) + \frac{1}{2}\tilde{\chi}_1^2 M_1''(\tilde{\chi}_0), \ldots.$$

The primary prerequisites for (1.1) are distinguished by their significant role in formulating the solution and their straightforward treatment of the recurrence connections. Next, we create an Adomian equation for the BVP that matches (3.1). The components $\tilde{\chi}_0, \tilde{\chi}_2, \tilde{\chi}_2, \ldots$ are found iteratively by

$$\begin{aligned}\tilde{\chi}(t,\theta) = \tilde{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta))\\ +\int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\\ \times\left[B(\lambda)\Im(\tilde{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\chi}(s,\theta))ds\right] d\lambda.\end{aligned} \tag{4.2}$$

We construct this Adomian equation in such a way that the boundary condition is automatically satisfied by the final solution. The following are obtained recurrence relations:

$$
\begin{aligned}
\tilde{\chi}_0(t,\theta) &= \tilde{h}(t,\theta),\\
&\vdots\\
\tilde{\chi}_{k+1}(t,\theta) &= -\frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\mathfrak{I}\left(\tilde{\chi}_k(\lambda,\theta)\right)+\int_0^\lambda \vartheta(\lambda,s)P_k ds\\
&\quad+\int_0^T \vartheta_1(\lambda,s)P_k ds\Big]d\lambda+\frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[B(\lambda)\mathfrak{I}\left(\tilde{\chi}_k(\lambda,\theta)\right)+\int_0^\lambda \vartheta(\lambda,s)P_k ds\\
&\quad+\int_0^T \vartheta_1(\lambda,s)P_k ds\Big]d\lambda, \quad k\geqslant 0.
\end{aligned}
\tag{4.3}
$$

The above relation is the only one utilized in the solution of BVP. If the series (4.1) is convergent uniformly, we can approximate the solution of FFCV-FIDE (1.1) by solving (4.2) and applying the initial condition, or by solving (4.3) and applying the boundary condition, and receiving the M terms

$$
\Phi_M(t,\theta)=\sum_{i=1}^{M-1}\tilde{\chi}_i(t,\theta). \tag{4.4}
$$

4.1. Convergence analysis

This section describes the convergence of the approximate solution for such an IVP that was previously discussed.

Theorem 4.1. *Let's assume that* (H1)-(H3) *are accurate. Take into consideration* $0<\Upsilon<1$ *as shown in*

$$
\Upsilon=\frac{(\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T)}{\Gamma(\eta+2)}LT^\eta\left(1+\frac{|\beta_2|}{|\beta_1+\beta_2|}\right)<1. \tag{4.5}
$$

Subsequently, the series (4.1) *converges uniformly to the BVP's solution* $\tilde{\chi}(t,\theta)$ *in* (1.1)*. Additionally, an approximate solution to* $\tilde{\chi}(t,\theta)$ *is given by the partial sum* (4.4)*.*

Keep in mind that since $\tilde{\varphi}(t,\theta)\in C(\Psi)$, $\tilde{\chi}_0(t,\theta)\in C(\Psi)$. Thus, for any $t\in\Psi$, there exists $\Theta\in R$ and $\Theta>0$ such that $|\tilde{\chi}_0(t,\theta)|\leqslant\Theta$. We now demonstrate that the i^{th} term in the series (4.1) meets the given requirement,

$$
|\tilde{\chi}_i(t,\theta)|\leqslant\Theta\Upsilon^i \text{ on } \Psi, \tag{4.6}
$$

where Υ was given in (4.5). For $i=1$, we get

$$
\begin{aligned}
|\tilde{\chi}_1(t,\theta)| &= \left|I^\eta\left[\tilde{h}(t,\theta)-\frac{\beta_2}{\beta_1+\beta_2}+B(t)\mathfrak{I}\left(\tilde{\chi}_0(t,\theta)\right)+\int_0^t\vartheta(t,\lambda)\mathfrak{I}\left(\tilde{\chi}_0(\lambda,\theta)\right)d\lambda+\int_0^T\vartheta_1(t,\lambda)\mathfrak{I}\left(\tilde{\chi}_0(\lambda,\theta)\right)d\lambda\right]\right|\\
&\leqslant \Theta_4+\Theta_2L\,|\tilde{\chi}_0(t,\theta)|\,I^\eta(1)+\Theta_1L\,|\tilde{\chi}_0(t,\theta)|\,I^\eta(t)+\Theta_1^*L\,|\tilde{\chi}_0(t,\theta)|\,I^\eta(t)\\
&= \Theta_4+\frac{\Theta_2L}{\Gamma(\eta+1)}|\tilde{\chi}_0(t,\theta)|\,t^\eta+\frac{\Theta_1L}{\Gamma(\eta+2)}|\tilde{\chi}_0(t,\theta)|\,t^{\eta+1}+\frac{\Theta_1^*L}{\Gamma(\eta+2)}|\tilde{\chi}_0(t,\theta)|\,t^{\eta+1}\leqslant\Upsilon\,|\tilde{\chi}_0(t,\theta)|\leqslant\Theta\Upsilon.
\end{aligned}
$$

Here, we consider (4.6) is true for $i=k-1$, i.e., $|\tilde{\chi}_{k-1}(t,\theta)|\leqslant\Theta\Upsilon^{k-1}$. Proceeding in the same way as previously, for $i=k$, we get

$$
\begin{aligned}
|\tilde{\chi}_k(t,\theta)| &= \Big|I^\eta\Big[\tilde{h}(t,\theta)-\frac{\beta_2}{\beta_1+\beta_2}+B(t)\mathfrak{I}\left(\tilde{\chi}_{k-1}(t,\theta)\right)+\int_0^t\vartheta(t,\lambda)\mathfrak{I}\left(\tilde{\chi}_{k-1}(\lambda,\theta)\right)d\lambda\\
&\quad+\int_0^t\vartheta_1(t,\lambda)\mathfrak{I}\left(\tilde{\chi}_{k-1}(\lambda,\theta)\right)d\lambda\Big]\Big|\leqslant\Upsilon\,|\tilde{\chi}_{k-1}(t,\theta)|\leqslant\Theta\Upsilon^k.
\end{aligned}
$$

We find the required outcome at (4.6) as an answer. Consequently, for every $t \in \Psi$,

$$\sum_{i=0}^{\infty} |\tilde{\chi}_i(t,\theta)| \leqslant \sum_{i=0}^{\infty} \Theta \Upsilon^i,$$

$\sum_{i=0}^{\infty} \Theta\Upsilon^i$ is a convergent series for $0 < \Upsilon < 1$. Then, $\sum_{i=0}^{\infty} \tilde{\chi}_i(t,\theta)$ is converges uniformly to the Weierstrass M-test. Therefore, again using the Weierstrass M-test, there is a uniform convergence of the series (4.1). As a result, the partial sum in (4.4) approximates the answer to (1.1).

5. An example

Consider the following FFCV-FIDE:

$$D^{\frac{1}{2}}\tilde{\chi}(t,\theta) = \tilde{\varphi}(t,\theta) + B(t)\tilde{\chi}(t,\theta) + \int_0^t \vartheta(t,\lambda)\tilde{\chi}(\lambda,\theta)d\lambda + \int_0^1 \vartheta_1(t,\lambda)\tilde{\chi}(\lambda,\theta)d\lambda,$$
$$2\tilde{\chi}(0,\theta) + \tilde{\chi}(1,\theta) = \tilde{\beta_3},$$

here, $\beta_1 = 2$, $\beta_2 = 1$, $\tilde{\beta_3} = (\theta-1, 1-\theta)$, where $\tilde{\varphi}(t,\theta) = 3t(\theta-1, 1-\theta)$, $B(t) = \frac{-(t^3)}{10}$, $\vartheta(t,\lambda) = \frac{-3\lambda t}{10}$, and $\vartheta_1(t,\lambda) = \frac{-6\lambda t}{10}$. Apply $I^{1/2}$ in Eq. (5.1). The preceding Eq. (5.1) has an equivalent form under type (i)-differentiability, which is

$$\begin{aligned}
\underline{\chi}(t,\theta) &= \underline{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\underline{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\underline{\chi}(s,\theta))ds \\
&\quad + \int_0^1 \vartheta_1(\lambda,s)\Im(\underline{\chi}(s,\theta))ds\Big] d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \\
&\quad \times \Big[B(\lambda)\Im(\underline{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\underline{\chi}(s,\theta))ds + \int_0^1 \vartheta_1(\lambda,s)\Im(\underline{\chi}(s,\theta))ds\Big] d\lambda \\
\bar{\chi}(t,\theta) &= \bar{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\bar{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\bar{\chi}(s,\theta))ds \\
&\quad + \int_0^1 \vartheta_1(\lambda,s)\Im(\bar{\chi}(s,\theta))ds\Big] d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \\
&\quad \times \Big[B(\lambda)\Im(\bar{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\bar{\chi}(s,\theta))ds + \int_0^1 \vartheta_1(\lambda,s)\Im(\bar{\chi}(s,\theta))ds\Big] d\lambda.
\end{aligned} \tag{5.1}$$

Let's now build $\underline{\chi}(t,\theta)$ as

$$\begin{aligned}
\underline{\chi}(t,\theta) &= \underline{h}(t,\theta) - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\underline{\chi}(\lambda,\theta)) \\
&\quad . + \int_0^\lambda \vartheta(\lambda,s)\Im(\underline{\chi}(s,\theta))ds + \int_0^1 \vartheta_1(\lambda,s)\Im(\underline{\chi}(s,\theta))ds\Big] d\lambda \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\underline{\chi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\underline{\chi}(s,\theta))ds + \int_0^1 \vartheta_1(\lambda,s)\Im(\underline{\chi}(s,\theta))ds\Big] d\lambda,
\end{aligned}$$

where

$$\tilde{h}(t,\theta) = \frac{\tilde{\beta_3}}{\beta_1+\beta_2} - \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^1 (1-\lambda)^{\eta-1}\tilde{\varphi}(\lambda,\theta)d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\tilde{\varphi}(\lambda,\theta)d\lambda,$$

here $\tilde{\beta_3} = (\theta-1, 1-\theta)$ is fuzzy value, $\beta_1 = 2$, and $\beta_2 = 1$,

$$\underline{h}(t,\theta) = \frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)}, \quad \bar{h}(t,\theta) = \frac{(1-\theta)}{3} - \frac{(1-\theta)}{\Gamma(5/2)} + \frac{3t^{3/2}(1-\theta)}{\Gamma(5/2)}.$$

We are now using the ADT,

$$\begin{aligned}
&\underline{\chi}_0(t,\theta) = \underline{h}(t,\theta),\\
&\underline{\chi}_0(t,\theta) = \underline{h}(t,\theta) = \frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)},\\
&\underline{\chi}_1(t,\theta) = -\frac{1}{3\Gamma(1/2)}\int_0^1 (1-\lambda)^{1/2}\left[\left(\frac{-t^3}{10}\right)\left(\frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)}\right)\right.\\
&\qquad\left. + \int_0^\lambda \left(\frac{-3\lambda t}{10}\right)\left(\frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)} + \frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) ds\right] d\lambda\\
&\qquad + \frac{1}{\Gamma(1/2)}\int_0^t (t-\lambda)^{1/2}\left[\left(\frac{-t^3}{10}\right)\left(\frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)} + \frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)}\right)\right.\\
&\qquad\left. + \int_0^\lambda \left(\frac{-3\lambda t}{10}\right)\left(\frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)} + \frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) ds\right] d\lambda,\\
&\tilde{\chi}_{k+1}(t,\theta) = -\frac{\beta_2}{\beta_1+\beta_2}\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\left[B(\lambda)\Im(\tilde{\chi}_k(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}_k(s,\theta))\, ds\right.\\
&\qquad\left. + \int_0^1 \vartheta_1(\lambda,s)\Im(\tilde{\chi}_k(s,\theta))\, ds\right] d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\chi}_k(\lambda,\theta))\\
&\qquad + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\chi}_k(s,\theta))\, ds + \int_0^1 \vartheta_1(\lambda,s)\Im(\tilde{\chi}_k(s,\theta))\, ds\Big] d\lambda.
\end{aligned}$$

In a similar manner, we may locate consecutive words and obtain the solution

$$\underline{\chi} = \sum_{n=0}^{\infty} \underline{\chi}_n = \frac{(\theta-1)}{3} - \frac{(\theta-1)}{\Gamma(5/2)} + \frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)} + \frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)} + \cdots.$$

Similarly, we may locate

$$\bar{\chi} = \sum_{n=0}^{\infty} \underline{\chi}_n = \frac{(1-\theta)}{3} - \frac{(1-\theta)}{\Gamma(5/2)} + \frac{3t^{3/2}(1-\theta)}{\Gamma(5/2)} + \frac{6t^{3/2}(1-\theta)}{\Gamma(5/2)} + \cdots.$$

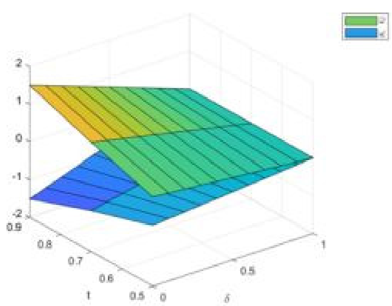

Figure 1: Approximate solutions for uncertainty θ and space variable t.

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