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Analysis of a parametric delay functional differential equation with nonlocal integral condition

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Abstract

This paper analyzes a nonlocal problem of a delay functional-differential equation with parameters. We confirm that there is at least one solution $x \in AC[0, T]$ to the problem. Furthermore, we provide the hypotheses that must be fulfilled for the solution's uniqueness. The analysis also implements the Hyers-Ulam stability of the problem and the continuous dependence of the unique solution on some parameters. We provide some exceptional cases and examples to illustrate our findings.

Keywords: Delay functional-differential equation, nonlocal condition, existence of solutions, Schauder fixed point theorem, Hyers-Ulam stability, continuous dependence.

2020 MSC: 34K20, 47H10, 34K05, 34A12.

1. Introduction

Functional equations have been extensively studied by numerous authors [7, 19, 41, 57], influencing and benefiting many fields with their use and techniques. Their remarkable applications have driven growth and development in various areas, including not only mathematics but also science, engineering, economics, epidemiology, computer science, biology, social and behavioral sciences [6, 16] . Understanding the future behavior of a particular phenomenon requires a thorough understanding of functional differential equations [17, 33, 43, 51]. Particularly, a family of mathematical models including time delays and parameters represented by parametric delay functional-differential equations finds wide use in simulating real-world processes which has been discussed by numerous authors (see [29, 31, 42, 50, 63]). Delay functions are widely employed to model the evolution of propagation and population dynamics [45, 48]. Economic systems, for instance, naturally involve delays due to decisions such as investment strategies and the dynamics of commodity markets spread over time periods [44, 46]. In particular, Dvalishvili et al. [23] construct a market relations model based on a controlled delay functional-differential equation.

Stability analysis is a highly representative field in mathematical sciences [10, 64]. In order to model a physical process, an equation or problem can be used if a small alteration to it results in a corresponding

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small alteration in the outcome. When this occurs, the equation or problem is said to be stable. The concept of stability in a functional equation emerges when the equation is replaced by an inequality, serving as a perturbation. The key inquiry of stability is: how do the solutions to inequality differ from those of the original functional equation ([35])? In a lecture delivered at the University of Wisconsin in 1940, Ulam [61, 62] initially introduced the topic of stability in functional equations by asking the following question: Given a group $(G, .)$, a metric group $(H, *)$ with metric $\rho(.,.)$, and a positive number ϵ, the question asks whether there exists a positive number $\delta > 0$ such that if a mapping $\psi : G \to H$ satisfies the condition

$$\rho(\psi(x.y), \psi(x) * \psi(y)) \leqslant \delta, \ \forall x, y \in G,$$

then there exists a homomorphism $\phi : G \to H$ such that

$$\rho(\psi(x), \phi(x)) \leqslant \epsilon, \ \forall x \in G?$$

In simpler terms, the question is whether a small deviation (controlled by δ) from the homomorphism property of ψ implies the existence of a homomorphism ϕ that closely approximates ψ within ϵ. If so, we classify the functional equation for homomorphisms as stable [36]. Hyers [34] then offered a partial solution for the problem in the context of Banach space in 1941 under the assumption that G and H are Banach spaces with $\delta = \epsilon$ and $\phi(x) = \lim_{n\to\infty} 2^{-n}\psi(2^n x)$. Subsequently, Rassias [53, 54] extended this conceptual framework by including variables in 1978, leading to its designation as Hyers-Ulam-Rassias stability.

The Hyers-Ulam stability of a differential equation was initially studied by Alsina and Ger [8] in 1998, they dealt with the differential equation $y'(t) = y(t)$. Furthermore, in 2004, Jung [37] has explored the Hyers-Ulam stability of the first order differential equation $\phi(t)y'(t) = y(t)$, he has published multiple papers concerning this type of equations (see [38, 39]). In the years from 2010 to 2015, several authors investigated the Hyers-Ulam stability of second and third order differential equations (see [2–4, 32, 47]). Recent research has implemented the Hyers-Ulam stability of various types of differential equations, including hypergeometric and Laguerre differential equations examined by Abdollahpour et al. [1, 5], as well as integro-differential equations investigated by Tunç et al. [59, 60].

Continuous dependence [49, 55], another important concept in stability theory, addresses the behavior of solutions in mathematical problems under varying conditions. It ensures that small changes in the initial conditions or parameters of a problem result in correspondingly small changes in the solution.

Nonlocal problems have been extensively studied by several authors in the last two decades (see [11, 24, 26, 30, 52, 56]), which is essential in the representation of real-life scenarios through mathematical models. Ensuring the trustworthiness of these models involves integrating the principles of Hyers-Ulam stability and continuous dependence. Hyers-Ulam stability assesses the problem's resilience to disturbances, while continuous dependence examines how small variations in parameters affect the unique solution of the problem.

The solvability of problems involving functional-differential equations has been analyzed using a variety of techniques, such as operator theory and fixed-point theorems. One approach involves formulating the problem as a fixed-point problem and applying the Schauder fixed-point theorem to demonstrate the existence of solutions. This method has been extensively explored in numerous publications and monographs (see [14, 15, 20]). Notably, Boucherif and Precup [12] investigated the existence of solutions to the nonlocal problem for the first-order differential equation

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big), \text{ a.e. } t \in [0,1], \quad x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0,$$

where t_k are given points with $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m < 1$ and $1 + \sum_{k=1}^{m} a_k \neq 0$. Furthermore, the authors in [27] studied the existence of at least one solution $x \in AC[0,1]$ of the nonlocal problem of the functional-differential equation

$$\frac{dx(t)}{dt} = f\left(t, \frac{dx(t)}{dt}\right), \text{ a.e. } t \in (0, 1],$$

with the nonlocal integral condition

$$x(0) + \int_0^1 x(s)ds = x_0.$$

Recently, El-Sayed et al. implemented the concepts of Hyers-Ulam stability and continuous dependence to confirm the stability of a functional integro-differential equation with a quadratic functional integro-differential constraint [25] and a delay tempered-fractal differential equation [28]. Additionally, András [9] applied the Hyers-Ulam stability for first-order differential systems with nonlocal initial conditions. Furthermore, Tunç and Biçer [58] applied the Hyers-Ulam-Rassias and the Hyers-Ulam stability for the first order delay functional differential equation of the form $x^{'}(t) = f(t, x(t), x(t-\tau))$, $\tau > 0$.

Motivated by the above results, we study the solvability of the nonlocal problem of the parametric delay functional-differential equation

$$\frac{dx(t)}{dt} = \sum_{i=1}^{k} f_i\left(t, \lambda_i \frac{d}{dt} x(\phi_i(t))\right), \text{ a.e. } t \in (0, T], \tag{1.1}$$

with the nonlocal integral condition

$$x(0) + \int_0^T g\left(s, x(s), \frac{dx(s)}{ds}\right) ds = x_0, \tag{1.2}$$

where $\lambda_i > 0$ are parameters, ϕ_i are delay functions, $i = 1, 2, \dots, k$, and x_0 is the initial data.

Our aim in this paper is to analyze the existence of solutions to the nonlocal problem (1.1)-(1.2) under suitable conditions, then we study the uniqueness of the solution. Additionally, we implement the Hyers-Ulam stability of the problem, identifying its resistance to perturbations. Moreover, we investigate the continuous dependence of the unique solution on the functions f_i, parameters λ_i, the initial data x_0, and the function g. Finally, in order to demonstrate our insights, we provide a few instances and special cases. The Schauder fixed-point theorem is applied in this work to determine the hypotheses for the solution's existence and uniqueness.

2. Main result

2.1. Formulation of the problem

Consider the nonlocal problem (1.1)-(1.2) under the following assumptions.

(i) $f_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition [18], i.e., it is measurable in $t \in [0, T]$ for all $x \in \mathbb{R}$ and continuous in $x \in \mathbb{R}$ for almost all $t \in [0, T]$.

(ii) There exist integrable functions $a_i \in L^1[0, T]$ and constants $b_i > 0$ such that $|f_i(t, x)| \leqslant |a_i(t)| + b_i|x|$.

(iii) $\phi_i : [0, T] \to [0, T]$ is continuous and increasing function such that $\phi_i(t) \leqslant t$.

(iv) $\sum_{i=1}^{k} b_i\lambda_i < 1$.

(v) $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, i.e., it is measurable in $t \in [0, T]$ for all $x, y \in \mathbb{R}$ and continuous in $x, y \in \mathbb{R}$ for almost all $t \in [0, T]$.

(vi) There exists a function $h \in L^1[0, T]$ and a constant $L > 0$ such that $|g(t, x, y)| \leqslant |h(t)| + L\big(|x| + |y|\big)$.

(vii) $LT < 1$.

The following lemma shows the equivalence between the problem (1.1)-(1.2) with its corresponding integral and functional equation.

Lemma 2.1. *Let x be a solution of* (1.1)-(1.2)*, then it can be given by the integral equation*

$$x(t) = x_0 - \int_0^T g\big(s, x(s), y(s)\big)ds + \int_0^t y(s)ds, \ t \in [0,T], \tag{2.1}$$

where $y(t)$ is the solution of the functional equation

$$y(t) = \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big), \ t \in [0,T]. \tag{2.2}$$

Proof. Let x be a solution of (1.1)-(1.2) and $\frac{dx(t)}{dt} = y \in L^1[0,T]$, then

$$x(t) = x(0) + \int_0^t y(s)ds,$$

using (1.2), we obtain (2.1)

$$x(t) = x_0 - \int_0^T g\big(s, x(s), y(s)\big)ds + \int_0^t y(s)ds \in AC[0,T],$$

and

$$x(\phi_i(t)) = x_0 - \int_0^T g\big(s, x(s), y(s)\big)ds + \int_0^{\phi_i(t)} y(s)ds,$$

hence

$$\frac{d}{dt}x(\phi_i(t)) = \phi_i'(t)y(\phi_i(t)), \tag{2.3}$$

using (2.3) in (1.1), we obtain (2.2)

$$y(t) = \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big), \ t \in [0,T].$$

Also, we can get back to (1.1)-(1.2) by differentiating (2.1) and using (2.2) and (2.3) as follows

$$\frac{dx(t)}{dt} = y(t), \text{ a.e. } t \in (0,T] = \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) = \sum_{i=1}^{k} f_i\Big(t, \lambda_i \frac{d}{dt}x\big(\phi_i(t)\big)\Big),$$

and the nonlocal integral condition (1.2) holds when substituting $t = 0$ and $y(s) = \frac{dx(s)}{ds}$ in (2.1). This proves the equivalence between the problem (1.1)-(1.2) to (2.1)-(2.2). □

2.2. Existence of solutions

In this part, we demonstrate the existence of at least one absolutely continuous solution $x \in AC[0,T]$ of (1.1)-(1.2). For this objective, we provide the following theorems.

Theorem 2.2. *Let the assumptions* (i)-(iv) *be satisfied, then* (2.2) *has at least one solution $y \in L^1[0,T]$.*

Proof. Define the closed ball Q_{r_1} and the operator F_1 associated with (2.2) by

$$Q_{r_1} := \big\{y \in \mathbb{R} : \|y\|_{L_1} \leqslant r_1\big\} \subset L^1[0,T], \text{ where } r_1 = \frac{\sum_{i=1}^{k} \|a_i\|_{L^1}}{1 - \sum_{i=1}^{k} b_i \lambda_i},$$

and

$$F_1 y(t) = \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big), \ t \in [0,T].$$

It is clear that Q_{r_1} is a nonempty, closed, bounded, and convex subset of $L^1[0,T]$. Let $y \in Q_{r_1}$, then for $t \in [0,T]$, we get

$$\begin{aligned}
|F_1 y(t)| &= \left| \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) \right| \\
&\leqslant \sum_{i=1}^{k} \left| f_i(t, \lambda_i \phi_i'(t) y(\phi_i(t))) \right| \\
&\leqslant \sum_{i=1}^{k} \Big[|a_i(t)| + b_i |\lambda_i \phi_i'(t) y(\phi_i(t))| \Big] \\
&= \sum_{i=1}^{k} |a_i(t)| + \sum_{i=1}^{k} b_i \lambda_i \phi_i'(t) |y(\phi_i(t))|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|F_1 y\|_{L^1} &:= \int_0^T |F_1 y(t)| dt \\
&\leqslant \int_0^T \sum_{i=1}^{k} |a_i(t)| dt + \int_0^T \sum_{i=1}^{k} b_i \lambda_i \phi_i'(t) |y(\phi_i(t))| dt \\
&= \sum_{i=1}^{k} \int_0^T |a_i(t)| dt + \sum_{i=1}^{k} \int_0^T b_i \lambda_i \phi_i'(t) |y(\phi_i(t))| dt \\
&= \sum_{i=1}^{k} \|a_i\|_{L^1} + \sum_{i=1}^{k} \left\{ b_i \lambda_i \int_0^T \phi_i'(t) |y(\phi_i(t))| dt \right\}.
\end{aligned}$$

Putting $\phi_i(t) = u \implies \phi_i'(t) dt = du$, then

$$\begin{aligned}
\|F_1 y\|_{L^1} &\leqslant \sum_{i=1}^{k} \|a_i\|_{L^1} + \sum_{i=1}^{k} b_i \lambda_i \int_{\phi_i(0)}^{\phi_i(T)} \phi_i'(t) |y(u)| \frac{du}{\phi_i'(t)} \\
&\leqslant \sum_{i=1}^{k} \|a_i\|_{L^1} + \sum_{i=1}^{k} b_i \lambda_i \int_0^T |y(u)| du \\
&= \sum_{i=1}^{k} \|a_i\|_{L^1} + \sum_{i=1}^{k} b_i \lambda_i \|y\|_{L^1} \leqslant \sum_{i=1}^{k} \|a_i\|_{L^1} + r_1 \sum_{i=1}^{k} b_i \lambda_i = r_1.
\end{aligned}$$

This proves that $F_1 : Q_{r_1} \to Q_{r_1}$ and the class $\{F_1 y(t)\}$ is uniformly bounded on Q_{r_1}. Let $y \in \Omega \subset Q_{r_1}$, then

$$\begin{aligned}
\|(F_1 y)_h - (F_1 y)\|_{L^1} &= \int_0^T |(F_1 y(t))_h - (F_1 y(t))| dt \\
&= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (F_1 y(\theta)) d\theta - (F_1 y(t)) \right| dt \\
&\leqslant \int_0^T \frac{1}{h} \int_t^{t+h} \left| (F_1 y(\theta)) - (F_1 y(t)) \right| d\theta dt \\
&= \int_0^T \frac{1}{h} \int_t^{t+h} \left| \sum_{i=1}^{k} f_i\big(\theta, \lambda_i \phi_i'(\theta) y(\phi_i(\theta))\big) - \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) \right| d\theta dt
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h}\sum_{i=1}^{k}\left|f_i\big(\theta,\lambda_i\phi_i'(\theta)y(\phi_i(\theta))\big)-f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)\right|d\theta dt\\
&=\sum_{i=1}^{k}\int_0^T \frac{1}{h}\int_t^{t+h}\left|f_i\big(\theta,\lambda_i\phi_i'(\theta)y(\phi_i(\theta))\big)-f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)\right|d\theta dt.
\end{aligned}$$

Using assumptions (i)-(ii), it follows that $f \in L^1[0,T]$, then

$$\frac{1}{h}\int_t^{t+h}\left|f_i\big(\theta,\lambda_i\phi_i'(\theta)y(\phi_i(\theta))\big)-f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)\right|d\theta \to 0 \ \text{ as } \ h\to 0.$$

This yields that $(F_1y(t))_h \to (F_1y(t))$ uniformly in $L^1[0,T]$. Thus, by Kolmogorov compactness criterion [21], $F_1(\Omega)$ is relatively compact, hence F_1 is compact operator. Let $\{y_n\}\subset Q_{r_1}$ such that $y_n \to y$, then

$$F_1y_n(t)=\sum_{i=1}^{k} f_i\big(t,\lambda_i\phi_i'(t)y_n(\phi_i(t))\big),\ n\in\mathbb{N},$$

and

$$\begin{aligned}
\lim_{n\to\infty} F_1y_n(t) &= \lim_{n\to\infty}\sum_{i=1}^{k} f_i\big(t,\lambda_i\phi_i'(t)y_n(\phi_i(t))\big)\\
&=\sum_{i=1}^{k}\lim_{n\to\infty} f_i\big(t,\lambda_i\phi_i'(t)y_n(\phi_i(t))\big)\\
&=\sum_{i=1}^{k} f_i\big(t,\lambda_i\phi_i'(t)\lim_{n\to\infty} y_n(\phi_i(t))\big)\\
&=\sum_{i=1}^{k} f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)=F_1y(t).
\end{aligned}$$

Thus, F_1 is continuous operator. Now all conditions of the Schauder fixed point Theorem [20] are satisfied, then F_1 has at least one fixed point $y\in Q_{r_1}$, hence (2.2) has at least one solution $y\in L^1[0,T]$. □

Theorem 2.3. *Let the assumptions* (i)-(vii) *be satisfied, then* (2.1) *has at least one continuous solution* $x\in C[0,T]$. *Consequently,* (1.1)-(1.2) *has at least one solution* $x\in AC[0,T]$.

Proof. Define the closed ball Q_{r_2} and the operator F_2 associated with (2.1) by

$$Q_{r_2} := \left\{x\in\mathbb{R} : \|x\|_C \leqslant r_2\right\}\subset C[0,T],\ \text{where } r_2=\frac{|x_0|+\|h\|_{L^1}+(L+1)r_1}{1-LT},$$

and

$$F_2x(t)=x_0-\int_0^T g\big(s,x(s),y(s)\big)ds+\int_0^t y(s)ds,\ t\in[0,T].$$

It is clear that Q_{r_2} is a nonempty, closed, bounded, and convex subset of $C[0,T]$. Let $x\in Q_{r_2}$, then for $t\in[0,T]$, we get

$$\begin{aligned}
|F_2x(t)| &= \left|x_0-\int_0^T g\big(s,x(s),y(s)\big)ds+\int_0^t y(s)ds\right|\\
&\leqslant |x_0|+\int_0^T \left|g\big(s,x(s),y(s)\big)\right|ds+\int_0^t |y(s)|ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant |x_0| + \int_0^T \big[|h(s)| + L|x(s) + y(s)|\big]\, ds + \int_0^T |y(s)|ds \\
&\leqslant |x_0| + \int_0^T |h(s)|ds + L\int_0^T |x(s)|ds + L\int_0^T |y(s)|ds + \int_0^T |y(s)|ds \\
&\leqslant |x_0| + ||h||_{L^1} + L\int_0^T \sup_{s\in[0,T]} |x(s)|ds + L||y||_{L^1} + ||y||_{L^1} \\
&= |x_0| + ||h||_{L^1} + LT||x||_C + (L+1)||y||_{L^1}.
\end{aligned}$$

Then, we have

$$||F_2x||_C \leqslant |x_0| + ||h||_{L^1} + LTr_2 + (L+1)r_1 = r_2.$$

This proves that the class $\{F_2x(t)\}$ is uniformly bounded on Q_{r_2}. Let $x \in Q_{r_2}$ and $t_1, t_2 \in [0, T]$, where $t_2 > t_1$ and $|t_2 - t_1| \leqslant \delta$, thus

$$\begin{aligned}
|F_2x(t_2) - F_2x(t_1)| &= \left| x_0 - \int_0^T g\big(s, x(s), y(s)\big)\, ds + \int_0^{t_2} y(s)ds - x_0 + \int_0^T g\big(s, x(s), y(s)\big)\, ds - \int_0^{t_1} y(s)ds \right| \\
&\leqslant \int_{t_1}^{t_2} |y(s)|ds \leqslant \epsilon.
\end{aligned}$$

This indicates that $F_2 : Q_{r_2} \to Q_{r_2}$ and the class $\{F_2x(t)\}$ is equi-continuous on Q_{r_2}. Thus, by the Arzela-Ascoli Theorem [13], $\{F_2x(t)\}$ is relatively compact, hence F_2 is compact operator. Let $\{x_n\} \subset Q_{r_2}$ such that $x_n \to x$, then

$$F_2x_n(t) = x_0 - \int_0^T g\big(s, x_n(s), y(s)\big)\, ds + \int_0^t y(s)ds, \ n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} F_2x_n(t) = x_0 - \lim_{n\to\infty} \int_0^T g\big(s, x_n(s), y(s)\big)\, ds + \int_0^t y(s)ds.$$

Using the Lebesgue dominated convergence Theorem [22] and assumptions (v)-(vi), we have

$$\begin{aligned}
\lim_{n\to\infty} F_2x_n(t) &= x_0 - \int_0^T \lim_{n\to\infty} g\big(s, x_n(s), y(s)\big)\, ds + \int_0^t y(s)ds \\
&= x_0 - \int_0^T g\big(s, \lim_{n\to\infty} x_n(s), y(s)\big)\, ds + \int_0^t y(s)ds \\
&= x_0 - \int_0^T g\big(s, x(s), y(s)\big)\, ds + \int_0^t y(s)ds = F_2x(t).
\end{aligned}$$

Thus, F_2 is a continuous operator. Then, by the Schauder fixed point Theorem, F_2 has at least one fixed point $x \in Q_{r_2}$, hence (2.1) has at least one solution $x \in C[0, T]$. Consequently, by Lemma 2.1, it follows that (1.1)-(1.2) has at least one solution $x \in AC[0, T]$, which completes the proof. □

3. Stability analysis of the problem

3.1. *Uniqueness of solution*

At this point, we confirm the existence of a unique solution of (1.1)-(1.2). To perform this, we require the following additional hypotheses.

(i)′ $f_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in [0, T]$ and satisfies the Lipschitz condition in $x \in \mathbb{R}$ such that

$$|f_i(t, x) - f_i(t, y)| \leqslant b_i|x - y| \text{ with constant } b_i > 0.$$

(ii)′ $f_i(t,0) \in L^1[0,T]$.

(iii)′ $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to R$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition in $x, y \in \mathbb{R}$ such that

$$|g(t,x,y) - g(t,v,w)| \leqslant L\big(|x-v| + |y-w|\big) \text{ with constant } L > 0.$$

(iv)′ $g(t,0,0) \in L^1[0,T]$.

Theorem 3.1. *Let the hypotheses* (iii)-(iv) *(of Theorem* 2.2*) and* (i)′-(ii)′ *be satisfied, then the solution of* (2.2)*,* $y \in L^1[0,T]$*, is unique.*

Proof. Hypotheses (i)-(ii) of Theorem 2.2 can be deduced from (i)′ and (ii)′ as follows, putting $y = 0$ in (i)′, we get $|f_i(t,x)| \leqslant b_i|x| + |f_i(t,0)|$, where $a_i(t) = f_i(t,0) \in L^1[0,T]$. Hence, we deduce that all assumptions of Theorem 2.2 are satisfied and (2.2) has at least one solution $y \in L^1[0,T]$. Now let u, v be two solutions of (2.2), then

$$\begin{aligned}
|u(t) - v(t)| &= \left|\sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) u(\phi_i(t))\big) - \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) v(\phi_i(t))\big)\right| \\
&\leqslant \sum_{i=1}^{k} \left|f_i\big(t, \lambda_i \phi_i'(t) u(\phi_i(t))\big) - f_i\big(t, \lambda_i \phi_i'(t) v(\phi_i(t))\big)\right| \\
&\leqslant \sum_{i=1}^{k} b_i \left|\lambda_i \phi_i'(t) u(\phi_i(t)) - \lambda_i \phi_i'(t) v(\phi_i(t))\right| \\
&= \sum_{i=1}^{k} b_i \lambda_i \phi_i'(t) \left|u(\phi_i(t)) - v(\phi_i(t))\right|.
\end{aligned}$$

Thus

$$\|u - v\|_{L^1} \leqslant \sum_{i=1}^{k} b_i \lambda_i \|u - v\|_{L^1}.$$

Since $\sum_{i=1}^{k} b_i \lambda_i < 1$, hence $u = v$ and the solution of (2.2), $y \in L^1[0,T]$, is unique. □

Theorem 3.2. *Let the hypotheses* (iii)-(iv) *and* (vii) *(of Theorem* 2.3*) and* (i)′-(iv)′ *be satisfied, then the solution of* (1.1)-(1.2)*,* $x \in AC[0,T]$*, is unique.*

Proof. Hypotheses (v)-(vi) of Theorem 2.3 can be deduced from (iii)′ and (iv)′ as follows, putting $v = w = 0$ in (iii)′, we get $|g(t,x,y)| \leqslant |g(t,0,0)| + L(|x| + |y|)$, where $h(t) = g(t,0,0) \in L^1[0,T]$. Hence, we deduce that all assumptions of Theorem 2.3 are satisfied and (2.1) has at least one solution $x \in C[0,T]$. Now let x_1, x_2 be two solutions of (2.1), then

$$\begin{aligned}
|x_2(t) - x_1(t)| &= \left|x_0 - \int_0^T g\big(s, x_2(s), y(s)\big) ds + \int_0^t y(s) ds - x_0 + \int_0^T g\big(s, x_1(s), y(s)\big) ds - \int_0^t y(s) ds\right| \\
&\leqslant \int_0^T \left|g\big(s, x_2(s), y(s)\big) - g\big(s, x_1(s), y(s)\big)\right| ds \leqslant L \int_0^T |x_2(s) - x_1(s)| ds.
\end{aligned}$$

Thus

$$\|x_2 - x_1\|_C \leqslant LT \|x_2 - x_1\|_C.$$

Since $LT < 1$, hence $x_1 = x_2$ and the solution of (2.1), $x \in C[0,T]$, is unique. Consequently, the solution of (1.1)-(1.2), $x \in AC[0,T]$, is unique, which completes the proof. □

3.2. Hyers-Ulam stability

Here, we implement the theory of Hyers-Ulam stability for the problem (1.1)-(1.2) attached with (2.1)-(2.2).

Definition 3.3 ([25, 28, 40]). Let the solution $x \in AC[0,T]$ of (1.1)-(1.2) exists uniquely. The nonlocal problem (1.1)-(1.2) is Hyers-Ulam stable, if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that for any solution $x_s \in AC[0,T]$ of (1.1)-(1.2) satisfying the inequality

$$\left| \frac{dx_s(t)}{dt} - \sum_{i=1}^{k} f_i\left(t, \lambda_i \frac{d}{dt} x_s(\phi_i(t))\right) \right| \leqslant \delta,$$

then $\|x - x_s\|_C \leqslant \epsilon$.

Theorem 3.4. *Let the hypotheses of Theorem 3.2 be satisfied, then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof. Let $\left| \frac{dx_s(t)}{dt} - \sum_{i=1}^{k} f_i\left(t, \lambda_i \frac{d}{dt} x_s(\phi_i(t))\right) \right| \leqslant \delta$, then

$$-\delta \leqslant \frac{dx_s(t)}{dt} - \sum_{i=1}^{k} f_i\left(t, \lambda_i \phi_i'(t) \frac{dx_s(\phi_i(t))}{d(\phi_i(t))}\right) \leqslant \delta, \qquad -\delta \leqslant y_s(t) - \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y_s(\phi_i(t))\big) \leqslant \delta.$$

Now consider

$$\begin{aligned}
&|y(t) - y_s(t)| \\
&= \left| \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) - y_s(t) \right| \\
&= \left| \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) - y_s(t) - \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y_s(\phi_i(t))\big) + \sum_{i=1}^{k} f_i(t, \lambda_i \phi_i'(t) y_s(\phi_i(t))) \right| \\
&\leqslant \sum_{i=1}^{k} \left| f_i\big(t, \lambda_i \phi_i'(t) y(\phi_i(t))\big) - f_i\big(t, \lambda_i \phi_i'(t) y_s(\phi_i(t))\big) \right| + \left| \sum_{i=1}^{k} f_i\big(t, \lambda_i \phi_i'(t) y_s(\phi_i(t))\big) - y_s(t) \right| \\
&\leqslant \sum_{i=1}^{k} b_i \left| \lambda_i \phi_i'(t) y(\phi_i(t)) - \lambda_i \phi_i'(t) y_s(\phi_i(t)) \right| + \delta = \sum_{i=1}^{k} b_i \lambda_i \phi_i'(t) \left| y(\phi_i(t)) - y_s(\phi_i(t)) \right| + \delta.
\end{aligned}$$

Thus

$$\|y - y_s\|_{L^1} \leqslant \sum_{i=1}^{k} b_i \lambda_i \|y - y_s\|_{L^1} + \delta T,$$

and

$$\|y - y_s\|_{L^1} \leqslant \frac{\delta T}{1 - \sum_{i=1}^{k} b_i \lambda_i}.$$

Since $\sum_{i=1}^{k} b_i \lambda_i < 1$, then $\|y - y_s\|_{L^1} \leqslant \epsilon^*$. Now

$$\begin{aligned}
|x(t) - x_s(t)| = \Bigg| x_0 - \int_0^T g\big(s, x(s), y(s)\big) ds + \int_0^t y(s) ds \\
- x_0 + \int_0^T g\big(s, x_s(s), y_s(s)\big) ds - \int_0^t y_s(s) ds \Bigg| \\
\leqslant \int_0^T \left| g\big(s, x(s), y(s)\big) - g\big(s, x_s(s), y_s(s)\big) \right| ds + \int_0^T \left| y(s) - y_s(s) \right| ds
\end{aligned}$$

$$\leqslant L\int_0^T \left[|x(s)-x_s(s)|+|y(s)-y_s(s)|\right]ds+\|y-y_s\|_{L^1}$$
$$\leqslant LT\|x-x_s\|_C+(L+1)\|y-y_s\|_{L^1}.$$

Then

$$\|x-x_s\|_C\leqslant LT\|x-x_s\|_C+(L+1)\epsilon^*,$$

and

$$\|x-x_s\|_C\leqslant \frac{(L+1)\epsilon^*}{1-LT}.$$

Since $LT<1$, then $\|x-x_s\|_C\leqslant \epsilon$. So, the problem (1.1)-(1.2) is Hyers-Ulam stable. □

3.3. Continuous dependency results

This portion investigates the continuous dependence of the unique solution of (1.1)-(1.2) on the functions f_i, parameters λ_i, the initial data x_0, and the function g.

Definition 3.5. The solution $x\in AC[0,T]$ of (1.1)-(1.2) depends continuously on the function $y\in L^1[0,T]$, if $\forall\epsilon>0$, $\exists\delta(\epsilon)>0$ such that $\|y-y^*\|_{L^1}\leqslant\delta\Rightarrow\|x-x^*\|_C\leqslant\epsilon$, where x^* is the unique solution of the integral equation

$$x^*(t)=x_0-\int_0^T g\big(s,x^*(s),y^*(s)\big)ds+\int_0^t y^*(s)ds,\ t\in[0,T]. \tag{3.1}$$

Theorem 3.6. *Let the hypotheses of Theorem 3.2 be fulfilled, then the solution $x\in AC[0,T]$ of (1.1)-(1.2) depends continuously on the function y.*

Proof. Let x and x^* be the two solutions of (2.1) and (3.1), respectively, then we have

$$|x(t)-x^*(t)|=\left|x_0-\int_0^T g\big(s,x(s),y(s)\big)ds+\int_0^t y(s)ds-x_0+\int_0^T g\big(s,x^*(s),y^*(s)\big)ds-\int_0^t y^*(s)ds\right|$$
$$\leqslant\int_0^T\left|g\big(s,x(s),y(s)\big)-g\big(s,x^*(s),y^*(s)\big)\right|ds+\int_0^t|y(s)-y^*(s)|ds$$
$$\leqslant L\int_0^T\left[|x(s)-x^*(s)|+|y(s)-y^*(s)|\right]ds+\|y-y^*\|_{L^1}\leqslant LT\|x-x^*\|_C+(L+1)\|y-y^*\|_{L^1}.$$

Thus

$$\|x-x^*\|_C\leqslant LT\|x-x^*\|_C+(L+1)\delta.$$

Hence

$$\|x-x^*\|_C\leqslant\frac{(L+1)\delta}{1-LT}=\epsilon.$$

Since $LT<1$, therefore, the solution of (1.1)-(1.2) depends continuously on y. □

Definition 3.7. The solution $y\in L^1[0,T]$ of the functional equation (2.2) depends continuously on the functions f_i and parameters λ_i, if $\forall\epsilon>0$, $\exists\delta(\epsilon)>0$ such that

$$\max\left\{|f_i(t,x)-f_i^*(t,x)|,\ |\lambda_i-\lambda_i^*|\right\}\leqslant\delta\Rightarrow\|y-y^*\|_{L^1}\leqslant\epsilon,$$

where y^* is the unique solution of the functional equation

$$y^*(t)=\sum_{i=1}^k f_i^*\big(t,\lambda_i^*\phi_i'(t)y^*(\phi_i(t))\big),\ t\in[0,T]. \tag{3.2}$$

Theorem 3.8. *Let the hypotheses of Theorem 3.1 be fulfilled, then the solution $y\in L^1[0,T]$ of (2.2) depends continuously on the functions f_i and parameters λ_i.*

Proof. Let y and y^* be the two solutions of (2.2) and (3.2), respectively, then we have

$$\begin{aligned}|y(t)-y^*(t)| &= \Big|\sum_{i=1}^{k} f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)-\sum_{i=1}^{k} f_i^*\big(t,\lambda_i^*\phi_i'(t)y^*(\phi_i(t))\big)\Big|\\ &\leqslant \sum_{i=1}^{k}\big|f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)-f_i^*\big(t,\lambda_i^*\phi_i'(t)y^*(\phi_i(t))\big)\big|\\ &\leqslant \sum_{i=1}^{k}\big|f_i\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)-f_i^*\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)\big|\\ &\quad+\sum_{i=1}^{k}\big|f_i^*\big(t,\lambda_i\phi_i'(t)y(\phi_i(t))\big)-f_i^*\big(t,\lambda_i^*\phi_i'(t)y^*(\phi_i(t))\big)\big|\\ &\leqslant \sum_{i=1}^{k}\delta+\sum_{i=1}^{k} b_i\big|\lambda_i\phi_i'(t)y(\phi_i(t))-\lambda_i^*\phi_i'(t)y^*(\phi_i(t))\big|\\ &= \delta k+\sum_{i=1}^{k} b_i|\phi_i'(t)||\lambda_i y(\phi_i(t))-\lambda_i y^*(\phi_i(t))+\lambda_i y^*(\phi_i(t))-\lambda_i^* y^*(\phi_i(t))|\\ &\leqslant \delta k+\sum_{i=1}^{k} b_i\phi_i'(t)\lambda_i|y(\phi_i(t))-y^*(\phi_i(t))|+\sum_{i=1}^{k} b_i\phi_i'(t)|\lambda_i-\lambda_i^*||y^*(\phi_i(t))|.\end{aligned}$$

Thus

$$\|y-y^*\|_{L^1}\leqslant \delta kT+\sum_{i=1}^{k} b_i\lambda_i\|y-y^*\|_{L^1}+\sum_{i=1}^{k} b_i\delta\|y^*\|_{L^1}.$$

Hence

$$\|y-y^*\|_{L^1}\leqslant \frac{\delta kT+\delta r_1\sum_{i=1}^{k} b_i}{1-\sum_{i=1}^{k} b_i\lambda_i}=\epsilon.$$

Since $\sum_{i=1}^{k} b_i\lambda_i<1$, then the solution of (2.2) depends continuously on f_i and λ_i. □

According to Theorem 3.6, we now obtain the following corollary.

Corollary 3.9. *Let the hypotheses of Theorem 3.6 be fulfilled, then the solution $x\in AC[0,T]$ of (1.1)-(1.2) depends continuously on the functions f_i and parameters λ_i.*

Definition 3.10. The solution $x\in AC[0,T]$ of (1.1)-(1.2) depends continuously on the initial data x_0 and the function g, if $\forall\epsilon>0$, $\exists\delta(\epsilon)>0$ such that

$$\max\big\{|x_0-x_0^*|,\ |g(t,x,y)-g^*(t,x,y)|\big\}\leqslant\delta\Rightarrow\|x-x^*\|_C\leqslant\epsilon,$$

where x^* is the unique solution of the integral equation

$$x^*(t)=x_0^*-\int_0^T g^*\big(s,x^*(s),y(s)\big)ds+\int_0^t y(s)ds,\ t\in[0,T]. \tag{3.3}$$

Theorem 3.11. *Let the hypotheses of Theorem 3.2 be fulfilled, then the solution $x\in AC[0,T]$ of (1.1)-(1.2) depends continuously on the initial data x_0 and the function g.*

Proof. Let x and x^* be the two solutions of (2.1) and (3.3), respectively, then we have

$$|x(t)-x^*(t)|=\Big|x_0-\int_0^T g\big(s,x(s),y(s)\big)ds+\int_0^t y(s)ds-x_0^*+\int_0^T g^*\big(s,x^*(s),y(s)\big)ds-\int_0^t y(s)ds\Big|$$

$$
\begin{aligned}
&\leqslant |x_0 - x_0^*| + \int_0^T \big|g\big(s, x(s), y(s)\big) - g^*\big(s, x^*(s), y(s)\big)\big| ds \\
&\leqslant \delta + \int_0^T \big|g\big(s, x(s), y(s)\big) - g^*\big(s, x(s), y(s)\big)\big| ds + \int_0^T \big|g^*\big(s, x(s), y(s)\big) - g^*\big(s, x^*(s), y(s)\big)\big| ds \\
&\leqslant \delta + \delta T + L \int_0^T \big|x(s) - x^*(s)\big| ds \leqslant (1+T)\delta + LT\|x - x^*\|_C.
\end{aligned}
$$

Thus

$$\|x - x^*\|_C \leqslant \frac{(1+T)\delta}{1 - LT} = \epsilon.$$

Since $LT < 1$, then the solution of (1.1)-(1.2) depends continuously on x_0 and g. □

4. Special cases and examples

Corollary 4.1. *Let the hypotheses of Theorem 2.3 be satisfied with $\phi_i(t) = \gamma_i t$, where $\gamma_i \in (0, 1]$, then the functional-differential equation*

$$\frac{dx(t)}{dt} = \sum_{i=1}^{k} f_i\left(t, \lambda_i \frac{d}{dt} x(\gamma_i t)\right), \text{ a.e. } t \in (0, T],$$

under the nonlocal condition (1.2), has at least one solution $x \in AC[0, T]$. Consequently, under the hypotheses of Theorem 3.2, it has a unique solution $x \in AC[0, T]$.

Corollary 4.2. *Let the hypotheses of Theorem 2.3 be satisfied with $\phi_i(t) = t^{\alpha_i}$, where $\alpha_i \geqslant 1$, then the functional-differential equation*

$$\frac{dx(t)}{dt} = \sum_{i=1}^{k} f_i\left(t, \lambda_i \frac{d}{dt} x(t^{\alpha_i})\right), \text{ a.e. } t \in (0, 1],$$

under the nonlocal condition (1.2), has at least one solution $x \in AC[0, 1]$. Consequently, under the hypotheses of Theorem 3.2, it has a unique solution $x \in AC[0, 1]$.

Corollary 4.3. *Let the hypotheses* (i)-(ii) *and* (iv) *of Theorem 2.3 be satisfied, then the initial value problem of the implicit differential equation*

$$\frac{dx(t)}{dt} = f_1\left(t, \lambda_1 \frac{d}{dt} x(\gamma t)\right), \text{ a.e. } t \in (0, T], \quad x(0) = x_0,$$

where $\gamma \in (0, 1]$, has at least one solution $x \in AC[0, T]$. Consequently, under the hypotheses (iv) *and* (i)′-(ii)′ *of Theorem 3.2, it has a unique solution $x \in AC[0, T]$.*

Example 4.4. Consider the following functional-differential equation

$$\frac{dx(t)}{dt} = \frac{e^{-t}}{t+4} + \frac{1}{4}\frac{d}{dt} x\left(\sin\left(\frac{\pi}{2}t\right)\right) + \frac{t}{5-t} + \frac{1}{2}\frac{d}{dt} x\left(\frac{1}{2}(t+1)\right), \text{ a.e. } t \in (0, 1], \tag{4.1}$$

with the nonlocal integral condition

$$x(0) + \int_0^1 \left((\frac{s}{2})^3 + \frac{1}{3}\left(x(s) + \frac{dx(s)}{ds}\right)\right) ds = 1. \tag{4.2}$$

The functional equation is given by

$$y(t) = \frac{e^{-t}}{t+4} + \frac{\pi}{8}\cos\left(\frac{\pi}{2}t\right) y\left(\sin\left(\frac{\pi}{2}t\right)\right) + \frac{t}{5-t} + \frac{1}{4} y\left(\frac{1}{2}(t+1)\right), \ t \in [0, 1].$$

Set

$$f_1\big(t,\lambda_1\phi_1'(t)y(\phi_1(t))\big) = \frac{e^{-t}}{t+4} + \frac{\pi}{8}\cos\left(\frac{\pi}{2}t\right)y\left(\sin\left(\frac{\pi}{2}t\right)\right),$$
$$f_2\big(t,\lambda_2\phi_2'(t)y(\phi_2(t))\big) = \frac{t}{5-t} + \frac{1}{4}y\left(\frac{1}{2}(t+1)\right),$$

and

$$g\big(t,x(t),y(t)\big) = \left(\frac{t}{2}\right)^3 + \frac{1}{3}\big(x(t)+y(t)\big).$$

We have $\phi_1(t) = \sin\left(\frac{\pi}{2}t\right) \leqslant 1$, $\phi_1'(t) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$, $\phi_2(t) = \frac{1}{2}(t+1) \leqslant 1$, $\phi_2'(t) = \frac{1}{2}$, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$, $x_0 = 1$, and $T = 1$. Hence, $f_1(t,0) = \frac{e^{-t}}{t+4} \in L^1[0,1]$, $f_2(t,0) = \frac{t}{5-t} \in L^1[0,1]$, $g(t,0,0) = \left(\frac{t}{2}\right)^3 \in L^1[0,1]$, and

$$\big|f_1(t,x) - f_1(t,y)\big| \leqslant \frac{\pi}{8}|x-y|, \quad \big|f_2(t,x)-f_2(t,y)\big| = \frac{1}{4}|x-y|, \quad \big|g(t,x,y)-g(t,v,w)\big| \leqslant \frac{1}{3}\big(|x-v|+|y-w|\big),$$

then $b_1 = \frac{\pi}{8}$, $b_2 = \frac{1}{4}$, $L = \frac{1}{3}$, $\sum_{i=1}^{2} b_i\lambda_i \approx 0.223175 < 1$, and $LT \approx 0.333333 < 1$. Obviously, all hypotheses of Theorem 3.2 are satisfied, then the solution of (4.1)-(4.2), $x \in AC[0,1]$, is unique.

Example 4.5. Consider the following functional-differential equation

$$\frac{dx(t)}{dt} = \frac{1}{3}e^{-t^3}\cos(t) + \frac{1}{5}\frac{d}{dt}x\left(\frac{1}{3}t\right), \text{ a.e. } t \in (0,3], \tag{4.3}$$

with the nonlocal integral condition

$$x(0) + \int_0^3 \left(e^s\sin(2s) + \frac{1}{6}\left(x(s) + \frac{dx(s)}{ds}\right)\right)ds = 1. \tag{4.4}$$

The functional equation is given by

$$y(t) = \frac{1}{3}e^{-t^3}\cos(t) + \frac{1}{15}y\left(\frac{1}{3}t\right), \ t \in [0,3].$$

Set

$$f_1\big(t,\lambda_1\phi_1'(t)y(\phi_1(t))\big) = \frac{1}{3}e^{-t^3}\cos(t) + \frac{1}{15}y\left(\frac{1}{3}t\right),$$

and

$$g\big(t,x(t),y(t)\big) = e^t\sin(2t) + \frac{1}{6}\big(x(t)+y(t)\big).$$

We have $\phi_1(t) = \frac{1}{3}t$, $\gamma_1 = \frac{1}{3}$, $\lambda_1 = \frac{1}{5}$, $x_0 = 1$, and $T = 3$. Hence, $f_1(t,0) = \frac{1}{3}e^{-t^3}\cos(t) \in L^1[0,3]$, $g(t,0,0) = e^t\sin(2t) \in L^1[0,3]$, and

$$\big|f_1(t,x)-f_1(t,y)\big| = \frac{1}{15}|x-y|, \quad \big|g(t,x,y)-g(t,v,w)\big| \leqslant \frac{1}{6}\big(|x-v|+|y-w|\big),$$

then $b_1 = \frac{1}{15}$, $L = \frac{1}{6}$, $b_1\lambda_1 \approx 0.013333 < 1$, and $LT = 0.5 < 1$. Therefore, by applying to Corollary 4.1, the solution of (4.3)-(4.4), $x \in AC[0,3]$, is unique.

Example 4.6. consider the next nonlocal problem

$$\frac{dx(t)}{dt} = \frac{t}{t^2+1} + \frac{1}{3}\frac{d}{dt}x(t^2) + \frac{t}{3-t^5} + \frac{1}{8}\frac{d}{dt}x(t^3), \text{ a.e. } t \in (0,1], \tag{4.5}$$

$$x(0) + \int_0^1 \left(5s^3 + s^2 + s + 1 + \frac{1}{5}\left(x(s) + \frac{dx(s)}{ds}\right)\right)ds = 2. \tag{4.6}$$

The functional equation is given by

$$y(t) = \frac{t}{t^2+1} + \frac{2}{3}ty(t^2) + \frac{t}{3-t^5} + \frac{3}{8}t^2y(t^3), \ t \in [0,1]. \tag{4.7}$$

Set

$$f_1\big(t, \lambda_1\phi_1'(t)y(\phi_1(t))\big) = \frac{t}{t^2+1} + \frac{2}{3}ty\big(t^2\big),$$
$$f_2\big(t, \lambda_2\phi_2'(t)y(\phi_2(t))\big) = \frac{t}{3-t^5} + \frac{3}{8}t^2y\big(t^3\big),$$

and

$$g\big(t, x(t), y(t)\big) = 5t^3 + t^2 + t + 1 + \frac{1}{5}\big(x(t) + y(t)\big).$$

We have $\phi_1(t) = t^2$, $\alpha_1 = 2$, $\phi_2(t) = t^3$, $\alpha_2 = 3$, $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{8}$, $x_0 = 2$, and $T = 1$. Hence, $f_1(t,0) = \frac{t}{t^2+1} \in L^1[0,1]$, $f_2(t,0) = \frac{t}{3-t^5} \in L^1[0,1]$, $g(t,0,0) = 5t^3 + t^2 + t + 1 \in L^1[0,1]$, and

$$\big|f_1(t,x) - f_1(t,y)\big| \leqslant \frac{2}{3}|x-y|, \ \big|f_2(t,x) - f_2(t,y)\big| \leqslant \frac{3}{8}|x-y|, \ \big|g(t,x,y) - g(t,v,w)\big| \leqslant \frac{1}{5}\big(|x-v| + |y-w|\big),$$

then $b_1 = \frac{2}{3}$, $b_2 = \frac{3}{8}$, $L = \frac{1}{5}$, $\sum_{i=1}^{2} b_i\lambda_i \approx 0.269097 < 1$, and $LT = 0.2 < 1$. Obviously, all hypotheses of Corollary 4.2 are satisfied, then the solution of (4.5)-(4.6), $x \in AC[0,1]$, is unique.

5. Conclusion

Understanding the stability of problems involving functional and differential equations is crucial for ensuring the predictability and reliability of mathematical models that represent real-life phenomena. In this study, we delve into the problem of the delay functional-differential equation with parameters (1.1) under the nonlocal integral equation (1.2). We establish the existence of solutions to the problem (1.1)-(1.2). We have outlined the necessary hypotheses that ensure the uniqueness of the solution. Furthermore, our study involves a rigorous analysis of the problem through the implementation of the Hyers-Ulam stability to the problem and the continuous dependence of the unique solution on key variables, including the functions f_i, parameters λ_i, the initial data x_0, and the function g. We presented various examples and special cases to illustrate our work. The solvability study presented here could be applied to future investigations into the stability of various types of initial value, boundary value, and constrained problems. Interested researchers could subsequently extend this concept to different kinds of functional-differential equations of fractional order.

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