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Unified approach to nonlinear Caputo fractional derivative boundary value problems: extending the upper and lower solutions method

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Abstract

The upper and lower solutions approach has been extended in this research to address nonlinear Caputo fractional derivative boundary value problems (FDBVPs). This study proposes generalized findings that unify the existence criteria of specific FDBVPs that have previously been handled separately in the literature. This includes both Dirichlet FDBVPs and Neumann FDBVPs, which are treated as special cases. In addition, we extend the results presented in [A. Batool, I. Talib, M. B. Riaz, C. Tunç, Arab J. Basic Appl. Sci., **29** (2022), 249–256], [A. Batool, I. Talib, R. Bourguiba, I. Suwan, T. Abdeljawad, M. B. Riaz, Int. J. Nonlinear Sci. Numer. Simul., **24** (2023), 2145–2154] and [D. Franco, D. O'Regan, Arch. Inequal. Appl., **1** (2003), 413–419]. To assess the validity of the established results, two examples are considered for examination.

Keywords: Upper and lower solutions, fractional derivative differential equations, Caputo fractional derivative, Dirichlet boundary conditions, Neumann boundary conditions.

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1. Introduction

In the last few decades, fractional derivative differential equations (FDDEs) have been the focus of substantial research due to their potential to model complex phenomena more effectively. The usefulness of the physical phenomena modeled with FDDEs has been demonstrated across a wide range of technological and scientific fields including aerodynamics, physics, ecology, biology, electron-analytic chemistry, physics, and a broad range of other disciplines. The applications of FDDEs are extensive, and numerous studies have investigated their usefulness in diverse areas. As an illustration FDDEs allow for the modeling of phenomena with memory effects, enabling more accurate representations of systems influenced by past events. In Mathematical Epidemiology, FDDEs capture complex disease dynamics with memory

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effects and non-local interactions, improving the accuracy of epidemic modeling [12]. They have also found applications in fluid mechanics [14, 15] and heat and diffusion phenomena [16, 17], showcasing their versatility and broad utility in various research domains.

The wide applications and usefulness of the differential equations modeled with fractional-order derivatives motivated the researchers to develop new criteria for investigating the existence of solutions of FDDEs corresponding to various boundary conditions (BCs). For instance, Ahmad et al. developed the existence results for generalized Caputo FDDEs with generalized fractional integral BCs, see [1]. Moreover, Ntouyas and Etemad in [22] focused on the existence of solutions of fractional derivative differential inclusions corresponding to sum and integral BCs by using the endpoint results for multi-functions. Muthaiah and Baleanu in [21] studied the existence and uniqueness results for FDDEs that involved generalized fractional integrals in the BCs. Recent advancements in fractional calculus have significantly expanded our understanding of complex systems. Bohner et al. conducted a qualitative analysis of Caputo fractional integro-differential equations with constant delays, focusing on stability and boundedness properties using Lyapunov functions [6]. Further extending the field, Subramanian et al. investigated systems of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. Their work provided insights into the behavior of more complex fractional systems with multiple interacting components [25].

The upper and lower solutions (ULSs) approach is an effective approach that has been broadly employed to investigate the existence of solutions of both integer-order and noninteger-order differential equations corresponding to various initial and BCs. The works of some of them are presented as follows: Franco and Regan in [11] developed the new existence criteria for the boundary value problems (BVPs) of second order by introducing the idea of coupled ULSs. The ULSs approach was utilized by Asif et al. [26] to establish the existence of solutions for nonlinear coupled systems of second order having coupled boundary conditions that are generalized and involve nonlinearity. Using the ULSs approach, maximal and minimal solutions of fractional-order difference equations defined in the perspective of Caputo fractional derivative were shown to exist for initial conditions by Chen et al. in [8]. Shi and Zhang in [23] provided conditions that are sufficient to ensure the existence of solutions for Caputo FDDEs of order $1 < \delta \leqslant 2$ with Dirichlet BCs by using the ULSs technique. Liu and Jia in [19] studied some new existence results for Caputo FDDEs with generalized integral BCs by employing the ULSs approach. Lin et al. in [18] investigated existence of solutions for Caputo FDDEs with periodic type BCs by using the ULSs method. In later times, Jeelani et al. established novel findings aimed at examining the presence of positive solutions for Riemann-Liouville FDDEs by using the ULSs approach together with classical fixed point theorems in a cone, see [13]. Additionally, we make reference to the publications by Mosa and Eloe [2] as well as Cabada and Samoza [7] in order to investigate the existence outcomes of BVPs with Neumann BCs, Dirichlet BCs, and periodic BCs using the ULSs approach.

Despite the significant advancements in the study of FDBVPs, several critical gaps remain in our understanding. For instance, in [4, 5], the authors introduced generalized existence results for FDBVPs using the coupled ULSs approach, including periodic and anti-periodic FDBVPs as specific cases. However, the conditions imposed on the boundary functions in these works are insufficient to unify the existence criteria for Dirichlet and Neumann FDBVPs. Our work addresses these gaps and extends previous results in several important ways.

- We extend the coupled ULSs approach to Dirichlet and Neumann boundary conditions, which were not addressed in previous works such as [5] that focused on periodic and anti-periodic conditions.
- Unlike studies such as [4] that were restricted to problems of order $0 < \delta < 1$, our work significantly broadens the scope by addressing higher-order problems, thus expanding the applicability of the theoretical framework.
- We extend the methodology used for integer-order problems in [11] to the fractional-order domain. This represents a non-trivial advancement in fractional differential equations, as it bridges the gap between integer-order and fractional-order boundary value problems.

To extend the applicability of the ULSs technique to FDDEs, it is necessary to have information on the extreme-points when dealing with the Caputo fractional-order derivative. Significant progress in this area was made by Shi and Zhang [23] as they presented the extremum results, when $1 < \delta < 2$. These results were further extended and improved in [3].

Motivated by the aforementioned studies and getting inspiration by the works presented in [3–5, 11, 23], we consider the following generalized FDDEs that involved Caputo fractional derivative

$$D_C^{\delta} y(t) = w(t, y(t)),\ t \in [0,1], \tag{1.1}$$

with nonlinear generalized BCs

$$\begin{cases} f_1(y(0), y(1), y'(0)) = 0, \\ f_2(y(0), y(1), y'(1)) = 0, \end{cases} \tag{1.2}$$

where, $w : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, and D_C^{δ} is the Caputo fractional derivative of order $1 < \delta \leqslant 2$, defined as [10]

$$D_C^{\delta} y(t) = \frac{1}{\Gamma(2-\delta)} \int_0^t (t-r)^{1-\delta} y''(r) dr. \tag{1.3}$$

The problem (1.1) with BCs (1.2) generalizes some certain FDBVPs, for instance if $f_1(t_1, t_2, t_3) = c_1 - t_1$ and $f_2(t_1, t_2, t_3) = c_2 - t_2$ with $c_1, c_2 \in \mathbb{R}$, then (1.1) is the Dirichlet FDBVPs, with

$$y(0) = c_1, y(1) = c_2. \tag{1.4}$$

If $f_1(t_1, t_2, t_3) = t_3 - c_1$, and $f_2(t_1, t_2, t_3) = c_2 - t_3$, then (1.1) is the Neumann FDBVPs, with

$$y'(0) = c_1, \quad y'(1) = c_2. \tag{1.5}$$

Using the ULSs approach to investigate the problem (1.1) with BCs (1.2) represents a novel approach not yet explored in existing literature dealing with similar topics. While the problem (1.1) with BCs (1.4) has been addressed in [23] using the ULSs technique, as far as we are aware, it has not been investigated with generalized BCs (1.2) using the same method. Furthermore, we expand upon the results proposed in [11] for solving FDDEs under generalized BCs (1.2) by implementing the extremum results presented in [3, 23]. Another notable aspect of our research is the development of a general approach for examining the existence of solutions for problems (1.1)-(1.2), with these findings also applicable to the existence criteria of problem (1.1)-(1.4) and problem (1.1)-(1.5). Additionally, we extend the results introduced in [4, 5].

The rest of the paper is organized as follows. Section 2 reviews some essential definitions. In Section 3, a theorem and a related corollary of the extremum principle in the context of the Caputo fractional derivative, which is vital for applying the ULSs approach, are presented. An existence result for generalized FDBVPs, along with its proof, is provided in Section 4. To demonstrate how the theoretical findings can be applied, an example is given in Section 5. The conclusion is drawn in Section 6.

2. Preliminary results

The ULSs approach requires a few key definitions, which are reviewed below in this section.

Definition 2.1 ([9])**.** A function $\phi \in C^2[0,1]$ is said to be a lower solution of (1.1), if the following inequality is satisfied:

$$D_C^{\delta} \phi(t) \geqslant w(t, \phi(t)),\ t \in [0,1]. \tag{2.1}$$

Similarly, a function $\psi \in C^2[0,1]$ is said to be an upper solution of (1.1), if it satisfies the following inequality

$$D_C^{\delta}\psi(t) \leqslant w(t,\psi(t)),\ t \in [0,1]. \tag{2.2}$$

In light of the aforementioned, the assumption will be $\phi(t) \leqslant \psi(t)$, $t \in [0,1]$. For $y_1, y_2 \in C^1[0,1]$ with $y_1(t) \leqslant y_2(t)$ for all $t \in [0,1]$, define the following set as

$$[y_1, y_2] = \{v \in C^1[0,1] : y_1(t) \leqslant v(t) \leqslant y_2(t),\ \text{for all } t \in [0,1]\}.$$

For the Dirichlet case, the condition for being a solution of the problem (1.1) and (1.4) is to lie between a lower and upper solutions is as follows

$$\phi(0) \leqslant c_1 \leqslant \psi(0),\quad \phi(1) \leqslant c_2 \leqslant \psi(1). \tag{2.3}$$

The following idea was used in [11] to cover the various prospects for the boundary functions f_1 and f_2 and to unify the treatment of various integer-order boundary value problems. We use this concept to extend the results for FDDEs with generalized nonlinear BCs without considering the assumptions on the boundary functions.

Definition 2.2 ([11])**.** The well ordered functions, $\phi, \psi \in C^2[0,1]$ are called coupled ULSs for the problem (1.1)-(1.2), if the inequalities (2.1)-(2.2) are satisfied along with the following inequalities:

$$\begin{cases} \max\{f_1(\psi(0),\psi(1),\psi'(0)), f_1(\psi(0),\phi(1),\psi'(0))\} \leqslant 0, \\ \text{and} \\ \min\{f_1(\phi(0),\phi(1),\phi'(0)), f_1(\phi(0),\psi(1),\phi'(0))\} \geqslant 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \max\{f_2(\psi(0),\psi(1),\psi'(1)), f_2(\phi(0),\psi(1),\psi'(1))\} \leqslant 0, \\ \text{and} \\ \min\{f_2(\phi(0),\phi(1),\phi'(1)), f_2(\psi(0),\phi(1),\phi'(1))\} \geqslant 0. \end{cases} \tag{2.5}$$

Remark 2.3. It is worth noticing that the classical concepts can be generalized by using the Definition 2.2. For example, considering the Dirichlet case we obtain from (2.4) and (2.5) that (2.3) holds.

Subsequently, we incorporate the following highly valuable lemma that plays a crucial role in establishing the main findings. Define the following

$$C_\circ[0,1] = \{y \in C[0,1] : y(0) = 0\}.$$

Lemma 2.4. *Let* $R : C^1[0,1] \to C_\circ[0,1] \times \mathbb{R} \times \mathbb{R}$ *be a linear operator. Then the inverse* $R^{-1} : C_\circ[0,1] \times \mathbb{R} \times \mathbb{R} \to C^1[0,1]$ *exists if and only if* $Ry(t) = 0 \Rightarrow y(t) = 0$.

Proof. First we assume that $Ry(t) = 0 \Rightarrow y(t) = 0$, and show that R^{-1} exists. Let $Ry(t_1) = Ry(t_2)$, where $t_1, t_2 \in [0,1]$. Since R is linear, we have

$$R(y(t_1) - y(t_2)) = 0 \ \Rightarrow\ y(t_1) - y(t_2) = 0 \text{ (by hypothesis) } \Rightarrow\ y(t_1) = y(t_2).$$

Since for a linear operators, if $Ry(t_1) = Ry(t_2) \Rightarrow y(t_1) = y(t_2)$, then there exists a mapping $R^{-1} : C_\circ[0,1] \times \mathbb{R} \times \mathbb{R} \to C^1[0,1]$, which maps every $y_\circ(t) \in C_\circ[0,1] \times \mathbb{R} \times \mathbb{R}$ onto that $y(t) \in C^1[0,1]$ for which $Ry(t) = y_\circ(t)$.

Conversely, suppose that R^{-1} exists and show that $Ry(t) = 0 \Rightarrow y(t) = 0$. As R^{-1} exists, so $R(y(t_1)) = R(y(t_2)) \Rightarrow y(t_1) = y(t_2)$ with $y(t_2) = 0 \Rightarrow R(y(t_1)) = R0 = 0$, which finally implies $y(t_1) = 0$. □

Theorem 2.5 ([24]). *A closed subspace of* $C(X, \mathbb{R})$ *on a compact metric space* X *is compact if and only if it satisfies the conditions of being bounded and equicontinuous.*

Theorem 2.6 ([24]). *If a compact operator* P *maps a nonempty, closed, bounded, and convex subset* Y *of a Banach space* X *to itself that is* $P : Y \to Y$*, then* P *must have at least one fixed point in* Y*.*

3. Extremum results

Determining the behavior of the fractional-order derivatives at the extreme points is necessary for the extension of the ULSs technique to FDDEs. So in [20], the behaviour of Caputo fractional derivative was studied at extreme points, when $0 < \delta < 1$. However, the extremum results presented in [20] are not sufficient to extend the ULSs approach for FDBVPs, when $1 < \delta < 2$. In order to overcome this problem, the authors in [23] presented the extremum results for the fractional derivative in Caputo sense, when $1 < \delta < 2$. These results were further improved and extended in [3]. By introducing some modifications in the statement and proof of the results proposed in [3], the following extremum result is presented in Caputo sense.

Theorem 3.1. *Let* $y \in C^2[0, 1]$ *gains its maximum at* $t_0 \in (0, 1)$*, then*

$$D_C^\delta y(t_0) \leqslant \frac{t_0^{-\delta}}{\Gamma(2-\delta)}[(\delta-1)(y(0)-y(t_0)) - t_0 y'(0)], \text{ for all } 1 < \delta < 2.$$

Proof. For the proof, we refer the reader to study [5]. □

Corollary 3.2. *Let* $y \in C^2[0, 1]$ *gains its maximum at* $t_0 \in (0, 1)$*, and* $y'(0) \geqslant 0$*. Then* $D_C^\delta y(t_0) \leqslant 0$*, for all* $1 < \delta < 2$*.*

Proof. Since $y(t_0) \geqslant y(0), t_0 > 0$, and $y'(0) \geqslant 0$, then Theorem 3.1 implies $D_C^\delta y(t_0) \leqslant 0$, for all $1 < \delta < 2$. □

4. Main result

In this section, the ULSs approach is applied to establish the existence result which is generalized in its nature for studying the existence of solutions of the problems (1.1)-(1.2), (1.1)-(1.4), and (1.1)-(1.5).

Theorem 4.1. *Assume that the functions* ψ, ϕ*, are the coupled ULSs of the FDBVPs (1.1)-(1.2), and* f_1 *and* f_2 *are monotone functions that specify the boundary conditions. The function* f_1 *is monotone non-decreasing in the third variable and* f_2 *is non-increasing in the third variable. Moreover the following functions are monotone in* $[\phi(1), \psi(1)]$ *and* $[\phi(0), \psi(0)]$*, respectively,*

$$f_{1_\phi}(t) := f_1(\phi(0), t, \phi'(0)), \quad f_{1_\psi}(t) := f_1(\psi(0), t, \psi'(0)),$$

and

$$f_{2_\phi}(t) := f_2(t, \phi(1), \phi'(1)), \quad f_{2_\psi}(t) := f_2(t, \psi(1), \psi'(1)).$$

Then the problem defined by (1.1)-(1.2) has at least one solution, such that

$$\phi(t) \leqslant y(t) \leqslant \psi(t), \ t \in [0, 1].$$

Proof. We define an appropriately modified problem by introducing the following function:

$$b(t, x_1) := \max\{\phi(t), \min\{x_1, \psi(t)\}\}, \tag{4.1}$$

then the modified problem is

$$\begin{cases} D_C^{\delta} y(t) - \mu y(t) = W^*(t, y(t)), \ t \in [0,1], \ \mu > 0, \\ y(0) = f_1^*(y(0), y(1), y'(0)), \\ y(1) = f_2^*(y(0), y(1), y'(1)), \end{cases} \tag{4.2}$$

where

$$W^*(t, y(t)) = \begin{cases} w(t, \psi(t)) - \mu\psi(t), & \text{if } \psi(t) < y(t), \\ w(t, y(t)) - \mu y(t), & \text{if } \phi(t) \leqslant y(t) \leqslant \psi(t), \\ w(t, \phi(t)) - \mu\phi(t), & \text{if } y(t) < \phi(t), \end{cases}$$

and

$$\begin{cases} f_1^*(t_1, t_2, t_3) = b(0, t_1 + f_1(t_1, t_2, t_3)), \\ f_2^*(t_1, t_2, t_3) = b(1, t_2 + f_2(t_1, t_2, t_3)). \end{cases}$$

Since (4.2) is the modified problem of (1.1)-(1.2), so solving it leads towards the solution of (1.1)-(1.2) that lies between ϕ and ψ. To be unambiguous, we will split the proof into steps.

Step 1: Determining the fixed point of the operator, $R^{-1}S : C^1[0,1] \to C^1[0,1]$, which is defined by the composition of the mapping defined below is equivalent to the solution of (4.2).

$$\begin{cases} R : C^1[0,1] \to C_\circ[0,1] \times \mathbb{R} \times \mathbb{R}, \\ \text{and} \\ S : C^1[0,1] \to C_\circ[0,1] \times \mathbb{R} \times \mathbb{R}, \end{cases}$$

which can be defined as

$$\begin{cases} [Ry](t) = \left(y(t) - y(0) - \left(\mu_{RL} J^{\delta} y\right)(t), y(0), y(1) \right), \\ [Sy](t) = \left({}_{RL}J^{\delta}(W^*(t, y(t)), f_1^*(y(0), y(1), y'(0)), f_2^*(y(0), y(1), y'(1)) \right), \end{cases}$$

where

$${}_{RL}J^{\delta} y(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-r)^{\delta-1} y(r) dr \tag{4.3}$$

is the fractional integral operator of order $1 < \delta < 2$, in the context of Riemann-Liouville see [10]. Also, ϕ and ψ are our coupled lower and upper solutions, respectively, and are elements of C^2-space defined on a closed interval $[0,1]$. Moreover, ϕ and ψ are bounded by the statement of the boundedness theorem. Since y lies between two continuous bounded functions, it implies the continuity and boundedness of y. Additionally, w is depending on y which implies that w is bounded and uniformly continuous on $[0,1] \times \mathbb{R}$. Hence, W^* is uniformly continuous. Moreover the functions f_1^*, f_2^* and Riemann-Liouville fractional integral are continuous so this will cause the continuity of $[Sy]$ on $[0,1]$. Furthermore, the class $\{Sy : y \in C^1[0,1]\}$ is uniformly bounded as well as equicontinuous, as for every $\epsilon > 0$, there exist a $\delta > 0$, such that

$$|Sy(t_1) - Sy(t_2)| < \epsilon, \ \forall y \in S, \ \text{ whenever } |y(t_1) - y(t_2)| < \delta, \ \forall t_1, t_2 \in [0,1].$$

Without compromising generality, suppose that $t_1 \leqslant t_2$, we have

$$\begin{aligned}|Sy(t_1) - Sy(t_2)| &= \Big(({}_{RL}J^{\delta}(W^*(t_1, y(t_1)), f_1^*(y(0), y(1), y'(0)), f_2^*(y(0), y(1), y'(1))) \\ &\quad - ({}_{RL}J^{\delta}(W^*(t_2, y(t_2)), f_1^*(y(0), y(1), y'(0)), f_2^*(y(0), y(1), y'(1)))\Big) \\ &= \left| \Big(\frac{1}{\Gamma(\delta)} \int_0^{t_1} (t_1 - s)^{\delta-1} W^*(s, y(s)) ds - \int_0^{t_2} (t_2 - s)^{\delta-1} W^*(s, y(s)) ds\Big) \right| \\ &\leqslant \frac{1}{\Gamma(\delta)} \left(\int_0^{t_1} \left|(t_1 - s)^{\delta-1} W^*(s, y(s)) ds\right| - \int_0^{t_2} \left|(t_2 - s)^{\delta-1} W^*(s, y(s)) ds\right| \right) \\ &= \frac{1}{\Gamma(\delta)} \left((\left| \int_0^{t_1} ((t_1 - s)^{\delta-1} - (t_2 - s)^{\delta-1}) W^*(s, y(s)) ds \right| \right. \\ &\quad \left. - \int_{t_1}^{t_2} \left|(t_2 - s)^{\delta-1} W^*(s, y(s)) ds\right| \right) \rightarrow 0 \text{ as } t_1 \rightarrow t_2.\end{aligned}$$

Consequently, by the statement of Arzelá-Ascoli Theorem 2.5, the relative compactness of the class $\{Sy : y \in C^1[0, 1]\}$ is ensured. In addition R^{-1} exists by Lemma 2.4 and is continuous. Next, we show the existence of at least one fixed point $y(t)$ of $R^{-1}S$ as follows. Since

$$\begin{aligned}[Sy](t) &= ({}_{RL}J^{\delta} W^*(t, y(t)), f_1^*(y(0), y(1), y'(0)), f_2^*(y(0), y(1), y'(1))) \\ &= \left({}_{RL}J^{\delta}\left(D_C^{\delta} y(t)) - \mu y(t)\right), y(0), y(1))\right) = (y(t) - y(0) - (\mu_{RL}J^{\delta} y)(t), y(0), y(1)) = [Ry](t),\end{aligned}$$

which further implies that

$$R^{-1}[Sy](t) = R^{-1}[Ry](t) = y(t).$$

Therefore, the relative compactness of the class $\{Sy : y \in C^1[0, 1]\}$ and the existence of the continuous inverse of R implies the compactness of the operator $R^{-1}S$. Consequently, the statement of Schauder's fixed point Theorem 2.6 ensures the existence of at least one fixed point $y(t)$ of $R^{-1}S$ which is the solution of problem (4.2).

Step 2: Now we will show that, if $y(t)$ is a solution of (4.2), then it must lie in a region bounded by the coupled ULSs that are well ordered, in such a way that $\phi(t) \leqslant y(t) \leqslant \psi(t), t \in [0, 1]$. Our claim is, $y(t) \leqslant \psi(t)$ for all $t \in [0, 1]$. On contrary, we suppose that $y(t) \nleqslant \psi(t)$, then $y - \psi$ acquires at some $t_0 \in [0, 1]$ (by definitions of f_1^* and f_2^*, we obtain that $t_0 \in (0, 1)$) a positive maximum. So $(y - \psi)'(t_0) = 0$. Corollary 3.2 implies $D_C^{\delta}(y - \psi)(t_0) \leqslant 0$. Ultimately, we get a contradiction

$$\begin{aligned}0 \geqslant D_C^{\delta}(y - \psi)(t_0) &\geqslant W^*(t_0, y(t_0)) + \mu y(t_0) - w(t_0, \psi(t_0)) \\ &= w(t_0, \psi(t_0)) - \mu\psi(t_0) + \mu y(t_0) - w(t_0, \psi(t_0)) = \mu\Big(y(t_0) - \psi(t_0)\Big) > 0.\end{aligned}$$

Consequently, $y(t) \leqslant \psi(t)$ for all $t \in [0, 1]$. Likewise, it can be proved that $\phi \leqslant y$ on $[0, 1]$.

Step 3: The boundary conditions (1.2) need to be satisfied by y as we claim that y is a solution of the modified problem. Based on previous steps where we have demonstrated that y is a solution to our problem which lies between the lower and upper solution, so with respect to the first boundary condition, this indicates that

$$\phi(0) \leqslant y(0) \leqslant \psi(0). \tag{4.4}$$

For this, it is sufficient to show that,

$$\phi(0) \leqslant y(0) + f_1\left(y(0), y(1), y'(0)\right)\right) \leqslant \psi(0). \tag{4.5}$$

Thus,

$$y(0) = f_1^*(y(0), y(1), y'(0)) = y(0) + f_1(y(0), y(1), y'(0)).$$

On the contrary, assume the following

$$y(0) + f_1\left(y(0), y(1), y'(0)\right)\right) > \psi(0).$$

Then using (4.1), yields

$$y(0) = \psi(0). \tag{4.6}$$

In the last step, we have shown that, $y(t) \leqslant \psi(t)$ and $y(t) - \psi(t) \in C^1[0,1]$. So, these results together with (4.6) give

$$y'(0) \leqslant \psi'(0).$$

If f_{1_ψ} is monotone nonincreasing, then we obtain

$$y(0) + f_1(y(0), y(1), y'(0)) = \psi(0) + f_1(\psi(0), y(1), y'(0)) \leqslant \psi(0) + f_1(\psi(0), \phi(1), \psi'(0)).$$

Since ϕ and ψ are coupled ULSs, we get the contradiction

$$y(0) + f_1(y(0), y(1), y'(0)) \leqslant \psi(0).$$

If f_{1_ψ} is monotone nondecreasing, then we have

$$y(0) + f_1(y(0), y(1), y'(0)) \leqslant \psi(0) + f_{1_\psi}(\psi(1)) = \psi(0) + f_1(\psi(0), \psi(1), \psi'(0)),$$

again a contradiction is obtained, hence

$$y(0) + f_1(y(0), y(1), y'(0)) \leqslant \psi(0).$$

In order to show that $\phi(0) \leqslant y(0) + f_1(y(0), y(1), y'(0))$, the boundary function f_{1_ϕ} will be used. For the second boundary condition, we have to show that,

$$\phi(1) \leqslant y(1) + f_1\left(y(0), y(1), y'(1)\right)\right) \leqslant \psi(1).$$

On the contrary, assume the following

$$y(1) + f_2\left(y(0), y(1), y'(1)\right)\right) > \psi(1).$$

Then

$$y(1) = f_2^*(y(0), y(1), y'(1)) = \psi(1).$$

In the Step 2, we have shown that $y(t) \leqslant \psi(t)$ and $y(t) - \psi(t) \in C^1[0,1]$. So, altogether these results imply $y'(1) \geqslant \psi'(1)$. If f_{2_ψ} is monotone nonincreasing, then we obtain

$$y(1) + f_2(y(0), y(1), y'(1)) \leqslant \psi(1) + f_{2_\psi}(\phi(0)) = \psi(1) + f_2(\phi(0), \psi(1), \psi'(1)).$$

Since ϕ and ψ are coupled ULSs, we get the contradiction

$$y(1) + f_2(y(0), y(1), y'(1)) \leqslant \psi(1).$$

If f_{2_ψ} is monotone nondecreasing, then we have

$$y(1) + f_2(y(0), y(1), y'(1)) \leqslant \psi(1) + f_2(\psi(0), \psi(1), \psi'(1)),$$

again a contradiction is obtained, hence

$$y(1) + f_2(y(0), y(1), y'(1)) \leqslant \psi(1).$$

In order to show that $\phi(1) \leqslant y(1) + f_2(y(0), y(1), y'(1))$, the boundary function f_{2_ϕ} will be used. Therefore, the problem (1.1)-(1.2) has at least one solution, such that

$$\phi(t) \leqslant y(t) \leqslant \psi(t), \ t \in [0, 1].$$

A conclusive proof has been made. □

5. Validation of theoretical results

This section provides two problems that validates the proposed theoretical results.

Problem 5.1. Consider nonlinear FDDEs as follows

$$D_C^{\frac{3}{2}} y(t) = y^3(t) - \sin^2(t), \ t \in [0, 1], \tag{5.1}$$

with nonlinear BCs

$$\begin{cases} f_1(y(0), y(1), y'(0)) = y(0)\sin(y(0)) - \cos(y'(0))y(1), \\ f_2(y(0), y(1), y'(1)) = y(0)\tan(y(0)) - \cos(y'(1))y(1). \end{cases} \tag{5.2}$$

Let $\phi(t) = -t$ and $\psi(t) = t^2 + 2$ be the lower and upper solutions, respectively, that satisfy (2.1) and (2.2), as

$$D_C^{\frac{3}{2}} \phi(t) = 0 \geqslant w(t, \phi(t)) = -t^3 - \sin^2(t), \ t \in [0, 1],$$

and

$$D_C^{\frac{3}{2}} \psi(t) = 2.256758334 t^{\frac{1}{2}} \leqslant w(t, \psi(t)) = (t^2 + 2)^3 - \sin^2(t), \ t \in [0, 1].$$

Additionally, the functions $-t$ and $t^2 + 2$ are coupled ULSs of the problem (5.1)-(5.2), satisfying (2.4) because the given set of inequalities are satisfied if f_1 is monotone nondecreasing in the second variable.

$$\begin{cases} f_1\left(\psi(0), \psi(1), \psi'(0)\right) \leqslant 0, \\ \text{and} \\ f_1\left(\phi(0), \phi(1), \phi'(0)\right) \geqslant 0. \end{cases}$$

In a similar manner, if in the first variable f_2 is monotone nondecreasing then the set of inequalities mentioned as follows are satisfied:

$$\begin{cases} f_2\left(\psi(0), \psi(1), \psi'(1)\right) \leqslant 0, \\ \text{and} \\ f_2\left(\phi(0), \phi(1), \phi'(1)\right) \geqslant 0. \end{cases}$$

Also the functions

$$f_{1_\phi}(t) := f_1(\phi(0), t, \phi'(0)), \quad f_{1_\psi}(t) := f_1(\psi(0), t, \psi'(0)),$$

and

$$f_{2_\phi}(t) := f_2(t, \phi(1), \phi'(1)), \quad f_{2_\psi}(t) := f_2(t, \psi(1), \psi'(1)),$$

are monotone on $[\phi(1), \psi(1)]$ and $[\phi(0), \psi(0)]$, respectively.

This confirms that Theorem 4.1's assumptions hold true. In Consequence, the problem (5.1)-(5.2) has atleast one solution, such that $\phi(t) \leqslant y(t) \leqslant \psi(t)$, for all $t \in [0, 1]$.

Problem 5.2. The dynamics of a nonlinear spring system whose mathematical models include nonlinear restorative forces can be expressed as follows:

$$mD_C^\delta y(t) + ky + k_1 y^3 = 0,$$

where D_C^δ denotes the Caputo fractional derivative. In this context, the spring is classified as *hard* if $k_1 > 0$ and *soft* if $k_1 < 0$. For the special case where $\delta = 2$, $m = 1$, $k = 1$, and $k_1 = -1$, the model reduces to a well-known scenario discussed in [27]. However, we discuss its fractional framework, which significantly broadens its applicability and provides deeper insights into the system's behavior.

The fractional extension is provided below:

$$D_C^\delta y(t) = y^3(t) - y(t), \quad t \in [0, 1], \quad 1 < \delta \leqslant 2, \tag{5.3}$$

subject to the generalized BCs:

$$\begin{cases} f_1(y(0), y(1), y'(0)) = -y(0) + y(1)\cos(y'(0)), \\ f_2(y(0), y(1), y'(1)) = -y(1) + y(0)\cos(y'(1)). \end{cases} \tag{5.4}$$

Furthermore, let $\phi(t) = 1$ and $\psi(t) = 2$ be the lower and upper solutions, respectively, that satisfy (2.1) and (2.2), as

$$D_C^\delta \phi(t) = 0 \geqslant w(t, \phi(t)) = 1^3 - 1, \quad \text{where } 1 < \delta \leqslant 2,$$

and

$$D_C^\delta \psi(t) = 0 \leqslant w(t, \psi(t)) = 2^3 - 2, \quad \text{where } 1 < \delta \leqslant 2.$$

Additionally, the functions $\psi(t)$ and $\phi(t)$ are coupled ULSs of the problem (5.3)-(5.4), satisfying (2.4) because the given set of inequalities are satisfied if f_1 is monotone nondecreasing in the second variable,

$$\begin{cases} f_1\left(\psi(0), \psi(1), \psi'(0)\right) \leqslant 0, \\ \text{and} \\ f_1\left(\phi(0), \phi(1), \phi'(0)\right) \geqslant 0. \end{cases}$$

In a similar manner, if the first variable f_2 is monotone nondecreasing, then the set of inequalities mentioned as follows are satisfied:

$$\begin{cases} f_2\left(\psi(0), \psi(1), \psi'(1)\right) \leqslant 0, \\ \text{and} \\ f_2\left(\phi(0), \phi(1), \phi'(1)\right) \geqslant 0. \end{cases}$$

Also the functions

$$f_{1_\phi}(t) := f_1(\phi(0), t, \phi'(0)), \quad f_{1_\psi}(t) := f_1(\psi(0), t, \psi'(0)),$$

and

$$f_{2_\phi}(t) := f_2(t, \phi(1), \phi'(1)), \quad f_{2_\psi}(t) := f_2(t, \psi(1), \psi'(1)),$$

are monotone on $[\phi(1), \psi(1)]$ and $[\phi(0), \psi(0)]$, respectively. This confirms that Theorem 4.1's assumptions hold true. In Consequence, the problem (5.3)-(5.4) has atleast one solution, such that $\phi(t) \leqslant y(t) \leqslant \psi(t)$, for all $t \in [0, 1]$.

6. Conclusion

We solved the nonlinear FDDEs involving Caputo derivatives corresponding to the generalized nonlinear BCs. Instead of assuming monotone conditions on the boundary functions, we used the idea of coupled ULSs to develop generalized results that unified the treatment of certain FDBVPs, which were previously treated separately in the literature. The extremum principle for Caputo fractional-order derivatives has also been discussed. Building on these findings, we extended the ULSs approach to FDBVPs. Furthermore, we extended and improved upon the results presented in [4, 5, 11] in this study. Two examples were provided to illustrate how the developed outcomes can be put into practice. In the future, we are interested in extending the proposed results by considering nonlinearities in the first-order derivatives of the solution function.

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