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On soft refined 2-normed spaces

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Abstract

In this work, we introduce the concepts of refined soft 2-normed spaces and refined soft 2-inner product spaces, obtaining important results such as Cauchy-Schwarz Inequality, that each refined soft 2-inner product induces a refined soft 2-normed space and that a refined soft 2-normed space is induced by a soft 2-inner product if the refined soft 2-normed satisfies the Parallelogram law. For this, we present the definition of refined linearly dependent soft vectors in a soft vector space which also allows us to show that given a classical inner product space, then the standard 2-inner product induces a refined soft 2-inner product space. The results presented here improve considerably the work of Kadhim [D. A. Kadhim, J. Al-Qadisiyah Comput. Sci. Math., **6** (2014), 157–168] and open a line of research in the context of refined soft 2-normed space and refined soft 2-inner product space.

Keywords: Soft sets, soft vectors, soft linear space, refined soft 2-normed space, refined soft 2-inner product space.

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1. Introduction

Functional analysis is a branch of mathematical analysis that finds multiple applications in other sciences such as engineering, medicine, and physics; the last one of particular importance, since the transcendence of inner product spaces is well known, precisely the Hilbert spaces in quantum mechanics, since quantum observables are nothing more than self-adjoint operators in a Hilbert space. So venturing into territories related to functional analysis can bring important benefits within complex physical theories that want to explain the universe. In this sense, the study of concepts of functional analysis and its applications has always been of great interest to the most passionate researchers on different topics as those presented in the classical books [14, 16].

The notion of 2-normed space was developed by Gähler in 1964 [23], and since then other concepts have been developed in this context, such as the notion of 2-inner product spaces and 2-Hilbert space [15, 36], which is a space with a complete 2-inner product. New notions involving these concepts and some of their generalizations have recently been studied, for example, the theory of frames in the reference

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[13], atomic systems in 2-inner product spaces in the reference [21] and weak n-inner product spaces in the reference [30]. As a continuation of these investigations, in 2019, Kundu et. al [27] studied 2-normed spaces from the point of view of topological vector spaces, showing that a separating family of seminorms is induced from a 2-norm and this fact can be used as a one-way bridge between 2-normed spaces and topological vector spaces. Furthermore, they provided an alternative proof that every 2-normed space is a locally convex topological vector space and found a necessary and sufficient condition for the normability of a 2-normed space.

In everyday life, problems may arise where the information is ambiguous or uncertain, so these are not solved with traditional mathematical methods. In view of this, several mathematical tools such as fuzzy set theory and soft set theory have been proposed to deal with these types of problems. The concept of soft sets was first proposed by Molodtsov [31] in 1999 as a new mathematical approach to deal with fuzzy situations and imprecise data. As described by Molodtsov, soft set theory is a very useful mathematical tool for approaching the study of problems related to other sciences such as engineering, physics, economics, social science, and medical science, etc. Soft sets have been used by numerous scholars and researchers interested in uncertainty in both theoretical and applied fields. To solve decision making problems, Maji et al. [29] used soft sets in 2002, and they [28] defined a set of operations between soft sets in 2003. Over time, some of these operations were found to have shortcomings that motivated certain authors to change their definitions and adopt new classes of them for various reasons [2, 8, 33, 37]. The flexible theoretical framework provided by the aforementioned operations and the origin of new soft properties has allowed soft set theory to be significantly extended and at the same time applied directly or in hybrid form with other theories to address uncertainty in decision making processes, as can be seen in recent references [1, 10, 22, 24, 26, 32, 34, 35]. For the above, soft set theory represents a field of research in constant and rapid growth.

1.1. Research Gap

The 2-normed spaces and the 2-inner product spaces are mathematical structures that were introduced as generalizations of the linear normed spaces and the inner product spaces, respectively. These spaces have been studied in the context of soft set theory in reference [25], but in that study the mathematical rigor required to establish results related to functional analysis was lacking, for example, a Cauchy-Schwarz inequality was stated there without presenting a proof of it. Trying to provide new results involving soft 2-normed spaces and soft 2-inner product spaces, we discovered that the notion of soft linear independence considered in [25] is not appropriate to relate the 2-inner product spaces of classical theory with the 2-inner product spaces of soft set theory, something that is natural in this area of research. In fact, reviewing the definitions and results given in [25], we identify some fallacies, such as that Example 3.2, does not correspond to a classical 2-inner product, much less the induced application is a soft 2-inner product; we also detect the inconvenience of performing a formal proof of the Cauchy-Schwarz inequality on soft 2-inner product spaces. Our research work answers the following research questions:

(i) How to give a definition of linear independence that allows to successfully relate the classical theory of 2-normed spaces to new concepts of soft 2-normed spaces and soft 2-inner product spaces?

(ii) How appropriate are these new concepts for formally proving the Cauchy-Schwarz inequality and the Parallelogram law in this context?

1.2. Motivation

The range of applications of soft set theory has motivated the study of it and its use in the development of new research in various fields has progressed rapidly because this theory is free of many of the difficulties of concern in the usual theoretical approaches. Thus, research related to soft sets has been carried out in several directions. In particular, as far as mathematics and its applications are concerned, several topics have been explored, including the following: Supra soft topological ordered spaces [6], two new forms of ordered soft separation axioms [7], sum of soft topological ordered spaces [9], soft parametric somewhat-open sets and applications via soft topologies [11]. Although the development of the previous investigations seems to be a repetition of classical results, this is not so, since it has been shown that there are divergences between classical properties and soft properties, as can be seen in references [3–5, 8, 12]. Motivated by all the above, it is that, in this manuscript, we intend to establish a theory of 2-normed spaces and 2-inner product spaces using soft sets, which opens a door for future research and applications based on it.

1.3. Main Contributions

In this paper, we have reviewed the notions of soft 2-normed spaces and soft 2-inner product spaces with the main objective of obtaining a formal proof of the Cauchy-Schwarz inequality in this theoretical framework. For this purpose, we have given another definition of linearly dependent soft vectors that allows us to prove the above inequality. In addition, we have studied the most important properties of 2-normed spaces and 2-inner product spaces in the context of soft set theory. In this sense, the results of this work open the study of soft 2-Hilbert spaces and other topics of interest in soft functional analysis. Finally, we have shown that a refined soft 2-normed space is induced by a refined soft 2-inner product if the refined soft 2-normed space satisfies the Parallelogram law, which is one of the most important results of this paper.

This manuscript was designed as follows. Section 2 corresponds to the preliminaries, where we study all the theory of 2-normed spaces and 2-inner product spaces. In Section 3, we cover all the theory related with soft sets which will be useful later to develop the theory of refined soft 2-normed spaces and refined soft 2-inner product spaces. In Sections 4 and 5, we revisit the work done in [25], making important observations such as that with the definition of linearly dependent soft vectors considered in [17] and cited by [25], the standard 2-inner product does not induce a soft 2-inner product. Furthermore, in Section 5, we present the proof of the Cauchy-Schwarz Inequality for refined soft 2-inner product spaces.

2. Preliminaries related to crisp set theory

Throughout this work, let X be vector space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers.

Definition 2.1 ([15])**.** A **2-norm** on X is a mapping $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following conditions:

1. $\|x, y\| \geqslant 0$ for all $x, y \in X$ and $\|x, y\| = 0$ if and only if x and y are linearly dependent in X;
2. $\|x, y\| = \|y, x\|$ for all $x, y \in X$;
3. $\|x, \alpha \cdot y\| = |\alpha|\|x, y\|$ for all $x, y \in X$ and for all scalar α;
4. $\|x, y + z\| \leqslant \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

A vector space X with a 2-norm $\|\cdot,\cdot\|$ on X is said to be a **2-normed space** and is denoted by $(X, \|\cdot,\cdot\|)$.

Definition 2.2 ([15])**.** A **2-inner product** on X is a mapping $\langle \cdot,\cdot \,|\, \cdot\rangle : X \times X \times X \to \mathbb{K}$ which satisfies the following conditions:

1. $\langle x, x \,|\, y\rangle \geqslant 0$ for all $x, y \in X$ and $\langle x, x \,|\, y\rangle = 0$ if and only if x and y are linearly dependent on X;
2. $\langle x, x \,|\, y\rangle = \langle y, y \,|\, x\rangle$ for all $x, y \in X$;
3. $\langle x, y \,|\, z\rangle = \overline{\langle y, x \,|\, z\rangle}$ for all $x, y, z \in X$, where $\overline{\lambda}$ means the conjugate complex of $\lambda \in \mathbb{K}$ whenever $\mathbb{K} = \mathbb{C}$;
4. $\langle \alpha \cdot x, y \,|\, z\rangle = \alpha \cdot \langle x, y \,|\, z\rangle$ for all $x, y, z \in X$ and for all scalar α;
5. $\langle x_1 + x_2, y \,|\, z\rangle = \langle x_1, y \,|\, z\rangle + \langle x_2, y \,|\, z\rangle$ for all $x_1, x_2, y, z \in X$.

A vector space X with a 2-inner product $\langle\cdot,\cdot\,|\,\cdot\rangle$ on X is said to be a **2-inner product space** and is denoted by $(X, \langle\cdot,\cdot\,|\,\cdot\rangle)$.

Example 2.3 ([15])**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, the mapping $\langle\cdot,\cdot\,|\,\cdot\rangle : X \times X \times X \to \mathbb{K}$ defined as*

$$\langle x, y\,|\,z\rangle = \begin{vmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} = \langle x,y\rangle\langle z,z\rangle - \langle x,z\rangle\langle z,y\rangle,$$

*for all $x, y, z \in X$, is a 2-inner product on X, which is called the **standard 2-inner product**. Consequently, the concept of 2-inner product makes sense, because whenever we have a classical inner product on a vector space, we can generate a 2-inner product on the same space.*

In any given 2-inner product space $(X, \langle\cdot,\cdot\,|\,\cdot\rangle)$ we can define a 2-norm on X by $\|x, z\| = \langle x, x\,|\,z\rangle^{\frac{1}{2}}$, for all $x, z \in X$.

3. Preliminaries related to soft set theory

Throughout this paper, X denotes any non-empty set (possibly without algebraic structure), $\mathcal{P}(X)$ the power set of X and A a non-empty set of parameters.

Definition 3.1 ([31])**.** A **soft set** on X is a pair (F, A) where F is a mapping given by $F : A \to \mathcal{P}(X)$.

In this way, we can see a soft set as

$$(F, A) := G_F^A = \{(\lambda, F(\lambda)) : \ \lambda \in A,\ F(\lambda) \in \mathcal{P}(X)\},$$

where G_F^A is the graph of F with respect to A.

Note that a soft set is determined by knowing $F(\lambda)$ for all $\lambda \in A$. Therefore, it is common to find ourselves in the literature that $F : A \to \mathcal{P}(X)$ is called a soft set on X, but it should not be a cause for confusion.

Example 3.2. *Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{\mathbb{Z}_4 = \lambda_1,\ \mathbb{Z}_6 = \lambda_2,\ \mathbb{Z}_8 = \lambda_3\}$. If $F : A \to \mathcal{P}(X)$ describes the generating elements of the cyclic group. Then, it is easy to see that $F(\lambda_1) = \{1, 3\}$, $F(\lambda_2) = \{1, 5\}$, $F(\lambda_3) = \{1, 3, 5, 7\}$. Hence (F, A) is a soft set on X seen as follows*

$$(F, A) = \{(\mathbb{Z}_4, \{1, 3\}), (\mathbb{Z}_6, \{1, 5\}), (\mathbb{Z}_8, \{1, 3, 5, 7\})\}.$$

Definition 3.3 ([28])**.** A soft set (F, A) on X is said to be:

1. A **null soft set** on X if $F(\lambda) = \emptyset$ for all $\lambda \in A$, and in this case we write $(F, A) := \Phi$.
2. A **non-null soft set** on X if $F(\lambda) \neq \emptyset$ for some $\lambda \in A$.
3. An **absolute soft set** on X if $F(\lambda) = X$ for all $\lambda \in A$, and in this case we write $(F, A) := \check{X}$. This convention of absolute soft set will be adopted throughout the present work.

Definition 3.4 ([18])**.** A **soft element** on X is a function $\epsilon : A \to X$. Now if $\epsilon(\lambda) \in F(\lambda)$ for all $\lambda \in A$, the soft element ϵ is said to belong to the soft set (F, A) on X, which we will denote by $\epsilon \,\widetilde{\in}\, F$. In this sense, given $\lambda \in A$, $F(\lambda)$ can be expressed as $F(\lambda) = \{\epsilon(\lambda) : \epsilon \,\widetilde{\in}\, F\}$.

We will denote the collection of all the soft elements of a soft set (F, A) by $SE((F, A))$; this is,

$$SE((F, A)) := \{\epsilon : \epsilon \,\widetilde{\in}\, F\} = \{\epsilon : \epsilon(\lambda) \ \in \ F(\lambda),\ \forall \lambda \in A\}.$$

Definition 3.5 ([18, 19])**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, A be a non-empty set of parameters and consider the set

$$\mathfrak{B}(\mathbb{K}) := \{B \in \mathcal{P}(\mathbb{K}) : \ B \neq \emptyset,\ B \text{ is bounded}\},$$

then a mapping $F : A \to \mathfrak{B}(\mathbb{K}) \subset \mathcal{P}(\mathbb{K})$ is called a **soft $\mathbb{K}$-set**. This is denoted by (F, A). Furthermore, if for all $\lambda \in A$ it is satisfied that $F(\lambda)$ is a singleton, then by identifying (F, A) with its corresponding soft element, we will call this soft element, **soft $\mathbb{K}$-number**.

We will denote the set of all the soft $\mathbb{R}$-numbers (resp. soft $\mathbb{C}$-numbers) or soft real numbers (resp. soft complex numbers) by $\mathbb{R}(A)$ (resp. $\mathbb{C}(A)$). In addition, we use the symbols $\hat{\alpha}$, $\hat{\beta}$, etc, to denote soft $\mathbb{K}$-numbers such that behave as constants, this is, $\hat{\alpha}(\lambda) = \alpha$ for all $\lambda \in A$. In particular, $\hat{0}$ and $\hat{1}$ are, respectively, the soft $\mathbb{K}$-numbers where $\hat{0}(\lambda) = 0$ and $\hat{1}(\lambda) = 1$, for all $\lambda \in A$. Furthermore, we denote by $\mathbb{Z}(A)$ and $\mathbb{Q}(A)$ the set of all the soft $\mathbb{Z}$-numbers and $\mathbb{Q}$-numbers, respectively.

Definition 3.6 ([18])**.** The set $\{\alpha \in \mathbb{R}(A) : \alpha(\lambda) \geqslant 0, \ \forall \lambda \in A\}$ is called the set of all non-negative soft real numbers and is denoted by $\mathbb{R}(A)^*$.

Definition 3.7 ([20])**.** For two soft real numbers $\alpha : A \to \mathbb{R}$ and $\beta : A \to \mathbb{R}$ we define:

1. $\alpha \tilde{\leqslant} \beta$, if $\alpha(\lambda) \leqslant \beta(\lambda)$, for all $\lambda \in A$;
2. $\alpha \tilde{<} \beta$, if $\alpha(\lambda) < \beta(\lambda)$, for all $\lambda \in A$.

Definition 3.8 ([19])**.** Let (F, A) be a soft complex set (number). The real and imaginary parts of (F, A), denoted by $\operatorname{Re} F$ and $\operatorname{Im} F$, respectively, are defined by

$$\operatorname{Re} F(\lambda) = \{\operatorname{Re}(z) : z \in F(\lambda)\},$$

and

$$\operatorname{Im} F(\lambda) = \{\operatorname{Im}(z) : z \in F(\lambda)\},$$

for every $\lambda \in A$.

It is clear that if (F, A) is a soft complex set (resp. number), then $\operatorname{Re} F$ and $\operatorname{Im} F$ are soft real sets (resp. numbers) on A.

Definition 3.9 ([19])**.** Let F, G be two soft complex sets. For every $\lambda \in A$, the following operations are defined:

1. The sum $(F + G, A)$ is defined as $(F + G)(\lambda) = \{u + v : u \in F(\lambda), v \in G(\lambda)\}$;
2. The difference $(F - G, A)$ is defined as $(F - G)(\lambda) = \{u - v : u \in F(\lambda), v \in G(\lambda)\}$;
3. The product $(F \cdot G, A$ is defined as $(F \cdot G)(\lambda) = \{u \cdot v : u \in F(\lambda), v \in G(\lambda)\}$;
4. The division $\left(\frac{F}{G}, A\right)$ is defined as $\left(\frac{F}{G}\right)(\lambda) = \left\{\frac{u}{v} : u \in F(\lambda), v \in G(\lambda)\right\}$, provided $0 \notin G(\lambda)$;
5. The complex conjugate of (F, A), denoted by $(\overline{F}, A)$, is defined as $\overline{F}(\lambda) = \{\overline{u} : u \in F(\lambda)\}$, where $\overline{u}$ is the conjugate of the ordinary complex number u;
6. The modulus of (F, A), denoted by $(|F|, A)$, is defined as $|F|(\lambda) = \{|u| : u \in F(\lambda)\}$, where $|u|$ is the modulus of the ordinary complex number u;
7. For any scalar (real or complex) k, the scalar multiplication of k by (F, A), denoted by $(k \cdot F, A)$, is defined as $(k \cdot F)(\lambda) = \{k \cdot u : u \in F(\lambda)\}$.

Remark 3.10. The operations given in Definition 3.9 are simply denoted by $F + G$, $F - G$, $F \cdot G$, $\frac{F}{G}$, $\overline{F}$, $|F|$ and $k \cdot F$, respectively. It is easy to check that "+" and "·" are commutative and associative; also $\hat{0}$ and $\hat{1}$ are respectively additive and multiplicative identity on the set $\mathcal{C}(A)$ of all soft complex sets over the set of parameters A.

Remark 3.11. If $F, G \in \mathbb{C}(A)$ (i.e. are soft complex numbers), then $F + G$, $F - G$, $F \cdot G$, $\frac{F}{G}$, $\overline{F}$ and $k \cdot F$ are soft complex numbers, while $|F|$ is a non-negative soft real number. In this case, $F(\lambda) = \{F(\lambda)\}$ is a singleton, with $F(\lambda) \in \mathbb{C}$ for all $\lambda \in A$. Thus, $\operatorname{Re} F = \{\operatorname{Re} F(\lambda)\}$ and $\operatorname{Im} F = \{\operatorname{Im} F(\lambda)\}$ for all $\lambda \in A$. Therefore, $F \in \mathbb{C}(A)$ is defined by $F(\lambda) = \{\operatorname{Re} F(\lambda) + i \operatorname{Im} F(\lambda)\}$ for all $\lambda \in A$. On the other hand, $[\operatorname{Re} F + \hat{\imath} \operatorname{Im} F](\lambda) = \{\operatorname{Re} F(\lambda) + i \operatorname{Im} F(\lambda)\} = F(\lambda)$ for all $\lambda \in A$. From this last discussion, we conclude that if $F \in \mathbb{C}(A)$, then F can be written as $F = \operatorname{Re} F + \hat{\imath} \operatorname{Im} F$.

Note that if we assign to X a structure of vector space, it is interesting to think of the possible structure of $F(\lambda)$ as subset of X for all $\lambda \in A$. This motivates the following important definition.

Definition 3.12 ([17]). Let X be a $\mathbb{K}$-vector space (typically $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), A be nonempty set of parameters and (F, A) be a soft set on X. The soft set (F, A) is said to be a **soft $\mathbb{K}$-vector space** on X if $F(\lambda)$ is a vector subspace of X for all $\lambda \in A$.

The importance of the above definition is that it allows us to relate the usual linear algebra to soft set theory. Furthermore, it gives us the tools to define important concepts in the classical functional analysis, such as norm, inner product, Banach space, Hilbert space, among others; but from this context.

Definition 3.13 ([17]). Let (F, A) be soft $\mathbb{K}$-vector space.

1. A soft element of (F, A) is said to be a **soft vector** of (F, A). Similarly, a soft element $\epsilon : A \to \mathbb{K}$ is said to be a **soft scalar**, where $\mathbb{K}$ is the scalar field.
2. A soft vector x of (F, A) is said to be a **null soft vector** if $x(\lambda) = \theta$, $\forall \lambda \in A$, where θ is the zero element of X. This will be denoted by Θ.

Definition 3.14 ([17]). Let (F, A) be a soft $\mathbb{K}$-vector space on X and x, y be two soft vectors of (F, A) and k be a soft scalar. Then the addition $x + y$ of x with y, and scalar multiplication $k \cdot x$ of k and x are defined by $(x + y)(\lambda) = x(\lambda) + y(\lambda)$, $(k \cdot x)(\lambda) = k(\lambda) \cdot x(\lambda)$. Evidently, $x + y$, $k \cdot x$ are soft vectors of (F, A).

Theorem 3.15 ([17]). *Let X be a $\mathbb{K}$-vector space, A be a nonempty set of parameters and (F, A) be a soft $\mathbb{K}$-vector space on X. Then*

1. $\hat{0} \cdot x = \Theta$, *for all* $x \tilde{\in} F$;
2. $k \cdot \Theta = \Theta$, *for all soft scalar* k;
3. $(-\hat{1}) \cdot x = -x$, *for all* $x \tilde{\in} F$.

4. Refined soft 2-normed space

In this section we establish some properties that improve the results given in [25]. In 2014, Kadhim [25] introduced the definition of soft 2-normed spaces based on the results from soft vector spaces given in [17] where the definition of linearly dependent soft vectors in a soft vector space is as follows:

Let X be a $\mathbb{K}$-vector space, A be a nonempty set of parameters and (F, A) be a soft $\mathbb{K}$-vector space on X. A finite set of soft vectors $\{x_1, x_2, \ldots, x_n\}$ of (F, A) is said to be linearly dependent in F if there exist soft scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all $\hat{0}$, such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \Theta$.

However, with this definition it is not possible to show that the mapping of Example 2.3 defines a soft 2-inner product in the sense of Definition 3.1 in [25], as shown in the following example:

Example 4.1. *Consider $X = \mathbb{R}^2$ as a vector space over $\mathbb{R}$ and let $A = \{1, 2\}$ be the set of parameters. Then with the following soft vectors*

$$y(\lambda) = \begin{cases} (1,0) \quad , \quad \lambda = 1, \\ \\ (1,2) \quad , \quad \lambda = 2. \end{cases} ; \quad z(\lambda) = \begin{cases} (2,0) \quad , \quad \lambda = 1, \\ \\ (2,1) \quad , \quad \lambda = 2. \end{cases}$$

and knowing that

$$\langle y, y \mid z \rangle = \begin{vmatrix} \langle y, y \rangle & \langle y, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle y, y \rangle \langle z, z \rangle - \langle y, z \rangle \langle z, y \rangle = \|y\|^2 \|z\|^2 - |\langle y, z \rangle|^2,$$

we have

$$\langle y, y \mid z \rangle(\lambda) = \|y(\lambda)\|^2 \|z(\lambda)\|^2 - |\langle y(\lambda), z(\lambda) \rangle|^2,$$

and therefore,

$$\langle y, y \mid z\rangle(2) = \|y(2)\|^2\|z(2)\|^2 - |\langle y(2), z(2)\rangle|^2 = \|(1,2)\|^2\|(2,1)\|^2 - |\langle(1,2),(2,1)\rangle|^2 = 9,$$

but the set of soft vectors $\{y, z\}$ is linearly dependent in the soft vector space $\hat{X}$ on the set of parameters A, which contradicts the first condition of the definition of soft 2-inner product given in [25]. Note that in particular we want that the standard 2-inner product induces a soft 2-inner product, see Example 5.2.

In view of the previous example, we give the definition of refined linearly dependent soft vectors which is more suitable to show results in the 2-normed spaces and 2-inner spaces in the context of soft set theory such as Cauchy-Schwarz inequality.

Definition 4.2. Let X be a $\mathbb{K}$-vector space, A be a nonempty set of parameters and (F, A) be a soft $\mathbb{K}$-vector space on X. A finite set of soft vectors $\{x_1, x_2, \dots, x_n\}$ of (F, A) is said to be refined linearly dependent in F if there exist soft scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\alpha_i(\lambda) \neq 0$ for some $i \in \{1, \dots, n\}$ and each $\lambda \in A$, such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \Theta$.

An arbitrary set H of soft vectors of F is said to be refined linearly dependent in F if there exists a finite subset of H which is refined linearly dependent in F.

Remark 4.3. It is clear that the notion of refined linearly dependent given in Definition 4.2 is distinct from the notion of linearly dependent given in [17]. Moreover, every set of refined linearly dependent soft vectors is linearly dependent in the sense of [17], but the converse, in general, is not true.

Lemma 4.4. *A set of soft vectors $\{x_1, x_2, \dots, x_n\}$ is refined linearly dependent in a soft vector space (F, A) on X if and only if $\{x_1(\lambda), x_2(\lambda), \dots, x_n(\lambda)\}$ is linearly dependent in X for all $\lambda \in A$.*

Proof. Let is suppose that $\{x_1, x_2, \dots, x_n\}$ is refined linearly dependent in (F, A) on X. Then in view of the definition 4.2, there exist soft scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\alpha_i(\lambda) \neq 0$ for some $i \in \{1, \dots, n\}$ and each $\lambda \in A$, such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \Theta$. So for all $\lambda \in A$ we have that $\alpha_1(\lambda)x_1(\lambda) + \alpha_2(\lambda)x_2(\lambda) + \dots + \alpha_n(\lambda)x_n(\lambda) = \theta$ and obviously the first part of the Lemma is true.

Conversely, suppose that $\{x_1(\lambda), x_2(\lambda), \dots, x_n(\lambda)\}$ is linearly dependent in X for all $\lambda \in A$. Then there exist scalars $\alpha_{1,\lambda}, \alpha_{2,\lambda}, \dots, \alpha_{n,\lambda}$ such that $\alpha_{1,\lambda}x_1(\lambda) + \alpha_{2,\lambda}x_2(\lambda) + \dots + \alpha_{n,\lambda}x_n(\lambda) = \theta$. So we can define the soft scalars $\alpha_i(\lambda) := \alpha_{i,\lambda}$ for all $i \in \{1, \dots, n\}$ and for all $\lambda \in A$. Therefore we trivially obtain that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \Theta$.

□

Definition 4.5. Let X be a $\mathbb{K}$-vector space, A be a nonempty set of parameters and (F, A) be a soft $\mathbb{K}$-vector space on X. A finite set of soft vectors $\{x_1, x_2, \dots, x_n\}$ of (F, A) such that $x_i(\lambda) \neq \theta$ for any $\lambda \in A$ and any $i \in \{1, 2 \dots, n\}$, is said to be refined linearly independent in F if $\{x_1(\nu), x_2(\nu), \dots, x_n(\nu)\}$ is linearly independent in X for some $\nu \in A$.

Definition 4.6. Let $\check{X}$ be an absolute soft $\mathbb{K}$-vector space. A **refined soft 2-norm** on $\check{X}$ is a mapping $\|\cdot,\cdot\| : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$ which satisfies the following conditions:

(2N1) $\|x, y\| \,\widetilde{\geqslant}\, \hat{0}$ for all $x, y \,\widetilde{\in}\, \check{X}$ and $\|x, y\| = \hat{0}$ if and only if x and y are refined linearly dependent in $\check{X}$;
(2N2) $\|x, y\| = \|y, x\|$ for all $x, y \,\widetilde{\in}\, \check{X}$;
(2N3) $\|x, \alpha \cdot y\| = |\alpha|\|x, y\|$ for all $x, y \,\widetilde{\in}\, \check{X}$ and for all soft scalar α;
(2N4) $\|x, y + z\| \,\widetilde{\leqslant}\, \|x, y\| + \|x, z\|$ for all $x, y, z \,\widetilde{\in}\, \check{X}$.

The absolute soft $\mathbb{K}$-vector space $\check{X}$ with a refined soft 2-norm $\|\cdot,\cdot\|$ on $\check{X}$ is said to be a **refined soft 2-normed space** and it is denoted by $(\check{X}, \|\cdot,\cdot\|, A)$ or $(\check{X}, \|\cdot,\cdot\|)$.

Remark 4.7. If $\|\cdot,\cdot\|$ is a soft 2-norm on X, then from (2N3), we have $\|x,\Theta\| = \|x,\hat{0}\cdot\Theta\| = |\hat{0}|\cdot\|x,\Theta\| = \hat{0}$. Now, by (2N4), we obtain that $\hat{0} = \|x,\Theta\| = \|x,y-y\| \,\widetilde{\leqslant}\, \|x,y\| + \|x,-y\| = \|x,y\| + \|x,y\|$, which implies that $\|x,y\| \,\widetilde{\geqslant}\, \hat{0}$ for all $x,y \,\widetilde{\in}\, \check{X}$. Also, we have

$$\|x,y+\alpha x\| \,\widetilde{\leqslant}\, \|x,y\| + \|x,\alpha x\| = \|x,y\| + |\alpha|\|x,x\| = \|x,y\| + |\alpha|\cdot\hat{0} = \|x,y\|$$

and

$$\|x,y\| = \|x,y+\Theta\| = \|x,y+\alpha x-\alpha x\| \,\widetilde{\leqslant}\, \|x,y+\alpha x\| + |\alpha|\|x,x\| = \|x,y+\alpha x\|,$$

which tells us that $\|x,y+\alpha x\| = \|x,y\|$ for all $x,y \,\widetilde{\in}\, \check{X}$ and for all soft scalar α.

Theorem 4.8. *In Definition 4.6, condition* (2N4) *can be replaced by the following condition:*
(2N4′) $\|x+z,y+z\| \,\widetilde{\leqslant}\, \|x,y\| + \|y,z\| + \|z,x\|$.

Proof. By (2N1), (2N2) and (2N4), we have

$$\begin{aligned}\|x+z,y+z\| &\,\widetilde{\leqslant}\, \|x+z,y\| + \|x+z,z\| = \|y,x+z\| + \|z,x+z\| \\ &\,\widetilde{\leqslant}\, \|y,x\| + \|y,z\| + \|z,x\| + \|z,z\| = \|x,y\| + \|y,z\| + \|z,x\|,\end{aligned}$$

that is, in a refined soft 2-normed space, condition (2N4′) is satisfied.

Conversely, suppose that a mapping $\|\cdot,\cdot\| : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$ with the conditions (2N1), (2N2), (2N3) and (2N4′) is given. Then, for arbitrary soft points x, y, z of $\check{X}$ and each soft scalar α, we obtain the inequalities

$$\begin{aligned}\|x+y,z\| &= \|(x+y-z-\alpha y) + (z+\alpha y), (-\alpha y) + (z+\alpha y)\| \\ &\,\widetilde{\leqslant}\, \|x+y-z-\alpha y, -\alpha y\| + \|-\alpha y, z+\alpha y\| + \|z+\alpha y, x+y-z-\alpha y\| \\ &\,\widetilde{\leqslant}\, \|x+y-z,\alpha y\| + \|z,\alpha y\| + \|x+y,z+\alpha y\| \\ &\,\widetilde{\leqslant}\, |\alpha|\{\|x+y-z,y\| + \|z,y\|\} + \|x+y,z+\alpha y\|.\end{aligned}$$

Also, since

$$\|z+\alpha y,\alpha x+\alpha y\| \,\widetilde{\leqslant}\, \|z,\alpha x\| + \|\alpha x,\alpha y\| + \|\alpha y,z\|$$

for any soft scalar α with $\alpha(\lambda) \neq 0$ for all $\lambda \in A$, we have the inequality

$$\|x+y,z+\alpha y\| \,\widetilde{\leqslant}\, \|x,z\| + \|z,y\| + |\alpha|\|x,y\|.$$

According to this, for each α with $\alpha(\lambda) \neq 0$ for all $\lambda \in A$, we deduce the relation

$$\|x+y,z\| \,\widetilde{\leqslant}\, |\alpha|\{\|x+y-z,y\| + \|z,y\| + \|x,y\|\} + \|x,z\| + \|z,y\|.$$

Taking limit as $\alpha \to \hat{0}$ (i.e. $\alpha(\lambda) \to 0, \forall\lambda \in A$), we obtain the inequality

$$\|x+y,z\| \,\widetilde{\leqslant}\, \|x,z\| + \|z,y\|. \qquad \square$$

Example 4.9. *Consider the set of soft real numbers* $\mathbb{R}(A)$ *and let* $\check{X} = \mathbb{R}(A)^2 = \mathbb{R}(A) \times \mathbb{R}(A)$. *The mapping* $\|\cdot,\cdot\| : \check{X} \times \check{X} \to \mathbb{R}(A)$ *defined by the formula* $\|x,y\| = |x_1y_2 - x_2y_1|$, *where* $x = (x_1,x_2)$, $y = (y_1,y_2)$, *and* $|\cdot|$ *denotes the modulus of soft real numbers, is a refined soft 2-norm on* $\check{X}$ *and hence* $(\check{X},\|\cdot,\cdot\|,A)$ *or* $(\check{X},\|\cdot,\cdot\|)$ *is a refined soft 2-normed space.*

Lemma 4.10. *Let* $(\check{X},\|\cdot,\cdot\|,A)$ *be a refined soft 2-normed space. Then, for all* $x,y \,\widetilde{\in}\, \check{X}$ *and* $\lambda \in A$, *we have* $\|x,y\|(\lambda) = 0$ *if and only if* $x(\lambda)$ *and* $y(\lambda)$ *are linearly dependent in* X *for each* $\lambda \in A$.

Proof. Let α be the soft scalar defined by

$$\alpha(\mu) = \begin{cases} 0, & \text{if} \quad \mu \neq \lambda, \\ 1, & \text{if} \quad \mu = \lambda. \end{cases}$$

Note that:

(i) If $\mu \neq \lambda$, then $(\alpha \cdot y)(\mu) = \alpha(\mu) \cdot y(\mu) = 0 \cdot y(\mu) = \theta \in X$;

(ii) If $\mu = \lambda$, then $(\alpha \cdot y)(\mu) = \alpha(\mu) \cdot y(\mu) = 1 \cdot y(\mu) = y(\mu)$.

Now, by (2N3) we have $\|x, \alpha \cdot y\| = |\alpha| \cdot \|x, y\|$, which implies that $\|x, y\|(\lambda) = 0 \Longleftrightarrow |\alpha|\|x, y\| = \hat{0} \Longleftrightarrow \|x, \alpha \cdot y\| = \hat{0} \Longleftrightarrow x$ and $\alpha \cdot y$ are refined linearly dependent on $\check{X}$; but by Lemma 4.4, this is equivalent to $x(\lambda)$ and $(\alpha \cdot y)(\lambda)$ are linearly dependent in X for each $\lambda \in A \Longleftrightarrow x(\lambda)$ and $y(\lambda)$ are linearly dependent in X for each $\lambda \in A$. □

Proposition 4.11. *Any parametrized family of crisp 2-norms $\{\|\cdot, \cdot\|_\lambda : \lambda \in A\}$ on a crisp vector space X can be considered as a refined soft 2-norm on the soft vector space $\check{X}$.*

Proof. Suppose that $\check{X}$ is an absolute soft $\mathbb{K}$-vector space over a field K, A is a non-empty set of parameters and $\{\|\cdot, \cdot\|_\lambda : \lambda \in A\}$ is a family of crisp 2-norms on the vector space X. We affirm that the mapping $\|\cdot, \cdot\| : \check{X} \times \check{X} \to \mathbb{R}(A)$ defined by $\|x, y\|(\lambda) = \|x(\lambda), y(\lambda)\|_\lambda$, $\forall \lambda \in A$, $\forall x, y \widetilde{\in} \check{X}$ is a soft 2-norm on $\check{X}$. Indeed, let us verify that conditions (2N1), (2N2), (2N3) and (2N4) of a refined soft 2-norm are satisfied.

(2N1). By Lemma 4.10 and Lemma 4.4, we have

$$\begin{aligned} \|x, y\| = \hat{0} \Longleftrightarrow \|x, y\|(\lambda) &= 0, \ \forall \lambda \in A \\ &\Longleftrightarrow x(\lambda) \text{ and } y(\lambda) \text{ are linearly dependent in } X, \text{ for each } \lambda \in A \\ &\Longleftrightarrow x \text{ and } y \text{ are refined linearly dependent in } \check{X}. \end{aligned}$$

(2N2). For all $x, y \widetilde{\in} \check{X}$, we have $\|x, y\|(\lambda) = \|x(\lambda), y(\lambda)\|_\lambda = \|y(\lambda), x(\lambda)\|_\lambda = \|y, x\|(\lambda)$, $\forall \lambda \in A$. Therefore, $\|x, y\| = \|y, x\|$.

(2N3). For all $x, y \widetilde{\in} \check{X}$ and for each soft scalar α, we have

$$\|x, \alpha \cdot y\|(\lambda) = \|x(\lambda), \alpha(\lambda) \cdot y(\lambda)\|_\lambda = |\alpha(\lambda)|\|x(\lambda), y(\lambda)\|_\lambda = (|\alpha|\|x, y\|)(\lambda), \ \forall \lambda \in A.$$

Hence, $\|x(\lambda), \alpha \cdot y\| = |\alpha|\|x, y\|$.

(2N4). For all $x, y, z \widetilde{\in} \check{X}$,

$$\begin{aligned} (\|x, y\| + \|x, z\|)(\lambda) = \|x, y\|(\lambda) + \|x, z\|(\lambda) &= \|x(\lambda), y(\lambda)\|_\lambda + \|x(\lambda), z(\lambda)\|_\lambda \\ &\widetilde{\geqslant} \|x(\lambda), y(\lambda) + z(\lambda)\|_\lambda = \|x, y + z\|(\lambda), \ \forall \lambda \in A. \end{aligned}$$

Thus, $\|x, y + z\| \widetilde{\leqslant} \|x, y\| + \|x, z\|$.

This shows that $\|\cdot, \cdot\|$ is a refined soft 2-norm on $\check{X}$ and so $(\check{X}, \|\cdot, \cdot\|)$ is a refined soft 2-normed space. □

Proposition 4.12. *Any crisp 2-norm $\|\cdot, \cdot\|_X$ on a crisp vector space X can be extended to a refined soft 2-norm on the soft vector space $\check{X}$.*

Proof. First, we will construct the absolute soft $\mathbb{K}$-vector space $\check{X}$ through a non-empty set of parameters A. Let us define a mapping $\|\cdot, \cdot\| : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$ by $\|x, y\|(\lambda) = \|x(\lambda), y(\lambda)\|_X$, $\forall \lambda \in A$, $\forall x, y \widetilde{\in} \check{X}$. Applying a procedure similar to that of the proof of Proposition 4.11, we get that $\|\cdot, \cdot\|$ is a refined soft 2-norm on $\check{X}$.

This refined soft 2-norm is generated using the crisp 2-norm $\|\cdot, \cdot\|_X$ and it is called the refined soft 2-norm generated by $\|\cdot\|_X$. □

Lemma 4.13. *Let* $(\check{X}, \|\cdot,\cdot\|, A)$ *be a refined soft* 2*-normed space. Then, the following condition is hold:*

(2N5) For each pair of vectors $\xi, \eta \in X$ *and* $\lambda \in A$, $\{\|x,y\|(\lambda) : \ x(\lambda) = \xi, \ y(\lambda) = \eta\}$ *is a singleton.*

Proof. Suppose that $(\xi, \eta) \in X \times X$ and $\lambda \in A$. Let $x, x', y, y' \in \check{X}$ be such that $x(\lambda) = \xi = x'(\lambda)$ and $y(\lambda) = \eta = y'(\lambda)$. Then, $(x - x')(\lambda) = \theta = (x' - x)(\lambda)$ and $(y - y')(\lambda) = \theta = (y' - y)(\lambda)$. Furthermore,

$$\|x,y\| - \|x',y'\| \tilde{\leqslant} \|x, y-y'\| + \|y', x-x'\|$$

and

$$\|x',y'\| - \|x,y\| \tilde{\leqslant} \|x', y'-y\| + \|y, x'-x\|,$$

which implies that

$$\begin{aligned}\|x,y\|(\lambda) - \|x',y'\|(\lambda) &\leqslant \|x, y-y'\|(\lambda) + \|y', x-x'\|(\lambda)\\ &= 0 + 0 = 0 \ \text{(by Lemma 4.10)}\end{aligned}$$

and

$$\begin{aligned}\|x',y'\|(\lambda) - \|x,y\|(\lambda) &\leqslant \|x', y'-y\|(\lambda) + \|y, x'-x\|(\lambda)\\ &= 0 + 0 = 0 \ \text{(again by Lemma 4.10)}.\end{aligned}$$

Thus,

$$\|x,y\|(\lambda) - \|x',y'\|(\lambda) \leqslant 0$$

and

$$\|x',y'\|(\lambda) - \|x,y\|(\lambda) \leqslant 0.$$

Therefore, $|\, \|x,y\|(\lambda) - \|x',y'\|(\lambda)\, | \leqslant 0$ and hence, $\|x,y\|(\lambda) = \|x',y'\|(\lambda)$. This completes the proof. □

The lemma above allow us to prove the following important theorem about refined soft 2-normed spaces.

Theorem 4.14. ***(Decomposition theorem)*** *Let* $(\check{X}, \|\cdot,\cdot\|, A)$ *be a refined soft* 2*-normed space. Then, for each* $\lambda \in A$*, the mapping* $\|\cdot,\cdot\|_\lambda : X \times X \to \mathbb{R}^+$ *defined by* $\|\xi,\eta\|_\lambda := \|x,y\|(\lambda)$ *where* $x, y \tilde{\in} \check{X}$ *are such that* $x(\lambda) = \xi$ *and* $y(\lambda) = \eta$*, is a 2-norm on* X.

Proof. By Lemma 4.13, we have for $\lambda \in A$, $\{\|x,y\|(\lambda) : \ x(\lambda) = \xi, \ y(\lambda) = \eta\}$ is a singleton, which implies that the mapping $\|\cdot,\cdot\|_\lambda : X \times X \to \mathbb{R}^+$ is well defined. Thus, from (2N1)-(2N4), it follows that $\|\cdot,\cdot\|_\lambda$ is a 2-norm on X for all $\lambda \in A$. □

Definition 4.15 ([36])**.** A sequence $\{x_n\}$ of soft elements in a soft normed space $(\check{X}, \|\cdot\|, A)$ is said to be **soft convergent** and soft converges to a soft element $x \tilde{\in} \check{X}$, if for every soft real number $\varepsilon \tilde{>} \hat{0}$ there exists a soft natural number N such that $\|x_n - x\|\,(\lambda) < \varepsilon(\lambda)$ whenever $n \geqslant N(\lambda)$, for all $\lambda \in A$. In such a case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, where x is called the **soft limit** of the sequence $\{x_n\}$ in the soft normed space $(\check{X}, \|\cdot\|, A)$.

Definition 4.16. A sequence $\{x_n\}$ of soft elements in a refined soft 2-normed space $(\check{X}, \|\cdot,\cdot\|, A)$ is said to be **soft convergent** to $x \tilde{\in} \check{X}$, if $\|x_n - x, y\| \to \hat{0}$ for every $y \tilde{\in} \check{X}$. In this case, we say that x is the **soft limit** of the sequence and that $\{x_n\}$ is **soft convergent** in the refined soft 2-normed space $(\check{X}, \|\cdot,\cdot\|, A)$.

Theorem 4.17. *If a sequence of soft elements in a refined soft 2-normed space is soft convergent, then its soft limit is unique.*

Proof. Suppose that $\{x_n\}$ is a sequence of soft elements in a refined soft 2-normed space $(\check{X}, \|\cdot,\cdot\|, A)$ such that x and y are two distinct soft limits of $\{x_n\}$. We select $z \widetilde{\in} \check{X}$ such that $\|x-y,z\| \neq \hat{0}$. Then, there exists at least one $\lambda_0 \in A$ such that $\|x-y,z\|(\lambda_0) > 0$. We choose a positive real number ε_{λ_0} satisfying $0 < \varepsilon_{\lambda_0} < \frac{1}{2}\|x-y,z\|(\lambda_0)$. Let ε be a soft real number defined by $\varepsilon(\lambda) = \varepsilon_{\lambda_0}$ for each $\lambda \in A$, so $\varepsilon \widetilde{>} \hat{0}$. Since $\|x_n - x, z\| \to \hat{0}$ and $\|x_n - y, z\| \to \hat{0}$, there exists two soft natural numbers N_1 and N_2 such that $\|x_n - x, z\|(\lambda) < \varepsilon(\lambda) = \varepsilon_{\lambda_0}$ whenever $n \geqslant N_1(\lambda)$, for all $\lambda \in A$, and $\|x_n - y, z\|(\lambda) < \varepsilon(\lambda) = \varepsilon_{\lambda_0}$ whenever $n \geqslant N_2(\lambda)$, for all $\lambda \in A$. If $N = \max\{N_1, N_2\}$, where the maximum of these soft natural numbers is taken as component wise, then $\|x_n - x, z\|(\lambda) < \varepsilon_{\lambda_0}$ and $\|x_n - y, z\|(\lambda) < \varepsilon_{\lambda_0}$ for all $n \geqslant N(\lambda)$, $\lambda \in A$. Then, by the triangle inequality, we get that

$$\begin{aligned}\|x-y,z\|(\lambda) &\leqslant \|x-x_n,z\|(\lambda) + \|x_n-y,z\|(\lambda) < \varepsilon_{\lambda_0} + \varepsilon_{\lambda_0} \\ &< \frac{1}{2}\|x-y,z\|(\lambda_0) + \frac{1}{2}\|x-y,z\|(\lambda_0) = \|x-y,z\|(\lambda_0),\ \forall \lambda \in A,\end{aligned}$$ □

which is a contradiction.

5. Refined soft 2-inner product space

Definition 5.1. Let $\check{X}$ be an absolute soft $\mathbb{K}$-vector space. A **refined soft 2-inner product** on $\check{X}$ is a mapping $\langle \cdot, \cdot \,|\, \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ which satisfies the following conditions:

(2I1) $\langle x, x \,|\, y \rangle \widetilde{\geqslant} \hat{0}$ for all $x, y \widetilde{\in} \check{X}$ and $\langle x, x \,|\, y \rangle = \hat{0}$ if and only if x and y are refined linearly dependent in $\check{X}$;
(2I2) $\langle x, x \,|\, y \rangle = \overline{\langle y, y \,|\, x \rangle}$ for all $x, y \widetilde{\in} \check{X}$;
(2I3) $\langle x, y \,|\, z \rangle = \overline{\langle y, x \,|\, z \rangle}$ for all $x, y, z \widetilde{\in} \check{X}$;
(2I4) $\langle \alpha \cdot x, y \,|\, z \rangle = \alpha \cdot \langle x, y \,|\, z \rangle$ for all $x, y, z \widetilde{\in} \check{X}$ and for all soft scalar α;
(2I5) $\langle \tilde{x_1} + \tilde{x_2}, y \,|\, z \rangle = \langle \tilde{x_1}, y \,|\, z \rangle + \langle \tilde{x_2}, y \,|\, z \rangle$ for all $\tilde{x_1}, \tilde{x_2}, y, z \widetilde{\in} \check{X}$.

The absolute soft $\mathbb{K}$-vector space $\check{X}$ with a refined soft 2-inner product $\langle \cdot, \cdot \,|\, \cdot \rangle$ on $\check{X}$ is said to be a **refined soft 2-inner product space** and is denoted by $(\check{X}, \langle \cdot, \cdot \,|\, \cdot \rangle, A)$ or $(\check{X}, \langle \cdot, \cdot \,|\, \cdot \rangle)$. Observe that, from Remark 3.11, it follows that $\langle x, y \,|\, z \rangle$ can be expressed as $\langle x, y \,|\, z \rangle = \mathrm{Re}\langle x, y \,|\, z \rangle + i\,\mathrm{Im}\langle x, y \,|\, z \rangle$.

Example 5.2. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. We know that by Example 2.3, the standard 2-inner product $\langle \cdot, \cdot \,|\, \cdot \rangle$ is defined on X by*

$$\langle x, y \,|\, z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle.$$

In this way, the mapping $\langle \cdot, \cdot \,|\, \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ defined by

$$\langle x, y \,|\, z \rangle(\lambda) = \langle x(\lambda), y(\lambda) \,|\, z(\lambda) \rangle = \langle x(\lambda), y(\lambda) \rangle \langle z(\lambda), z(\lambda) \rangle - \langle x(\lambda), z(\lambda) \rangle \langle z(\lambda), y(\lambda) \rangle, \text{ for all } \lambda \in A,$$

is a refined soft 2-inner product on the soft vector space $\check{X}$.

Now, we can consider $X = \ell_2$ with the classical inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$, where $x = \{\xi_i\}$, $y = \{\eta_i\}$ belong to ℓ_2. Then ℓ_2 is a 2-inner product space with respect to the standard 2-inner product

$$\langle x, y \,|\, z \rangle = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle,$$

where $x = \{\xi_i\}$, $y = \{\eta_i\}$, $z = \{\zeta_i\} \in \ell_2$. If x, y and z are soft elements of the absolute soft vector space $\check{X}$, then $x(\lambda) = \{\xi_i^\lambda\}$, $y(\lambda) = \{\eta_i^\lambda\}$ and $z(\lambda) = \{\zeta_i^\lambda\}$ are elements of ℓ_2 and the refined soft 2-inner product is given by

$$\langle x, y \,|\, z\rangle(\lambda) = \left(\sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\eta_i^{\lambda}}\right)\left(\sum_{i=1}^{\infty} |\zeta_i^{\lambda}|^2\right) - \left(\sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\zeta_i^{\lambda}}\right)\left(\sum_{i=1}^{\infty} \zeta_i^{\lambda} \overline{\eta_i^{\lambda}}\right)$$

Remark 5.3. From conditions (2-I3) and (2-I4), we have:

i. $\langle x, \Theta \,|\, z\rangle = \hat{0}$ and $\langle \Theta, y \,|\, z\rangle = \hat{0}$.

ii. $\langle x, \alpha \cdot y \,|\, z\rangle = \overline{\alpha} \cdot \langle x, y \,|\, z\rangle$.

Proposition 5.4. *Let $(\check{X}, \langle \cdot, \cdot \,|\, \cdot\rangle, A)$ be a refined soft 2-inner product space. Then, for all $x, y, z \,\widetilde{\in}\, \check{X}$ and all $\lambda \in A$, the following statements hold:*

1. $\langle z, z \,|\, x \pm y\rangle = \langle x, x \,|\, z\rangle + \langle y, y \,|\, z\rangle \pm \left[\langle x, y \,|\, z\rangle + \overline{\langle x, y \,|\, z\rangle}\right]$.
2. $\mathrm{Re}\langle x, y \,|\, z\rangle = \widehat{(\frac{1}{4})}\,[\langle z, z \,|\, x + y\rangle - \langle z, z \,|\, x - y\rangle]$.
3. $\mathrm{Im}\langle x, y \,|\, z\rangle = \widehat{(\frac{1}{4})}\,\left[\langle z, z \,|\, x + \hat{\imath} y\rangle - \langle z, z \,|\, x - \hat{\imath} y\rangle\right]$, *where $\hat{\imath}$ is the soft complex number $\hat{\imath}(\lambda) = i$ for all $\lambda \in A$.*
4. $\langle x, y \,|\, z\rangle = \widehat{(\frac{1}{4})}\,[\langle x + y, x + y \,|\, z\rangle - \langle x - y, x - y \,|\, z\rangle] + \widehat{(\frac{i}{4})}\,\left[\langle x + \hat{\imath} y, x + \hat{\imath} y \,|\, z\rangle - \langle x - \hat{\imath} y, x - \hat{\imath} y \,|\, z\rangle\right]$.
5. $\langle x, y \,|\, \Theta\rangle = \hat{0}$.

Proof. (1). Let $x, y, z \,\widetilde{\in}\, \check{X}$. Using conditions (2-I2)-(2-I5), we have

$$\begin{aligned}
\langle z, z \,|\, x \pm y\rangle &= \langle x \pm y, x \pm y \,|\, z\rangle = \langle x, x \pm y \,|\, z\rangle \pm \langle y, x \pm y \,|\, z\rangle \\
&= \overline{\langle x \pm y, x \,|\, z\rangle} \pm \overline{\langle x \pm y, y \,|\, z\rangle} \\
&= \overline{\langle x, x \,|\, z\rangle \pm \langle y, x \,|\, z\rangle} \pm \overline{\langle x, y \,|\, z\rangle \pm \langle y, y \,|\, z\rangle} \\
&= \langle x, x \,|\, z\rangle \pm \overline{\langle y, x \,|\, z\rangle} \pm \overline{\langle x, y \,|\, z\rangle} + \langle y, y \,|\, z\rangle \\
&= \langle x, x \,|\, z\rangle + \langle y, y \,|\, z\rangle \pm \left[\langle x, y \,|\, z\rangle + \overline{\langle x, y \,|\, z\rangle}\right].
\end{aligned}$$

(2) From part (1), it follows that

$$\langle x, y \,|\, z\rangle(\lambda) + \overline{\langle x, y \,|\, z\rangle}(\lambda) = \langle z, z \,|\, x + y\rangle(\lambda) - [\langle x, x \,|\, z\rangle + \langle y, y \,|\, z\rangle]\,(\lambda) \tag{5.1}$$

and

$$\langle x, y \,|\, z\rangle(\lambda) + \overline{\langle x, y \,|\, z\rangle}(\lambda) = \langle x, x \,|\, z\rangle(\lambda) + \langle y, y \,|\, z\rangle(\lambda) - \langle z, z \,|\, x - y\rangle(\lambda), \tag{5.2}$$

for all $\lambda \in A$ and all $x, y, z \,\widetilde{\in}\, \check{X}$. Adding equations (5.1) and (5.2), we get that

$$2\left[\langle x, y \,|\, z\rangle(\lambda) + \overline{\langle x, y \,|\, z\rangle}(\lambda)\right] = \langle z, z \,|\, x + y\rangle(\lambda) - \langle z, z \,|\, x - y\rangle(\lambda),$$

which implies that

$$\langle x, y \,|\, z\rangle(\lambda) + \overline{\langle x, y \,|\, z\rangle}(\lambda) = \frac{1}{2}\,[\langle z, z \,|\, x + y\rangle(\lambda) - \langle z, z \,|\, x - y\rangle(\lambda)]\,,$$

for all $\lambda \in A$ and all $x, y, z \,\widetilde{\in}\, \check{X}$. Thus,

$$2\,\mathrm{Re}\langle x, y \,|\, z\rangle(\lambda) = \frac{1}{2}\,[\langle z, z \,|\, x + y\rangle - \langle z, z \,|\, x - y\rangle]\,(\lambda)$$

and hence, $\mathrm{Re}\langle x, y \,|\, z\rangle(\lambda) = \frac{1}{4}\,[\langle z, z \,|\, x + y\rangle - \langle z, z \,|\, x - y\rangle]\,(\lambda)$, for all $\lambda \in A$ and all $x, y, z \,\widetilde{\in}\, \check{X}$. This shows that $\mathrm{Re}\langle x, y \,|\, z\rangle = \widehat{(\frac{1}{4})}\,[\langle z, z \,|\, x + y\rangle - \langle z, z \,|\, x - y\rangle]$.

(3) Since $\operatorname{Im} u = \operatorname{Re}(-iu)$ for each ordinary complex number u, we have

$$\operatorname{Im}\langle x, y \mid z\rangle(\lambda) = \operatorname{Re}[-i\langle x, y \mid z\rangle(\lambda)] \text{ for all } \lambda \in A \text{ and all } x, y, z \,\widetilde{\in}\, \check{X}.$$

Thus,

$$\operatorname{Im}\langle x, y \mid z\rangle = \operatorname{Re}[-\hat{\imath}\langle x, y \mid z\rangle] = \operatorname{Re}[\bar{\hat{\imath}}\langle x, y \mid z\rangle] = \operatorname{Re}[\hat{\imath}\,\overline{\langle y, x \mid z\rangle}] = \operatorname{Re}[\langle x, \hat{\imath}y \mid z\rangle],$$

for all $x, y, z \,\widetilde{\in}\, \check{X}$. Therefore, by (ii), we get that $\operatorname{Im}\langle x, y \mid z\rangle = \widehat{(\frac{1}{4})}\left[\langle z, z \mid x + \hat{\imath}y\rangle - \langle z, z \mid x - \hat{\imath}y\rangle\right]$ for all $x, y, z \,\widetilde{\in}\, \check{X}$.

(4) The proof follows from Remark 3.11 and parts (2) and (3).

(5) From part (4), for any soft scalar α, we have

$$\begin{aligned}
\langle x, y \mid \alpha z\rangle =& \widehat{\left(\frac{1}{4}\right)}\left[\langle \alpha z, \alpha z \mid x + y\rangle - \langle \alpha z, \alpha z \mid x - y\rangle\right] + \widehat{\left(\frac{i}{4}\right)}\left[\langle \alpha z, \alpha z \mid x + \hat{\imath}y\rangle - \langle \alpha z, \alpha z \mid x - \hat{\imath}y\rangle\right]\\
=& \widehat{\left(\frac{1}{4}\right)}\alpha\overline{\alpha}\left[\langle x + y, x + y \mid z\rangle - \langle x - y, x - y \mid z\rangle\right]\\
&+ \widehat{\left(\frac{i}{4}\right)}\alpha\overline{\alpha}\left[\langle x + \hat{\imath}y, x + \hat{\imath}y \mid z\rangle - \langle x - \hat{\imath}y, x - \hat{\imath}y \mid z\rangle\right]\\
=& \alpha\overline{\alpha}\left[\widehat{\left(\frac{1}{4}\right)}\left[\langle x + y, x + y \mid z\rangle - \langle x - y, x - y \mid z\rangle\right]\right.\\
&\left. + \widehat{\left(\frac{i}{4}\right)}\left[\langle x + \hat{\imath}y, x + \hat{\imath}y \mid z\rangle - \langle x - \hat{\imath}y, x - \hat{\imath}y \mid z\rangle\right]\right] = |\alpha|^2\langle x, y \mid \alpha z\rangle,
\end{aligned}$$

for all $x, y, z \,\widetilde{\in}\, \check{X}$. In particular, for $\alpha = \hat{0}$, we conclude that

$$\langle x, y \mid \Theta\rangle = \hat{0}.$$

□

Lemma 5.5. *Let $(\check{X}, \langle\cdot,\cdot \mid \cdot\rangle, A)$ be a refined soft 2-inner product space. Then, for all $x, y \,\widetilde{\in}\, \check{X}$ and for $\lambda \in A$, we have $\langle x, x \mid y\rangle(\lambda) = 0$ if and only if $x(\lambda)$ and $y(\lambda)$ are linearly dependent on X for each $\lambda \in A$.*

Proof. Considering the soft scalar α as in Lemma 4.13 and conditions (2-I2)-(2-I4), we obtain that $\langle \alpha \cdot y, \alpha \cdot y \mid x\rangle = |\alpha|^2 \cdot \langle x, x \mid y\rangle$, which implies that $\langle x, x \mid y\rangle(\lambda) = 0 \iff |\alpha|^2 \cdot \langle x, x \mid y\rangle = \hat{0} \iff \langle \alpha \cdot y, \alpha \cdot y \mid x\rangle = \hat{0} \iff x$ and $\alpha \cdot y$ are refined linearly dependent in $\check{X} \iff x(\lambda)$ and $(\alpha \cdot y)(\lambda)$ are linearly dependent in X for each $\lambda \in A \iff x(\lambda)$ and $y(\lambda)$ are linearly dependent in X for each $\lambda \in A$. Note that we have used Lemma 4.4 here. □

The following example shows that $\langle x, y \mid z\rangle$ can be equal to $\hat{0}$, but this does not imply that the soft vectors x, y, z are refined linearly dependent.

Example 5.6. *Consider the refined soft 2-inner product space defined in Example 5.2. Let x, y and z be three soft elements of $\check{X}$ such that for each $\lambda \in A$, $x(\lambda) = \{1, 2, 1, 0, \dots\} \in \ell_2$, $y(\lambda) = \{1, \frac{1}{2}, -2, 0, \dots\} \in \ell_2$ and $z(\lambda) = \{0, 1, -2, 0, \dots\} \in \ell_2$. Then,*

$$\langle x, y \mid z\rangle = \langle x, y\rangle\langle z, z\rangle - \langle x, z\rangle\langle z, y\rangle = \hat{0}.$$

On the other hand, suppose that α_1, α_2 and α_3 are soft scalars such that $\alpha_1 x + \alpha_2 y + \alpha_3 z = \Theta$. Then, we get the equations $\alpha_1(\nu) + \alpha_2(\nu) = 0$, $2\alpha_1(\nu) + \frac{1}{2}\alpha_2(\nu) + \alpha_3(\nu) = 0$ and $\alpha_1(\nu) - 2\alpha_2(\nu) - 2\alpha_3(\nu) = 0$ for all $\nu \in A$, which implies that $\alpha_1 = \alpha_2 = \alpha_3 = \hat{0}$. Therefore, x, y, z are refined linearly independent in $\check{X}$

Proposition 5.7. *Let* $\{\langle \cdot, \cdot \mid \cdot \rangle_\lambda : \lambda \in A\}$ *be any family of crisp 2-inner products on a crisp vector space* X. *Then the mapping* $\langle \cdot, \cdot \mid \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ *defined by* $\langle x, y \mid z \rangle(\lambda) = \langle x(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda$, *for all* $\lambda \in A$ *and all* $x, y, z \in \check{X}$ *is a refined soft 2-inner product on the soft vector space* $\check{X}$.

Proof. Let us verify that conditions (2I1), (2I2), (2I3), (2I4) and (2I5) of a refined soft 2-inner product are satisfied.

(2I1). For all $x, y \,\widetilde{\in}\, \check{X}$, we have $\langle x, x \mid y \rangle(\lambda) = \langle x(\lambda), x(\lambda) \mid y(\lambda) \rangle_\lambda \geqslant 0$, $\forall \lambda \in A$. Thus, $\langle x, x \mid y \rangle \,\widetilde{\geqslant}\, \hat{0}$. By Lemma 5.5, we have

$$\begin{aligned}
\langle x, x \mid y \rangle = \hat{0} &\iff \langle x, x \mid y \rangle(\lambda) = 0, \ \forall \lambda \in A \\
&\iff x(\lambda) \text{ and } y(\lambda) \text{ are linearly dependent in } X, \text{ for each } \lambda \in A \\
&\iff x \text{ and } y \text{ are refined linearly dependent in } \check{X}.
\end{aligned}$$

(2I2). For all $x, y \,\widetilde{\in}\, \check{X}$, we have $\langle x, x \mid y \rangle(\lambda) = \langle x(\lambda), x(\lambda) \mid y(\lambda) \rangle_\lambda = \langle y(\lambda), y(\lambda) \mid x(\lambda) \rangle_\lambda = \langle y, y \mid x \rangle(\lambda)$, $\forall \lambda \in A$. Therefore, $\langle x, x \mid y \rangle = \langle y, y \mid x \rangle$.

(2I3). For all $x, y, z \,\widetilde{\in}\, \check{X}$, we have

$$\begin{aligned}
\langle x, y \mid z \rangle(\lambda) = \langle x(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda &= \overline{\langle y(\lambda), x(\lambda) \mid z(\lambda) \rangle_\lambda} \\
&= \overline{\langle y, x \mid z \rangle(\lambda)} = \overline{\langle y, x \mid z \rangle}(\lambda), \ \forall \lambda \in A.
\end{aligned}$$

Hence, $\langle x, y \mid z \rangle = \overline{\langle y, x \mid z \rangle}$.

(2I4). For all $x, y, z \,\widetilde{\in}\, \check{X}$ and for each soft scalar α, we have

$$\begin{aligned}
\langle \alpha \cdot x, y \mid z \rangle(\lambda) &= \langle \alpha(\lambda) \cdot x(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda = \alpha(\lambda) \cdot \langle x(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda \\
&= \alpha(\lambda) \cdot \langle x, y \mid z \rangle(\lambda) = (\alpha \cdot \langle x, y \mid z \rangle)(\lambda), \ \forall \lambda \in A.
\end{aligned}$$

Thus, $\langle \alpha \cdot x, y \mid z \rangle = \alpha \cdot \langle x, y \mid z \rangle$.

(2I5). For all $x_1, x_2, y, z \,\widetilde{\in}\, \check{X}$,

$$\begin{aligned}
\langle x_1 + x_2, y \mid z \rangle(\lambda) &= \langle x_1(\lambda) + x_2(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda \\
&= \langle x_1(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda + \langle x_2(\lambda), y(\lambda) \mid z(\lambda) \rangle_\lambda \\
&= \langle x_1, y \mid z \rangle(\lambda) + \langle x_2, y \mid z \rangle(\lambda) = (\langle x_1, y \mid z \rangle + \langle x_2, y \mid z \rangle)(\lambda), \ \forall \lambda \in A.
\end{aligned}$$

Therefore, $\langle x_1 + x_2, y \mid z \rangle = \langle x_1, y \mid z \rangle + \langle x_2, y \mid z \rangle$.
This shows that $\langle \cdot, \cdot \mid \cdot \rangle$ is a refined soft 2-inner product on $\check{X}$. □

Proposition 5.8. *Any crisp 2-inner product* $\langle \cdot, \cdot \mid \cdot \rangle_X$ *on a crisp vector space* X *can be extended to a refined soft 2-inner product on the soft vector space* $\check{X}$.

Proof. First, we will construct the absolute soft $\mathbb{K}$-vector space $\check{X}$ through a non-empty set of parameters A. Let us define a mapping $\langle \cdot, \cdot \mid \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ by $\langle x, y \mid z \rangle(\lambda) = \langle x(\lambda), y(\lambda) \mid z(\lambda) \rangle_X$, $\forall \lambda \in A$, $\forall x, y, z \widetilde{\in} \check{X}$. Applying a procedure similar to that of the proof of Proposition 5.7, we conclude that $\langle \cdot, \cdot \mid \cdot \rangle$ is a refined soft 2-inner product on $\check{X}$.

This refined soft 2-inner product is generated using the crisp 2-inner product $\langle \cdot, \cdot \mid \cdot \rangle_X$ and it is called the refined soft 2-inner product generated by $\langle \cdot, \cdot \mid \cdot \rangle_X$. □

Lemma 5.9. *Let* $(\check{X}, \langle \cdot, \cdot \mid \cdot \rangle, A)$ *be a refined soft 2-inner product space. Then, the following condition is hold:*
(2-I6) *For each pair of vectors* $\xi, \eta \in X$ *and* $\lambda \in A$, $\{\langle x, x \mid y \rangle(\lambda) : \ x(\lambda) = \xi, \ y(\lambda) = \eta\}$ *is a singleton.*

Proof. Assume that $(\xi,\eta) \in X \times X$ and $\lambda \in A$. Let $x, x', y, y' \in \check{X}$ be such that $x(\lambda) = \xi = x'(\lambda)$ and $y(\lambda) = \eta = y'(\lambda)$. Since $(x - x')(\lambda) = \theta = (y' - y)(\lambda)$ and $\lambda \in A$ is arbitrary, we have $x - x' = \Theta = y' - y$, which implies that

$$\langle x - x', x \,|\, y\rangle = \langle \Theta, x \,|\, y\rangle = \hat{0} = \langle x', \Theta \,|\, y\rangle = \langle x', x - x' \,|\, y\rangle \text{ and}$$
$$\langle y', y' - y \,|\, x'\rangle = \langle y', \Theta \,|\, x'\rangle = \hat{0} = \langle y, \Theta \,|\, x'\rangle = \langle y, y' - y \,|\, x'\rangle.$$

Therefore, the result follows from the facts that

$$\langle x, x \,|\, y\rangle = \langle x - x', x \,|\, y\rangle + \langle x', x - x' \,|\, y\rangle + \langle x', x' \,|\, y\rangle$$

and

$$\langle x', x' \,|\, y'\rangle = \langle x', x' \,|\, y\rangle + \langle y', y' - y \,|\, x'\rangle + \langle y, y' - y \,|\, x'\rangle. \qquad \square$$

Proposition 5.10. ***(Cauchy-Schwarz Inequality)*** *If $(\check{X}, \langle \cdot, \cdot \,|\, \cdot\rangle, A)$ is a refined soft 2-inner product space, then*

$$|\langle x, y \,|\, z\rangle|^2 \,\tilde{\leqslant}\, \langle x, x \,|\, z\rangle \, \langle y, y \,|\, z\rangle,$$

for all $x, y, z \in \check{X}$.

Proof. Suppose that $x, y, z \tilde{\in} \check{X}$ are not the soft vector Θ, because if either of them is Θ, then by Remark 5.3(i) and Proposition 5.4(5), it follows that $\langle x, y \,|\, z\rangle = \hat{0}$ and $\langle x, x \,|\, z\rangle \, \langle y, y \,|\, z\rangle = \hat{0}$, so we obtain the equality trivially. Then, for each soft scalar α, we have

$$\begin{aligned}
\hat{0} \tilde{\leqslant} \langle x - \alpha y, x - \alpha y \,|\, z\rangle &= \langle x, x - \alpha y \,|\, z\rangle - \langle \alpha y, x - \alpha y \,|\, z\rangle \\
&= \overline{\langle x - \alpha y, x \,|\, z\rangle} - \overline{\langle x - \alpha y, \alpha y \,|\, z\rangle} \\
&= \overline{\langle x, x \,|\, z\rangle - \langle \alpha y, x \,|\, z\rangle} - \overline{\langle x, \alpha y \,|\, z\rangle - \langle \alpha y, \alpha y \,|\, z\rangle} \\
&= \langle x, x \,|\, z\rangle - \overline{\alpha} \langle x, y \,|\, z\rangle - \alpha \langle y, x \,|\, z\rangle - \alpha\overline{\alpha} \langle y, y \,|\, z\rangle \\
&= \langle x, x \,|\, z\rangle - \alpha[\langle y, x \,|\, z\rangle - \overline{\alpha} \langle y, y \,|\, z\rangle] - \overline{\alpha} \langle x, y \,|\, z\rangle.
\end{aligned}$$

Now, we consider the following two possible cases:
Case 1: y and z are not refined linearly dependent.
Case 2: y and z are refined linearly dependent.
In Case 1, $\langle y, y \,|\, z\rangle \tilde{>} \hat{0}$ and putting $\alpha = \dfrac{\langle x, y \,|\, z\rangle}{\langle y, y \,|\, z\rangle}$, we get that

$$\begin{aligned}
\hat{0} \,\tilde{\leqslant}& \langle x, x \,|\, z\rangle - \alpha\left[\langle y, x \,|\, z\rangle - \frac{\langle y, x \,|\, z\rangle}{\langle y, y \,|\, z\rangle} \langle y, y \,|\, z\rangle\right] - \frac{\langle y, x \,|\, z\rangle}{\langle y, y \,|\, z\rangle} \langle x, y \,|\, z\rangle \\
=& \langle x, x \,|\, z\rangle - \frac{\overline{\langle x, y \,|\, z\rangle}}{\langle y, y \,|\, z\rangle} \langle x, y \,|\, z\rangle.
\end{aligned}$$

Thus, $\dfrac{\overline{\langle x, y \,|\, z\rangle}}{\langle y, y \,|\, z\rangle} \langle x, y \,|\, z\rangle \tilde{\leqslant} \langle x, x \,|\, z\rangle$ and hence, $|\langle x, y \,|\, z\rangle|^2 = \overline{\langle x, y \,|\, z\rangle} \langle x, y \,|\, z\rangle \tilde{\leqslant} \langle x, x \,|\, z\rangle \langle y, y \,|\, z\rangle$.
In Case 2, $\langle y, y \,|\, z\rangle = \hat{0}$ and there exist soft scalars α_1 and α_2 with $\alpha_i(\lambda) \neq 0$ for some $i \in \{1, 2\}$ and each $\lambda \in A$, such that $\alpha_1 y + \alpha_2 z = \Theta$. If $\alpha_1(\lambda) \neq 0$ for each $\lambda \in A$, then we can write $y = \alpha z$ where α is the soft scalar defined by $\alpha(\lambda) = -\dfrac{\alpha_2(\lambda)}{\alpha_1(\lambda)}$, so $\langle x, y \,|\, z\rangle = \langle x, \alpha z \,|\, z\rangle = \bar{\alpha} \langle x, z \,|\, z\rangle = \bar{\alpha} \overline{\langle z, x \,|\, z\rangle} = \bar{\alpha} \bar{\hat{0}} = \bar{\alpha} \hat{0} = \hat{0}$. If $\alpha_2(\lambda) \neq 0$ for each $\lambda \in A$, then we can write $z = \alpha y$ where α is the soft scalar defined by $\alpha(\lambda) = -\dfrac{\alpha_1(\lambda)}{\alpha_2(\lambda)}$, so $\langle x, y \,|\, z\rangle = \langle x, y \,|\, \alpha y\rangle = |\alpha|^2 \langle x, y \,|\, y\rangle = |\alpha|^2 \, \hat{0} = \hat{0}$. Therefore, $|\langle x, y \,|\, z\rangle|^2 = \overline{\langle x, y \,|\, z\rangle} \langle x, y \,|\, z\rangle = \hat{0} = \langle x, x \,|\, z\rangle \, \hat{0} = \langle x, x \,|\, z\rangle \langle y, y \,|\, z\rangle$. $\square$

Proposition 5.11. *Let $(\check{X}, \langle \cdot, \cdot \mid \cdot \rangle, A)$ be a refined soft 2-inner product space. Then in view of Cauchy-Schwarz Inequality we can define the mapping $\|\cdot,\cdot\| : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)$, $\|x, y\| := \sqrt{\langle x, x \mid y \rangle}$, for all $x, y \,\widetilde{\in}\, \check{X}$, which is a refined soft 2-norm on $\check{X}$.*

Proof. Suppose that $h, k, \xi \,\widetilde{\in}\, \check{X}$ and let α be a soft scalar. Then $\|h, \xi\| = \sqrt{\langle h, h \mid \xi \rangle} \,\widetilde{\geqslant}\, \hat{0}$ by definition. Now $\|h, \xi\|^2 = \hat{0}$ if and only if $\langle h, h \mid \xi \rangle = \hat{0}$, and by (2I1) we have h and ξ are refined linearly dependent, so that (2N1) is satisfied. Also, by (2I2), we have (2N2) is satisfied, this is,

$$\|h, \xi\| = \sqrt{\langle h, h \mid \xi \rangle} = \sqrt{\langle \xi, \xi \mid h \rangle} = \|\xi, h\|.$$

On the other hand,

$$\|\alpha h, \xi\| = \sqrt{\langle \alpha h, \alpha h \mid \xi \rangle} = \sqrt{\alpha \bar{\alpha} \langle h, h \mid \xi \rangle} = \sqrt{|\alpha|^2 \langle h, h \mid \xi \rangle} = |\alpha| \sqrt{\langle h, h \mid \xi \rangle} = |\alpha| \|h, \xi\|,$$

which implies that (2N3) is satisfaced. Finally, we have

$$\begin{aligned} \|h + k, \xi\|^2 &= \langle h + k, h + k \mid \xi \rangle = \langle h, h \mid \xi \rangle + 2\,\mathrm{Re}(\langle h, k \mid \xi \rangle) + \langle k, k \mid \xi \rangle \\ &\widetilde{\leqslant} \langle h, h \mid \xi \rangle + 2|\langle h, k \mid \xi \rangle| + \langle k, k \mid \xi \rangle \\ &\widetilde{\leqslant} \langle h, h \mid \xi \rangle + 2\left(\langle h, h \mid \xi \rangle^{1/2} \langle k, k \mid \xi \rangle^{1/2}\right) + \langle k, k \mid \xi \rangle \\ &= \left(\langle h, h \mid \xi \rangle^{1/2} + \langle k, k \mid \xi \rangle^{1/2}\right)^2 = (\|h, \xi\| + \|k, \xi\|)^2, \end{aligned}$$

where we have used Cauchy-Schwarz Inequality. Thus, we conclude that the refined soft 2-inner product induces a refined soft 2-norm on $\check{X}$. □

Corollary 5.12. *Let $(\check{X}, \langle \cdot, \cdot \mid \cdot \rangle, A)$ be a refined soft 2-inner product space. Then for each $\lambda \in A$ the mapping $\|\cdot,\cdot\|_\lambda : X \times X \to \mathbb{R}^+$ defined by $\|\xi, \eta\|_\lambda := \|x, y\|(\lambda) = \sqrt{\langle x, x \mid y \rangle(\lambda)}$ where $x, y \in \check{X}$ are such that $x(\lambda) = \xi$ and $y(\lambda) = \eta$ is a refined soft 2-norm on X.*

Proof. It is clear by Theorem 4.14. □

Proposition 5.13. *Let $(\check{X}, \langle \cdot, \cdot \mid \cdot \rangle, A)$ be a refined soft 2-inner product space and $x, z \,\widetilde{\in}\, \check{X}$. Then,*

$$\|x, z\| = \sup\{|\langle x, y \mid z \rangle| : y \,\widetilde{\in}\, \check{X}, \|y, z\| = \hat{1}\}.$$

Proof. Let $x, z \,\widetilde{\in}\, \check{X}$ and $L = \{|\langle x, y \mid z \rangle| : y \,\widetilde{\in}\, \check{X}, \|y, z\| = \hat{1}\}$. We will show that $\|x, z\| = \sup L$. First, we will prove that $\sup L \,\widetilde{\leqslant}\, \|x, z\|$. Indeed, if $|\langle x, y \mid z \rangle| \in L$, then $\|y, z\| = \hat{1}$. Now, by Cauchy-Schwarz inequality, we have

$$|\langle x, y \mid z \rangle| \,\widetilde{\leqslant}\, \|x, z\| \|y, z\| = \|x, z\|,$$

which implies that $|\langle x, y \mid z \rangle| \,\widetilde{\leqslant}\, \|x, z\|$, so $\|x, z\|$ is an upper bound for L. Therefore, $\sup L \,\widetilde{\leqslant}\, \|x, z\|$.

On the other hand, we will verify that $\|x, z\| \,\widetilde{\leqslant}\, \sup L$, it is clear for the case $\|x, z\| = \hat{0}$, in other case, we put $y = \frac{1}{\|x,z\|} x$ and let us note that $\|y, z\| = \hat{1}$ and $\|x, z\| = |\langle x, y \mid z \rangle|$; this is,

$$\begin{aligned} \|y, z\| = \langle y, y \mid z \rangle &= \left\langle \frac{1}{\|x, z\|} x, \frac{1}{\|x, z\|} x \mid z \right\rangle = \frac{1}{\|x, z\|} \left(\overline{\frac{1}{\|x, z\|}}\right) \langle x, x \mid z \rangle \\ &= \frac{1}{\|x, z\|^2} \langle x, x \mid z \rangle = \frac{1}{\|x, z\|^2} \|x, z\|^2 = 1. \end{aligned}$$

and

$$|\langle x, y \mid z \rangle| = \left| \left\langle x, \frac{x}{\|x, z\|} \mid z \right\rangle \right| = \frac{1}{\|x, z\|} |\langle x, x \mid z \rangle| = \frac{1}{\|x, z\|} \|x, z\|^2 = \|x, z\|.$$

Thus $\|x,z\| \in L$ and hence $\|x,z\| \,\widetilde{\leqslant}\, \sup L$. This shows that $\|x,z\| = \sup L$. □

Proposition 5.14. *Let $(\check{X}, \langle\cdot,\cdot\,|\,\cdot\rangle, A)$ be a refined soft 2-inner product space. Then, for all $x, y, z \in \check{X}$ we have*

1. $\|x+y,z\|^2 + \|x-y,z\|^2 = 2\|x,z\|^2 + 2\|y,z\|^2$ *(Parallelogram law);*
2. $\langle x, y\,|\,z\rangle = \widehat{(\frac{1}{4})}\left\{\|x+y,z\|^2 - \|x+y,z\|^2 + \hat{\imath}\|x+\hat{\imath}y,z\|^2 - \hat{\imath}\|x-\hat{\imath}y,z\|^2\right\}$ *(Polarization identity).*

Proof. The proof of (1) is similar to the one given in [25]. Furthermore, (2) is an immediate consequence of Proposition 5.4. □

Remark 5.15. Given two soft $\mathbb{R}$-numbers x and y with $x \,\widetilde{<}\, y$, in the sense of Definition 3.7, we can define the following set:

$$(x,y) := \{z \in \mathbb{R}(A) : x \,\widetilde{<}\, z \,\widetilde{<}\, y\} = \{z \in \mathbb{R}(A) : (\forall \alpha \in A)(x(\alpha) < z(\alpha) < y(\alpha))\}.$$

Then it is easy to check that the collection of sets of the form (x,y) is a basis for a topology on $\mathbb{R}(A)$. Moreover, $\mathbb{Q}(A)$ is a dense subspace of $\mathbb{R}(A)$.

Theorem 5.16. *Let $(\check{X}, \|\cdot,\cdot\|, A)$ be a refined soft 2-normed space. Then, the refined soft 2-norm $\|\cdot,\cdot\|$ is induced by a refined 2-inner product if it satisfies the Parallelogram law.*

Proof. Let us first consider the real case, that is, $\check{X}$ is an absolute soft $\mathbb{R}$-vector space. For this case, we define the mapping

$$\varphi : SE(\check{X}) \times SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A),\ \ \varphi(x,y,z) = \widehat{(\frac{1}{4})}(\|x+y,z\|^2 - \|x-y,z\|^2) \text{ for all } x,y,z \in SE(\check{X}).$$

It is clear that $\|x,y\|^2 = \varphi(x,x,y)$ for all $x, y \in SE(\check{X})$. Now, we will prove that φ verifies the properties of a refined soft 2-inner product. Indeed, if $x, y, z \in SE(\check{X})$, then:
(2I1) $\varphi(x,x,y) = \|x,y\|^2 \,\widetilde{\geqslant}\, \hat{0}$. Furthermore, note that

$$\varphi(x,x,y) = \hat{0} \iff \|x,y\| = \hat{0} \iff \ \ x \text{ and } y \text{ are refined linearly dependent in } \check{X}.$$

(2I2)

$$\varphi(x,x,y) = \|x,y\|^2 = \|y,x\|^2 = \varphi(y,y,x).$$

(2I3) This is clear.
(2I5) We need to prove that $\varphi(x_1+x_2,y,z) = \varphi(x_1,y,z) + \varphi(x_2,y,z)$ for all $x_1, x_2, y, z \in SE(\check{X})$. Indeed,

$$\varphi(x_1+x_2,y,z) = \widehat{(\frac{1}{4})}(\|x_1+x_2+y,z\|^2 - \|x_1+x_2-y,z\|^2),$$

and by Parallelogram law, we have

$$\|x_1+x_2+y,z\|^2 = \hat{2}\|x_1+y,z\|^2 + \hat{2}\|x_2,z\|^2 - \|x_1-x_2+y,z\|^2;$$

$$\|x_1+x_2-y,z\|^2 = \hat{2}\|x_1,z\|^2 + \hat{2}\|x_2-y,z\|^2 - \|x_1-x_2+y,z\|^2.$$

Then,

$$\begin{aligned}\hat{4}\varphi(x_1+x_2,y,z) &= \|x_1+x_2+y,z\|^2 - \|x_1+x_2-y,z\|^2 \\ &= \hat{2}\|x_1+y,z\|^2 + \hat{2}\|x_2,z\|^2 - \hat{2}\|x_1,z\|^2 - \hat{2}\|x_2-y,z\|^2.\end{aligned}$$

Similarly,

$$\hat{4}\varphi(x_1+x_2,y,z) = \hat{4}\varphi(x_2+x_1,y,z) = \hat{2}\|x_2+y,z\|^2 + \hat{2}\|x_1,z\|^2 - \hat{2}\|x_2,z\|^2 - \hat{2}\|x_1-y,z\|^2.$$

So,

$$\hat{8}\varphi(x_1+x_2,y,z)=\hat{2}\|x_1+y,z\|^2+\hat{2}\|x_2+y,z\|^2-\hat{2}\|x_2-y,z\|^2-\hat{2}\|x_1-y,z\|^2.$$

Therefore,

$$\hat{8}\varphi(x_1+x_2,y,z)=\hat{8}\varphi(x_1,y,z)+\hat{8}\varphi(x_2,y,z);\text{ this is, }\varphi(x_1+x_2,y,z)=\varphi(x_1,y,z)+\varphi(x_2,y,z).$$

(2I4) We consider the set

$$L=\{\alpha\in\mathbb{R}(A):\alpha\varphi(x,y,z)=\varphi(\alpha x,y,z),\ \forall x,y,z\in SE(\check{X})\}.$$

We want to prove that $L=\mathbb{R}(A)$. In view of (2I5) is clear that $(\alpha\pm\beta)\in L$ for all $\alpha,\beta\in L$. Indeed,

$$\begin{aligned}(\alpha\pm\beta)\varphi(x,y,z)&=\alpha\varphi(x,y,z)\pm\beta\varphi(x,y,z)=\varphi(\alpha x,y,z)\pm\varphi(\beta x,y,z)\\&=\varphi((\alpha\pm\beta)x,y,z).\end{aligned}$$

Furthermore, for all $\alpha,\beta\in\mathbb{Z}(A)$ with $\beta(\lambda)\neq 0$ for all $\lambda\in A$, we have

$$\frac{\alpha}{\beta}\varphi(x,y,z)=\frac{1}{\beta}\varphi(\alpha x,y,z)=\frac{1}{\beta}\varphi\left(\beta\left(\frac{\alpha}{\beta}\right)x,y,z\right)=\varphi\left(\frac{\alpha}{\beta}x,y,z\right).$$

Hence $\dfrac{\alpha}{\beta}\in L$. Thus $\mathbb{Q}(A)\subseteq L$. Now, given $x,y,z\in SE(\check{X})$ we consider the mapping $l_{xyz}:\mathbb{R}(A)\to\mathbb{R}(A)$ defined by $l_{xyz}(\alpha)=\varphi(\alpha x,y,z)-\alpha\varphi(x,y,z)$. Furthermore, note that the mapping $\alpha\mapsto\|\alpha r,s\|$, $r,s\,\widetilde{\in}\,\check{X}$, is a continuous mapping in $\mathbb{R}(A)$. Indeed,

$$|\|\alpha r,s\|-\|\beta r,s\||=|(|\alpha|-|\beta|)\|x,y\||=||\alpha|-|\beta||\|x,y\|\ \widetilde{\leqslant}\ |\alpha-\beta|\|x,y\|$$

which shows that l_{xyz} is also a continuous mapping in $\mathbb{R}(A)$ for every $x,y,z\in SE(\check{X})$. Then, $L=\bigcap_{x,y,z\in SE(\check{X})} l^{-1}_{xyz}(\{\hat{0}\})$ is a closed subset of $\mathbb{R}(A)$ respect to the topology give in Remark 5.15. Therefore, $L=\mathbb{R}(A)$ and φ is a refined soft real 2-inner product.

On the other hand, if $\check{X}$ is an absolute soft $\mathbb{C}$-vector space. Note that the mapping

$$\gamma(x,y,z):=\widehat{(\frac{1}{4})}\left\{\|x+y,z\|^2-\|x+y,z\|^2+\hat{\imath}\|x+\hat{\imath}y,z\|^2-\hat{\imath}\|x-\hat{\imath}y,z\|^2\right\}=\varphi(x,y,z)+\hat{\imath}\varphi(x,\hat{\imath}y,z),$$

define a refined soft complex 2-inner product and it satisfies that $\|x,y\|^2=\gamma(x,x,y)$ for all $x,y\in SE(\check{X})$. The proof of this fact is easy to verify. □

6. Conclusions

In this research, we have revisited the notions of soft 2-normed spaces and soft 2-inner spaces with the main goal of obtaining a formal prove of the Cauchy-Schwarz inequality. In view of this, we have given another definition of linearly dependent soft vectors which allows us to show this inequality. Furthermore, we have studied the most important properties of the 2-normed spaces and 2-inner product spaces in the context of the soft set theory. In this sense, the results that we have obtained in this paper open the study about soft 2-Hilbert spaces and other topics interesting in the soft functional analysis. Finally, we have showed that a refined soft 2-normed space is induced by a refined 2-inner product if the refined soft 2-normed space satisfies the Parallelogram law, which is one of the most important results of this work. From the results presented, one can go deeper into the properties of soft 2-Hilbert spaces. The structure of soft frames is another important and interesting issue that can be addressed.

References

[1] M. Abu Saleem, *On soft covering spaces in soft topological spaces*, AIMS Math., **9** (2024), 18134–18142. 1
[2] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl., **57** (2009), 1547–1553. 1
[3] T. M. Al-shami, *Comment on "Soft mappings space"*, Sci. World J., **2019** (2019), 2 pages. 1.2
[4] T. M. Al-shami, *Investigation and corrigendum to some results related to g-soft equality and gf-soft equality relations*, Filomat, **33** (2019), 3375–3383.
[5] T. M. Al-shami, *Comments on some results related to soft separation axioms*, Afr. Mat., **31** (2020), 1105–1119. 1.2
[6] T. M. Al-shami, M. E. El-Shafei, *On supra soft topological ordered spaces*, Arab J. Basic Appl. Sci., **26** (2019), 433–445. 1.2
[7] T. M. Al-shami, M. E. El-Shafei, *Two new forms of ordered soft separation axioms*, Demonstr. Math., **53** (2020), 8–26. 1.2
[8] T. M. Al-shami, M. E. El-Shafei, *T-soft equality relation*, Turkish J. Math., **44** (2020), 1427–1441. 1, 1.2
[9] T. M. Al-shami, M. E. El-Shafei, B. A. Asaad, *Sum of soft topological ordered spaces*, Adv. Math.: Sci. J., **9** (2020), 4695–4710. 1.2
[10] T. M. Al-shami, A. Mhemdi, *A weak form of soft α-open sets and its applications via soft topologies*, AIMS Math., **8** (2023), 11373–11396. 1
[11] T. M. Al-shami, A. Mhemdi, *On soft parametric somewhat-open sets and applications via soft topologies*, Heliyon, **9** (2023), 15 pages. 1.2
[12] T. M. Al-shami, A. Tercan, A. Mhemdi, *New soft separation axioms and fixed soft points with respect to total belong and total non-belong relations*, Demonstr. Math., **54** (2021), 196–211. 1.2
[13] A. A. Arefijamaal, G. Sadeghi, *Frames in 2-inner product spaces*, Iran. J. Math. Sci. Inform., **8** (2013), 123–130. 1
[14] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, (2011). 1
[15] Y. J. Cho, P. C. S. Lin, S. S. Kim, A. Misiak, *Theory of 2-inner product spaces*, Nova Science Publishers, Huntington, NY, (2001). 1, 2.1, 2.2, 2.3
[16] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, (1990). 1
[17] S. Das, P. Majumdar, S. K. Samanta, *On soft linear spaces and soft normed linear spaces*, Ann. Fuzzy Math. Inform., **9** (2015), 91–109. 1.3, 3.12, 3.13, 3.14, 3.15, 4, 4.3
[18] S. Das, S. K. Samanta, *Soft real sets, soft real numbers and their properties*, J. Fuzzy Math., **20** (2012), 551–576. 3.4, 3.5, 3.6
[19] S. Das, S. K. Samanta, *On soft complex sets and soft complex numbers*, J. Fuzzy Math., **21** (2013), 195–216. 3.5, 3.8, 3.9
[20] S. Das, S. K. Samanta, *On soft metric spaces*, J. Fuzzy Math., **21** (2013), 707–734. 3.7
[21] B. Dastourian, M. Janfada, *Atomic systems in 2-inner product spaces*, Iran. J. Math. Sci. Inform., **13** (2018), 103–110. 1
[22] O. Ferrer, A. Sierra, J. Sanabria, *Soft frames in soft Hilbert spaces*, Mathematics, **9** (2021), 14 pages. 1
[23] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr., **28** (1964), 1–43. 1
[24] S. J. John, *Soft sets—theory and applications*, Springer, Cham, (2021). 1
[25] D. A. Kadhim, *On soft 2-inner product spaces*, J. Al-Qadisiyah Comput. Sci. Math., **6** (2014), 157–168. 1.1, 1.3, 4, 4.1, 5
[26] E. Korkmaz, M. Riaz, M. Deveci, S. Kadri, *A novel approach to fuzzy N-soft sets and its application for identifying and sanctioning cyber harassment on social media platforms*, Artif. Intell. Rev., **57** (2024), 22 pages. 1
[27] A. Kundu, T. Bag, Sk. Nazmul, *On metrizability and normability of 2-normed spaces*, Math. Sci. (Springer), **13** (2019), 69–77. 1
[28] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Comput. Math. Appl., **45** (2003), 555–562. 1, 3.3
[29] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in decision making problem*, Comput. Math. Appl., **44** (2002), 1077–1083. 1
[30] N. Minculete, R. Păltănea, *Weak n-inner product spaces*, Ann. Funct. Anal., **12** (2021), 22 pages. 1
[31] D. Molodtsov, *Soft set theory—first results*, Comput. Math. Appl., **37** (1999), 19–31. 1, 3.1
[32] F. Onat Bulak, H. Bozkurt, *Soft quasilinear operators in soft normed quasilinear spaces*, J. Intell. Fuzzy Syst., **45** (2023), 4847–4856. 1
[33] K. Qin, Z. Hong, *On soft equality*, J. Comput. Appl. Math., **234** (2010), 1347–1355. 1
[34] J. Sanabria, M. Álvarez, O. Ferrer, *Fuzzy set and soft set theories as tools for vocal risk diagnosis*, Appl. Comput. Intell. Soft Comput., **2023** (2023), 12 pages. 1
[35] J. Sanabria, K. Rojo, F. Abad, *A new approach of soft rough sets and a medical application for the diagnosis of Coronavirus disease*, AIMS Math., **8** (2023), 2686–2707. 1
[36] R. Thakur, S. K. Samanta, *Soft Banach algebra*, Ann. Fuzzy Math. Inform., **10** (2015), 397–412. 1, 4.15
[37] P. Zhu, Q. Wen, *Operations on soft sets revisited*, J. Appl. Math., **2013** (2013), 7 pages. 1