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$\overline{\varphi(x)}$-Tribonnaci polynomial, numbers, and its sum

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Abstract

This study presents a general third-order nabla difference operator that allows us to get $\overline{\varphi(x)}$-Tribonacci sequences, Tribonacci numbers, and their sum using the coefficients of different trigonometric functions and their inverse. In this section, we examined the numerical solutions and C^*-solutions of the $\overline{\varphi(x)}$-Tribonacci sequences for different functions. In addition, some interesting conclusions and theorems are obtained for the sum of the terms of the Tribonacci sequence. Also, we offer appropriate examples to show how to use MATLAB to demonstrate our results.

Keywords: Generalized nabla difference operator with trigonometric coefficients, generalized Tribonacci sequence, N^*-solution, C^*-solution, Tribonacci summation.

2020 MSC: 39A70, 39A10, 47B39, 65J10, 65Q10.

1. Introduction

Fibonacci numbers, often referred to as Natural Numbers, permeate various aspects of the natural world, from the arrangement of leaves on plants to the patterns seen in flower petals, pineapple bracts, and pineapple scales. The sequence $1, 1, 2, 3, 5, 8, \dots$, known as the Fibonacci sequence, has a unique property: each number is the sum of the previous two. If $f_\nu \nu = 0^\infty$ represents the Fibonacci sequence mathematically, then it follows that $f_0 = f_1 = 1$ and for $\nu \geqslant 2$, $f_\nu = f_{\nu-1} + f_{\nu-2}$. Furthermore, the ratio of consecutive Fibonacci terms converges to the irrational number $\frac{1+\sqrt{5}}{2}$, commonly known as the golden ratio. This ratio has important applications in architecture, science, and art. Further insights into the properties of Fibonacci numbers have been extensively explored in the literature, particularly in works such as [9].

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The Fibonacci sequence is used in many fields today and has led to the development of various conceptual and mathematical models aimed at explaining its meaning. Originally developed while studying the growth of rabbit populations, the Fibonacci numbers have fascinated art, nature, and mathematics enthusiasts alike. Over the centuries, researchers, particularly those associated with the Fibonacci Society, have tirelessly explored this concept, leading to groundbreaking advances and encouraging further research in related fields (see reference therein [3, 4, 11, 14]).

Fractional calculus, an esteemed field of mathematics, traces its origins back to 1695 when it was initially discussed in a series of correspondences. Fractional calculus, as highlighted by Miller [10], serves as a valuable tool extensively utilized across various domains, including electronics engineering and computer science. Despite its intricate mathematical underpinnings, the genesis of fractional calculus stemmed from the resolution of seemingly simple differentiation problems. While a first-order derivative signifies the slope of a function, the notion of a half-order derivative poses intriguing questions, paving the way for unexplored avenues in mathematical inquiry. Its significance extends to signal and image processing, mechanics, control theory, biology, chemistry, and economics, among others. Currently, numerous scholars are concurrently delving into fractional differentiation, contributing to the ongoing exploration and advancement of this field. L'Hopital's inquiry to Leibniz regarding the implications of fractional differentiation, specifically when the order is $\frac{1}{2}$, sparked significant contemplation. Leibniz's response marked a pivotal moment in the exploration of this mathematical concept in [6, 8]. One specific form of difference operator on $u(t)$ was presented by Jerzy Popenda [2] in 1984 and is defined as $\Delta_\alpha u(t) = u(t+1) - \alpha u(t)$. Miller and Rose [10] presented the discrete fractional derivative and its inverse, $\nabla_h^{-\nu} f(t)$, in 1989, which is similar to the Riemann-Liouville fractional derivative.

The literature on fractional calculus spans various applications and theoretical developments. [5] examined the theoretical foundations, particularly focusing on solving differential equations of fractional order, while [12] provide computational insights into the dynamics of fractional order systems, aiding in modeling and simulation efforts. The authors in [1] contribute to understanding solution properties, especially in weighted systems, and [13] offer a comparative analysis of fractional operators, highlighting their efficacy in solving differential equations. [20] tackles nonlocal boundary problems, shedding light on the behavior of fractional differential equations with nonlocal conditions, and [7] extends fractional calculus into financial mathematics, proposing novel option pricing models under uncertain conditions; for more applications by fractional operators, we refer readers to these works [15–18]. Together, these works advance both theoretical understanding and practical applications of fractional calculus across diverse domains, showcasing its versatility and significance in tackling complex problems and fostering innovation. As stated in reference [19], the operator Δ_α was expanded to include the generic (α,β)-difference operator, which is represented as $\Delta_{(\alpha,\beta)\ell} v(t) = \beta v(t+\ell) - \alpha v(t)$ for the real-valued functions $v(t)$. Motivated by this study, we introduce the third-order trigonometric nabla difference operator and its inverse by which we observe the $\overline{\varphi(x)}$-Tribonacci sequence, polynomials, its sum, and the conformable α-derivative of the $\overline{\varphi(x)}$-Fibonacci polynomials in this paper.

2. $\overline{\varphi(x)}$-Trinonacci polynomial, sequence and its series

In this section, we introduce a generalized third-order nabla-difference operator with the coefficients of trigonometric functions as

$$\underset{\overline{\varphi(x)}}{\nabla} w(x) = w(x) - \alpha_1 \sin(r_1 x) w(x-h) - \alpha_2 \sin(r_2 x) w(x-2h) - \alpha_3 \sin(r_3 x) w(x-3h),$$

which generates $\overline{\varphi(x)}$-Tribonacci polynomials, numbers, sequences, and its sums.

Definition 2.1. For $x, h \in \mathbb{R}^+$, $\overline{\varphi(x)}$-Tribonacci sequence is defined as:

$$\begin{aligned} &T_0 = 1, \quad T_1 = \alpha_1 \sin(r_1 x), \quad T_2 = T_1 \alpha_1 \sin(r_1(x-h)) + \alpha_2 \sin(r_2 x), \text{ and} \\ &T_n = \alpha_1 \sin(r_1 x_{n,1}) T_{n-1} + \alpha_2 \sin(r_2 x_{n,2}) T_{n-2} + \alpha_3 \sin(r_3 x_{n,3}) T_{n-3}, \quad n \geqslant 3. \end{aligned} \tag{2.1}$$

Table 1: Symbols and explanations used in this manuscript.

Symbol	explanations
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^+$	Set of all positive real numbers
$\mathbb{C}$	Set of all complex numbers
h	Shifting value (i.e., $h \in [0, \infty)$)
$x^{(t)}$	$x(x-h)(x-2h)(x-3h)\cdots(x-(t-1)h)$
$x-(n+r)h$	$x_{n,r}$, where n, r being any integers
C^*-solution	Closed form solution
N^*-solution	Numerical solution
$\overline{\varphi(x)}$-sequence	Tribonacci sequence arising from generic third order nabla difference equation with the coefficient of trigonometric functions
$\overline{\varphi(x)}$-equation	Generic third order nabla difference equation with the coefficient of trigonometric functions

Remark 2.2. Instead of sine function we can deal with any other trigonometric and product of any trigonometric functions in Definition 2.1.

Example 2.3.

(i) Taking $x = 6$, $\alpha_1 = 1$, $n = 3$, $\alpha_2 = 2$, $h = 0.2$, $\alpha_3 = 3$, $r_1 = 3$, $r_2 = 2$, and $r_3 = 1$ in (2.1), we get a Tribonacci sequence $\{1, -0.7510, -0.3277, 0.6885, \ldots\}$.

(ii) When $x = 6$, $\alpha_1 = 1$, $n = 3$, $\alpha_2 = 2$, $h = 0.3$, $\alpha_3 = 3$, $r_1 = 3$, $r_2 = 2$, and $r_3 = 1$ in (2.1), we have a Tribonacci sequence $\{1, 0.6134, 1.3859, 1.8588, \ldots\}$.

The $\overline{\varphi(x)}$-Tribonacci sequence that corresponds to each pair of $\overline{\varphi(x)} \in \mathbb{R}^3$ can also be obtained in a similar manner.

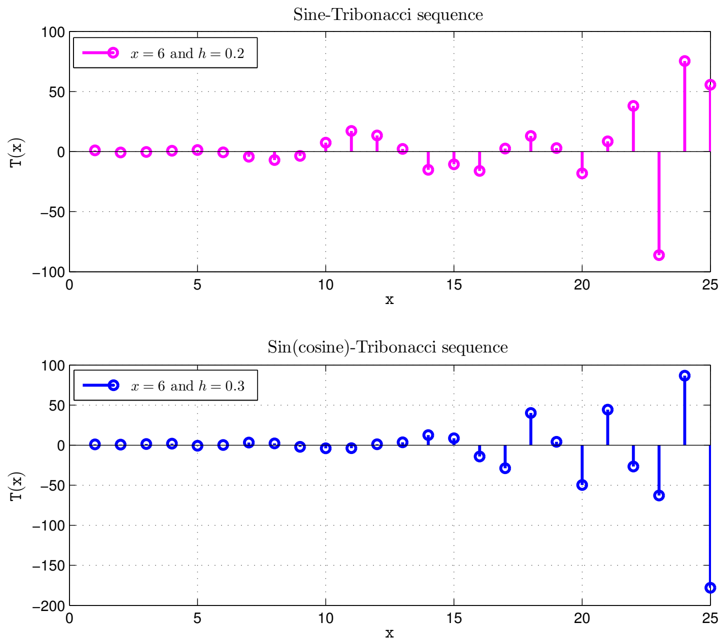

Figure 1: Sine and sin (cosine) Tribonacci sequences.

Remark 2.4. From the above Example 2.3, equation (2.1) yields the following sin (cosine)-Tribonacci numbers:

n	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
0	1									
1	1	-0.7510								
2	1	-0.7510	-0.3277							
3	1	-0.7510	-0.3277	0.6885						
4	1	-0.7510	-0.3277	0.6885	1.3632					
5	1	-0.7510	-0.3277	0.6885	1.3632	-0.5832				
6	1	-0.7510	-0.3277	0.6885	1.3632	-0.5832	-4.2323			
7	1	-0.7510	-0.3277	0.6885	1.3632	-0.5832	-4.2323	-7.0653		
8	1	-0.7510	-0.3277	0.6885	1.3632	-0.5832	-4.2323	-7.0653	-3.5141	
9	1	-0.7510	-0.3277	0.6885	1.3632	-0.5832	-4.2323	-7.0653	-3.5141	7.4179

Definition 2.5. For $x, h \in \mathbb{R}^+$, a generic third order nabla operator with the coefficients of trigonometric functions on $w(x)$, denoted as $\underset{\varphi(x)}{\nabla} w(x)$, is defined by

$$\underset{\varphi(x)}{\nabla} w(x) = w(x) - \alpha_1 \sin(r_1 x) w(x-h) - \alpha_2 \sin(r_2 x) w(x-2h) - \alpha_3 \sin(r_3 x) w(x-3h), \tag{2.2}$$

and its inverse is defined as

$$\text{if} \quad \underset{\varphi(x)}{\nabla} w(x) = y(x), \quad \text{then we write} \quad w(x) = \underset{\varphi(x)}{\overset{-1}{\nabla}} y(x). \tag{2.3}$$

Lemma 2.6. *Let $w(x)$ be a function of $x \in (-\infty, \infty)$. Then we obtain*

$$\underset{\varphi(x)}{\overset{-1}{\nabla}} a^{sx}\Big[1 - \frac{\alpha_1 \sin(r_1 x)}{a^{sh}} - \frac{\alpha_2 \sin(r_2 x)}{a^{2sh}} - \frac{\alpha_3 \sin(r_3 x)}{a^{3sh}}\Big] = a^{sx}. \tag{2.4}$$

Proof. Taking $w(x) = a^x$ in (2.2), we obtain

$$\underset{\varphi(x)}{\nabla} a^{sx} = a^{sx}\Big[1 - \frac{\alpha_1 \sin(r_1 x)}{a^{sh}} - \frac{\alpha_2 \sin(r_2 x)}{a^{2sh}} - \frac{\alpha_3 \sin(r_3 x)}{a^{3sh}}\Big].$$

Now (2.4) follows from (2.3). □

Remark 2.7. If $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in Lemma 2.6, then we obtain

$$\underset{\varphi(x)}{\overset{-1}{\nabla}} a^{sx}\Big[1 - \frac{\sin(r_1 x)}{a^{sh}} - \frac{\sin(r_2 x)}{a^{2sh}} - \frac{\sin(r_3 x)}{a^{3sh}}\Big] = a^{sx}. \tag{2.5}$$

Proposition 2.8. *If $w(x) = \underset{\varphi(x)}{\overset{-1}{\nabla}} y(x)$ be any function, x be any non-negative integer, $T_0 = 1, T_1 = \alpha_1 \sin(r_1 x)$, $T_2 = T_1 \alpha_1 \sin(r_1(x-h)) + \alpha_2 \sin(r_2 x)$, and*

$$F_{n+1} = \alpha_1 \sin(r_1 x_{n,0}) T_n + \alpha_2 \sin(r_2 x_{n,1}) T_{n-1} + \alpha_3 \sin(r_3 x_{n,2}) T_{n-2}, \textit{ for } i = 0, 1, 2, \ldots,$$

then C^*-solutions and N^*-solutions are equal, that is

$$\begin{aligned}&\mathtt{w}(x) - T_{n+1}\mathtt{w}(x_{n,-1}) - [T_n\alpha_2 \sin(r_2x_{n,0}) + T_{n-1}\alpha_3 \sin(r_3x_{n,1})]\mathtt{w}(x_{n,-2})\\ &- T_n\alpha_3 \sin(r_3x_{n,0})\mathtt{w}(x_{n,-3}) = \sum_{i=0}^{n} T_i y(x - ih).\end{aligned} \tag{2.6}$$

Proof. From (2.2) and (2.3), we arrive

$$\mathtt{w}(x) = y(x) + \alpha_1 \sin(r_1x)\mathtt{w}(x-h) + \alpha_2 \sin(r_2x)\mathtt{w}(x-2h) + \alpha_3 \sin(r_3x)\mathtt{w}(x-3h). \tag{2.7}$$

After changing x to $x - h$ in (2.7), and then putting the value of $\mathtt{w}(x - h)$, we observe

$$\begin{aligned}\mathtt{w}(x) &= y(x) + T_1y(x-h) + [T_1\alpha_1 \sin(r_1(x-h)) + \alpha_2 \sin(r_2x)]\mathtt{w}(x-2h)\\ &\times [T_1\alpha_2 \sin(r_2(x-2h)) + \alpha_3 \sin(r_3x-h)]\mathtt{w}(x-3h) + T_1\alpha_3 \sin(r_3(x-h))\mathtt{w}(x-4h),\end{aligned}$$

which gives

$$\begin{aligned}\mathtt{w}(x) &= T_0y(x) + T_1y(x-h) + T_2\mathtt{w}(x-2h) + [T_1\alpha_2 \sin(r_2(x-2h))\\ &+ \alpha_3 \sin(r_3x-h)]\mathtt{w}(x-3h) + T_1\alpha_3 \sin(r_3(x-h))\mathtt{w}(x-4h),\end{aligned} \tag{2.8}$$

where T_0, T_1, and T_2 are available in (2.1). After changing x to $x - 2h$ in (2.7), and then putting the value of $\mathtt{w}(x - 2h)$ into (2.8), we observe

$$\begin{aligned}\mathtt{w}(x) &= T_0y(x) + T_1y(x-h) + T_2y(x-2h) + T_3\mathtt{w}(x-3h) + [T_2\alpha_2 \sin(r_2(x-2h))\\ &+ T_1\alpha_3 \sin(r_3(x-h))]\mathtt{w}(x-4h) + [T_2\alpha_3 \sin(r_3(x-2h))]\mathtt{w}(x-5h),\end{aligned}$$

where T_0, T_1, T_2, and T_3 are find from the Definition 2.1. When repeating this process, we get (2.6). □

Corollary 2.9. *If* $\mathtt{w}(x) = \underset{\varphi(x)}{\overset{-1}{\nabla}} y(x)$ *be any function,* x *be any non-negative real number,* n *be an integer,* $T_0 = 1$, $T_1 = \sin(r_1x)$, $T_2 = T_1 \sin(r_1(x-h)) + \sin(r_2x), \dots$, *and*

$$F_{n+1} = \sin(r_1x_{n,0})T_n + \sin(r_2x_{n,1})T_{n-1} + \sin(r_3x_{n,2})T_{n-2},$$

then C^*-solutions and N^*-solutions are equal, which is observed by

$$\mathtt{w}(x) - T_{n+1}\mathtt{w}(x_{n,-1}) - [T_n \sin(r_2x_{n,0}) + T_{n-1} \sin(r_3x_{n,1})]\mathtt{w}(x_{n,-2}) - T_n \sin(r_3x_{n,0})\mathtt{w}(x_{n,-3}) = \sum_{i=0}^{n} T_i y(x - ih).$$

Proof. According to Theorem 2.8, the proof can be followed by taking $\alpha_1 = \alpha_2 = \alpha_3 = 1$. □

Corollary 2.10. *If* $\mathtt{w}(x)$ *is a* C^**-solution of the third order nabla difference equation with the coefficients of trigonometric functions* $\underset{\varphi(x)}{\nabla} \mathtt{w}(x) = a^x \left[1 - \frac{\alpha_1 \sin(r_1x)}{a^{sh}} - \frac{\alpha_2 \sin(r_2x)}{a^{2sh}} - \frac{\alpha_3 \sin(r_3x)}{a^{3sh}}\right]$, *then we obtain*

$$\begin{aligned}&a^{sx} - T_{n+1}a^{sx_{n,-1}} - [T_n\alpha_2 \sin(r_2x_{n,0}) + T_{n-1}\alpha_3 \sin(r_3x_{n,1})]a^{sx_{n,-2}} + T_n\alpha_3 \sin(r_3x_{n,0})a^{sx_{n,-3}}\\ &= \sum_{i=0}^{n} F_i a^{s(x-ih)}\left[1 - \frac{\alpha_1 \sin(r_1(x-ih))}{a^{sh}} - \frac{\alpha_2 \sin(r_2(x-ih))}{a^{2sh}} - \frac{\alpha_3 \sin(r_3(x-ih))}{a^{3sh}}\right].\end{aligned} \tag{2.9}$$

Proof. The proof of (2.9) can be obtained by assuming that $\mathtt{w}(x) = a^{sx}$ and, then using (2.4) to (2.6). □

An illustration of the verification of Corollary 2.10 is seen in the following example.

Example 2.11. Taking $x = 10$, $s = 2$, $h = 0.8$,$n = 3$, $a = 2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.9$, $\alpha_2 = 1.5$, $r_1 = 5$, $r_2 = 3$, and $r_3 = 2$ in (2.9), we get

$$2^{20} - T_4 2^{13.6} - [T_3(0.9)\sin(38) + T_2(1.5)\sin(16.8)]2^{12} - T_3(1.5)\sin(15.2)2^{10.4}$$
$$= \sum_{i=0}^{3} T_i 2^{(20-1.6i)}\left[1 - \frac{(0.2)\sin(50-4i)}{2^{1.6}} - \frac{(0.9)\sin(30-2.4i)}{2^{3.2}} - \frac{(1.5)\sin(20-1.6i)}{2^{4.8}}\right] = 1028539,$$

where $T_0 = 1$, $T_1 = -0.053$, $T_2 = -0.899$, $T_3 = 1.505$, and $T_3 = 0.069$.

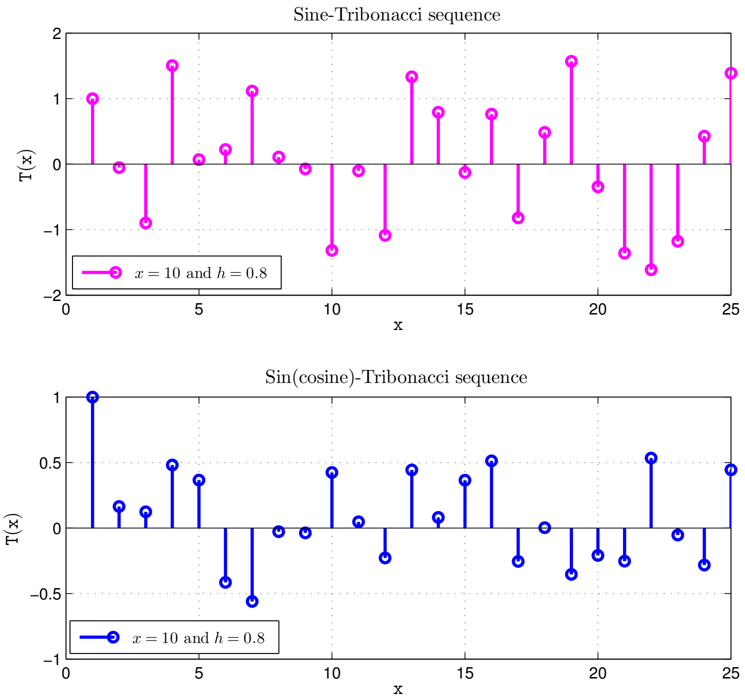

Figure 2: Sine and sin (cosine)-Tribonacci sequences.

Proposition 2.12. *Consider* $t \in \mathbb{N}(0)$, *and the third order nabla difference equation is solved in the closed form technique. Then,*

$$w(x) - \alpha_1 \sin(r_1 x)w(x_{0,1}) - \alpha_2 \sin(r_2 x)w(x_{0,2}) - \alpha_3 \sin(r_3 x)w(x_{0,3})$$
$$= \left[x^t - \alpha_1 \sin(r_1 x)(x_{0,1})^t - \alpha_2 \sin(r_2 x)(x_{0,2})^t - \alpha_3 \sin(r_3 x)(x_{0,3})^t\right],$$

is

$$\mathop{\nabla}\limits_{\varphi(x)}^{-1}\left[x^t - \alpha_1 \sin(r_1 x)(x_{0,1})^t - \alpha_2 \sin(r_2 x)(x_{0,2})^t - \alpha_3 \sin(r_3 x)(x_{0,3})^t\right] = x^t. \quad (2.10)$$

Proof. Taking $w(x) = x^t$ in (2.2) and using (2.3), we get (2.10). □

Corollary 2.13. *If* C^**-solution of the equation* (2.10) *is*

$$w(x) = \mathop{\nabla}\limits_{\varphi(x)}^{-1}\left[x^t - \alpha_1 \sin(r_1 x)(x-h)^t - \alpha_2 \sin(r_2 x)(x-2h)^t - \alpha_3 \sin(r_3 x)(x-3h)^t\right],$$

then

$$w(x)-T_{n+1}w(x_{n,-1})-[T_n\alpha_2\sin(r_2x_{n,0})+T_{n-1}\alpha_3\sin(r_3x_{n,1})]w(x_{n,-2})-T_n\alpha_3\sin(r_3x_{n,0})w(x_{n,-3})$$
$$=\sum_{i=0}^{n}F_i[(x-ih)^t-\alpha_1\sin(r_1(x-ih))(x_{i,-1})^t-\alpha_2\sin(r_2(x-ih))(x_{i,-2})^t$$
$$-\alpha_3\sin(r_3(x-ih))(x_{i,-3})^t]. \quad (2.11)$$

Proof. Taking $y(x)=x^t-\alpha_1\sin(r_1x)(x-h)^t-\alpha_2\sin(r_2x)(x-2h)^t-\alpha_3\sin(r_3x)(x-3h)^t$ in Theorem 2.8, we observe (2.11). □

An illustration of the verification of Corollary 2.13 is seen in the following example.

Example 2.14. Let $x=7.5$, $n=3$, $r_1=6$, $h=0.6$, $r_2=4$, $r_3=2$ $\alpha_1=0.3$, $t=3$, $\alpha_2=0.5$, and $\alpha_3=1.2$ in Corollary 2.13. Then,

$$\sum_{i=0}^{3}F_iy(7.5-0.6i)=w(7.5)-T_4w(5.1)-[T_3(0.5)\sin(22.8)+T_2(1.2)\sin(12.6)]w(7)$$
$$-T_3(1.2)\sin(11.4)w(3.9)=345.717,$$

where $y(x)=x^t-\alpha_1\sin(r_1x)(x-h)^t-\alpha_2\sin(r_2x)(x-2h)^t-\alpha_3\sin(r_3x)(x-3h)^t$, $T_0=1$, $T_1=-0.015$, $T_2=0.493$, $T_3=0.929$, and $T_4=0.522$.

Proposition 2.15. *If $w(x)$ is a C^*-solution of third order nabla difference equation with the coefficients of trigonometric functions*

$$w(x)-\alpha_1\sin(r_1x)w(x-h)-\alpha_2\sin(r_2x)w(x-2h)-\alpha_3\sin(r_3x)w(x-3h)$$
$$=x^ta^x-\alpha_1\sin(r_1x)(x-h)^ta^{x_{0,1}}-\alpha_2\sin(r_2x)(x-2h)^ta^{x-2h}-\alpha_3\sin(r_3x)(x-3h)^ta^{x-3h},$$

then we have the equality relation of C^* and N^*-solution observed by

$$w(x)-T_{n+1}w(x_{n,-1})-\alpha_2\sin(r_2x_{n,0})w(x_{n,-2})$$
$$=\sum_{i=0}^{n}F_i(x-ih)^ta^{x-ih}\Big[1-\alpha_1\frac{(x_{i,-1})^t}{a^{-h}}\sin(r_1(x-ih))$$
$$-\alpha_2\frac{\sin(r_2(x-ih))(x_{i,-2})^t}{a^{-2h}}-\alpha_3\frac{\sin(r_3(x-ih))(x_{i,-3})^t}{a^{-3h}}\Big]. \quad (2.12)$$

Proof. By taking

$$y(x)=[x^ta^x-\alpha_1\sin(r_1x)(x-h)^ta^{x_{0,1}}-\alpha_2\sin(r_2x)(x-2h)^ta^{x-2h}-\alpha_3\sin(r_3x)(x-3h)^ta^{x-3h}]$$

in Theorem 2.8 and applying (2.4), we observe (2.12). □

Corollary 2.16. *A C^*-solution of the third order C^*-equation*

$$\underset{\varphi(x)}{\nabla}w(x)=x^3a^x-\alpha_1\sin(r_1x)(x-h)^3a^{x_{0,1}}-\alpha_2\sin(r_1x)(x-2h)^3a^{x-3h}-3h)^3a^{x-3h}$$

is x^3a^x, *and hence we have*

$$\begin{aligned}
&x^3a^x - T_{n+1}x^3_{n,-1}a^{x_{n,-1}} - [T_n\alpha_2 \sin(r_2(x-nh)) \\
&\quad + T_{n-1}\alpha_3 \sin(r_3x_{n,1})]x^3_{n,-2}a^{x_{n,-2}} - T_{n+1}\alpha_3 \sin(r_3(x-nh))x^3_{n,-3}a^{x_{n,-3}} \\
&\quad = \sum_{i=0}^{n} F_i(x-ih)^3a^{x-ih} - [\alpha_1 \sin(r_1(x-ih))(x_{i,-1})^3a^{x_{i,-1}} \\
&\quad - \alpha_2 \sin(r_2(x-ih))](x_{i,-2})^3a^{x_{i,-2}} - \alpha_3 \sin(r_3(x-ih))(x_{i,-3})^3a^{x_{i,-3}}.
\end{aligned} \tag{2.13}$$

Proof. By taking $t = 3$ in Theorem 2.15, we get (2.13). □

An illustration of the verification of Corollary 2.16 is seen in the following example.

Example 2.17. Let $x = 7.5$, $h = 0.6$, $\alpha_1 = 0.3$, $n = 3$, $\alpha_2 = 0.5$, $a = 3$, $\alpha_3 = 1.2$, $r_1 = 6$, $r_2 = 4$, and $r_3 = 2$, in Corollary 2.16. Then, we obtain

$$\sum_{i=0}^{3} F_i y(7.5-0.6i) = w(7.5) - T_4w(5.1) - [T_3(0.5)\sin(22.8) + T_2(1.2)\sin(12.6)]w(7) - T_3(1.2)\sin(11.4)w(3.9) = 282.983,$$

where $y(x) = x^3a^x - \sum_{r=1}^{3} \alpha_i \sin(r_ix)(x-ih)^3a^{x-ih}$, $T_0 = 1$, $T_1 = 0.255$, $T_2 = -0.535$, $T_3 = 0.844$, and $T_4 = 0.360$.

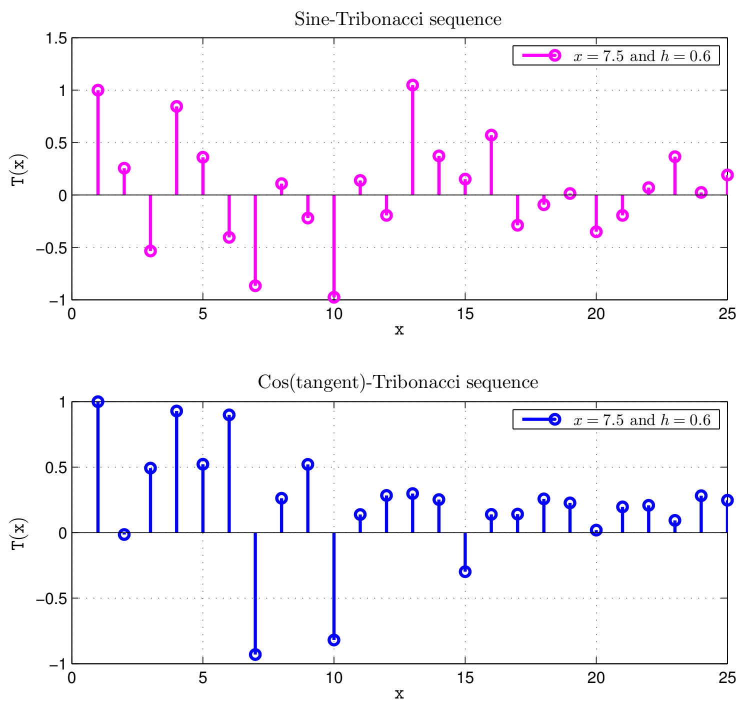

Figure 3: Sine and Cos (tangent)-Tribonacci sequences.

Corollary 2.18. *A* C^**-solution of the third order nabla difference equation*

$$w(x) - \alpha_1 \sin(r_1x)w(x-h) - \alpha_2 \sin(r_2x)w(x-2h) - \alpha_3 \sin(r_3x)w(x-3h)$$

$$= x^t e^{-x} - \alpha_1 \sin(r_1 x)(x-h)^t e^{-(x-h)} - \alpha_2 \sin(r_2 x)(x-2h)^t e^{-(x-2)} - \alpha_3 \sin(r_3 x)(x-3h)^t e^{-(x-3)},$$

is given by

$$\begin{aligned} & w(x) - T_{n+1} w(x_{n,-1}) - [T_n \alpha_2 \sin(r_2 x_{n,0}) + T_{n-1}\alpha_3 \sin(r_3 x_{n,1})] w(x_{n,-2}) \\ & - T_n \alpha_3 \sin(r_3 x_{n,0}) w(x_{n,-3}) \sum_{i=0}^{n} F_i e^{-(x-ih)} [(x-ih)^t - \alpha_1 \sin(r_1(x-ih)) \\ & \times (x_{i,-1})^t e^h - \alpha_2 \sin(r_2(x-ih))[x_{i,-2}]^t e^{2h} - \alpha_3 \sin(r_3(x-ih))[x_{i,-3}]^t e^{3h}]. \end{aligned} \tag{2.14}$$

Proof. Taking $a = e^{-1}$ in (2.12), we get (2.14). □

Proposition 2.19. *Let* $w(x)$ *be a* C^**-solution of the* $\overline{\varphi(x)}$*-equation*

$$\begin{aligned} & w(x) - \alpha_1 \sin(r_1 x) w(x-h) - \alpha_2 \sin(r_2 x) w(x-2h) - \alpha_3 \sin(r_3 x) w(x-3h) = x^{(t)} a^x \\ & - \alpha_1 \sin(r_1 x)(x-h)^{(t)} a^{x_{0,1}} - \alpha_2 \sin(r_2 x)(x-2h)^{(t)} a^{x-2h} - \alpha_3 \sin(r_3 x)(x-3h)^{(t)} a^{x-3h}, \end{aligned}$$

then we have

$$\begin{aligned} & w(x) - T_{n+1} w(x_{n,-1}) - [T_n \alpha_2 \sin(r_2 x_{n,0}) + T_{n-1}\alpha_3 \sin(r_3 x_{n,1})] w(x_{n,-2}) - T_n \alpha_3 \sin(r_3 x_{n,0}) w(x_{n,-3}) \\ & = \sum_{i=0}^{n} F_i a^{x-ih} [(x-ih)^{(t)} - \alpha_1 \sin(r_1(x-ih))[x_{i,-1}]^{(t)} a^{-h} \\ & \quad - \alpha_2 \sin(r_2(x-ih))[x_{i,-2}]^{(t)} a^{-2h} - \alpha_3 \sin(r_3(x-ih))[x_{i,-3}]^{(t)} a^{-3h}]. \end{aligned} \tag{2.15}$$

Proof. Substituting $w(x) = x^{(t)} a^x$ in Theorem 2.8 and applying (2.4), leads the proof. □

Corollary 2.20. *If* $w(x)$ *is the* C^**-solution of the equation* (2.15)*, then we have*

$$\begin{aligned} & x^{(2)} a^x - T_{n+1} x^{(2)}_{n,-1} a^{x_{n,-1}} - [T_n \alpha_2 \sin(r_2 x_{n,0}) \\ & \quad + T_{n-1} \alpha_3 \sin(r_3 x_{n,1})] x^{(2)}_{n,-2} a^{x_{n,-2}} - T_n \alpha_2 \sin(r_2 x_{n,0}) x^{(2)}_{n,-2} a^{x_{n,-2}} \\ & = \sum_{i=0}^{n} F_i a^{x-ih} [(x-ih)^{(2)} - \alpha_1 \sin(r_1(x-ih))[x_{i,-1}]^{(2)} a^{-h} \\ & \quad - \alpha_2 \sin(r_2(x-ih))[x_{i,-2}]^{(2)} a^{-2h} - \alpha_3 \sin(r_3(x-ih))[x_{i,-3}]^{(2)} a^{-3h}]. \end{aligned}$$

Proof. Substituting $t = 2$ in Theorem 2.19, we observe the proof. □

An illustration of the verification of Corollary 2.20 is seen in the following example.

Example 2.21. Let $x = 11.5$, $h = 0.6$, $a = 2$, $n = 3$, $\alpha_1 = 0.35$, $\alpha_2 = 0.75$, $\alpha_3 = 1.26$, $r_1 = 4$, $r_2 = 5$, and $r_3 = 2$, in Corollary 2.20. Then, we obtain

$$\begin{aligned} & w(11.5) - T_4 w(9.1) - [T_3(0.75)\sin(48.5) + T_2(1.26)\sin(20.6)] w(8.5) \\ & - T_3(1.26)\sin(19.4) w(7.9) = \sum_{i=0}^{3} F_i y(11.5 - 0.6i) = 420166, \end{aligned}$$

where $y(x) = x^{(2)} a^x - \sum_{r=1}^{3} \alpha_i \sin(r_i x)(x-ih)^{(2)} a^{x-ih}$, $T_0 = 1$, $T_1 = 0.316$, $T_2 = 0.570$, $T_3 = -1.347$, and $T_4 = 0.059$.

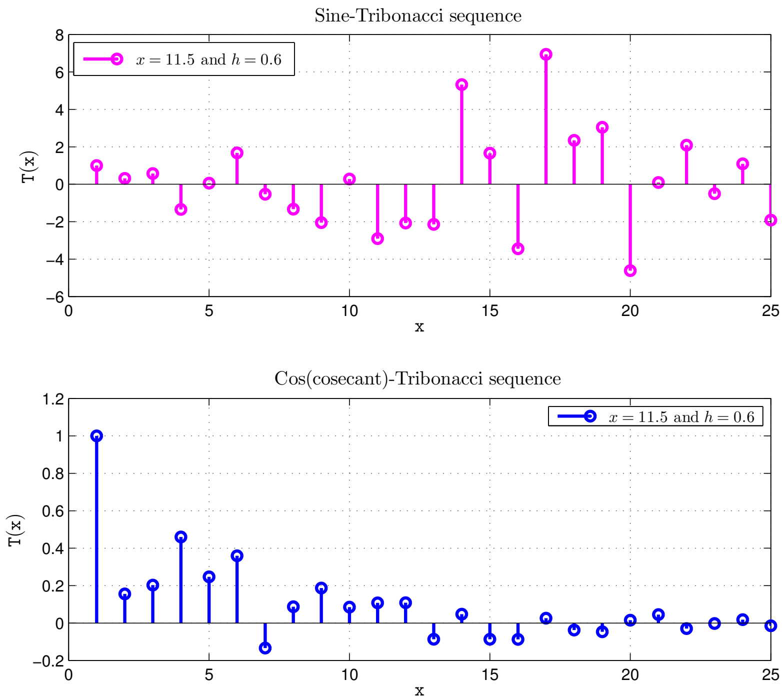

Figure 4: Sine and cos (cosecant)-Tribonacci sequences.

Corollary 2.22. *Let* $w(x)$ *be the* C^**-solution and* N^**-solution of generic third order nabla difference equation with the coefficients of trigonometric functions*

$$\begin{aligned} w(x) - \alpha_1 \sin(r_1 x) w(x-h) - \alpha_2 \sin(r_2 x) w(x-2h) - \alpha_3 \sin(r_3 x) w(x-3h) \\ = e^{-x}[x^{(t)} - \alpha_1 \sin(r_1 x)(x-h)^{(t)} e^{h} - \alpha_2 \sin(r_2 x)(x-2h)^{(t)} e^{2h} - \alpha_2 \sin(r_2 x) w(x-2h) \\ - \alpha_3 \sin(r_3 x) w(x-3h) = e^{-x}[x^{(t)} - \alpha_1 \sin(r_1 x)(x-h)^{(t)} e^{h} - \alpha_3 \sin(r_3 x)(x-3h)^{(t)} e^{3h}]. \end{aligned}$$

Then, we have

$$\begin{aligned} & w(x) - T_{n+1} w(x_{n,-1}) - [T_n \alpha_2 \sin(r_2 x_{n,0}) + T_{n-1} \alpha_3 \sin(r_3 x_{n,1})] \\ & = \sum_{i=0}^{n} F_i e^{-(x-ih)}[(x-ih)^{(t)} - \alpha_1 \sin(r_1(x-ih))[x_{i,-1}]^{(t)} e^{h} \\ & \quad - \alpha_2 \sin(r_2(x-ih))[x_{i,-2}]^{(t)} e^{2h} - \alpha_3 \sin(r_3(x-ih))[x_{i,-3}]^{(t)} e^{3h}]. \end{aligned} \tag{2.16}$$

Proof. Substituting $a = e^{-1}$ in equation (2.19), we get the equation (2.16). □

Corollary 2.23. *A* C^* *and* N^**-solutions of the* $\overline{\varphi(x)}$*-difference equation*

$$\underset{\varphi(x)}{\nabla} w(x) = e^{-x}[x^{(3)} - \alpha_1 \sin(r_1 x)(x-h)^{(3)} e^{h} - \alpha_2 \sin(r_2 x)(x-2h)^{(3)} e^{2h}],$$

is $x^{(3) e^{-x}}$ *and hence we have*

$$\begin{aligned} & w(x) - T_{n+1} w(x_{n,-1}) - [T_n \alpha_2 \sin(r_2 x_{n,0}) + T_{n-1} \alpha_3 \sin(r_3 x_{n,1})] \\ & = \sum_{i=0}^{n} F_i e^{-(x-ih)}[(x-ih)^{(3)} - \alpha_1 \sin(r_1(x-ih))[x_{i,-1}]^{(3)} e^{h} - \alpha_2 \sin(r_2(x-ih))[k-(i+2h)]^{(3)} e^{2h}]. \end{aligned} \tag{2.17}$$

Proof. Substituting $t = 3$ in Corollary 2.22, leads equation (2.17). □

An illustration of the verification of Corollary 2.22 is seen in the following example.

Example 2.24. Let $x = 6.5$, $h = 0.7$ $n = 3$, $a = 2$, $\alpha_1 = 0.7$, $\alpha_2 = 0.9$, $\alpha_3 = 1.2$, $r_1 = 5$, $r_2 = 3$, and $r_3 = 1$ in Corollary 2.23. Then, we obtain

$$\sum_{i=0}^{3} F_i y(6.5 - 0.7i) = w(6.5) - T_4 w(3.7) - [T_3(0.9)\sin(13.2) + T_2(1.2)\sin(4.4)]w(3) - T_3(1.2)\sin(4.4)w(3.3) = 128260,$$

where $y(x) = x^{(3)}e^{-x} - \sum_{r=1}^{3} \alpha_i \sin(r_i x)(x - ih)^{(3)} e^{-x-ih}$, $T_0 = 1$, $T_1 = 0.619$, $T_2 = 0.258$, $T_3 = -0.230$, and $T_4 = -0.252$.

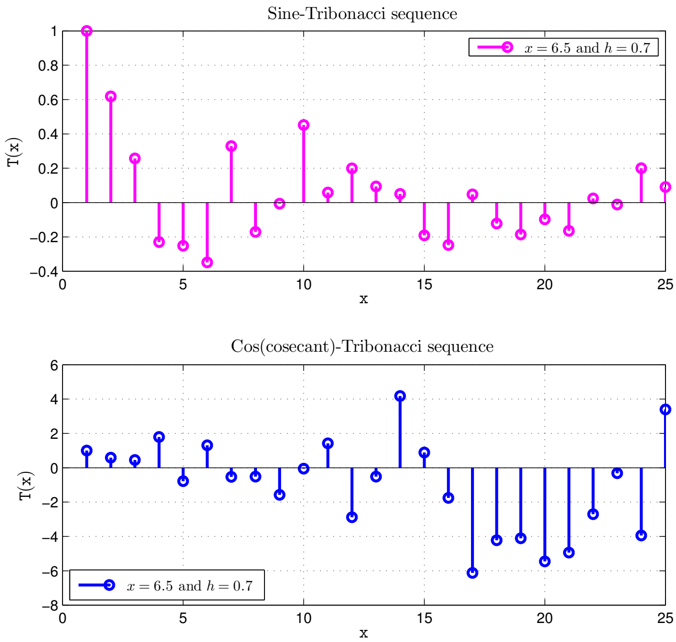

Figure 5: Sine and cos (cosecant)-Tribonacci sequences.

Corollary 2.25. *Consider the negative exponential function of* $x \in (-\infty, \infty)$ *with the coefficient* s. *Then, we observe*

$$\begin{aligned} &e^{-sx} - T_{n+1}e^{-sx_{n,-1}} - [T_n\alpha_2\sin(r_2x_{n,0}) + T_{n-1}\alpha_3\sin(r_3x_{n,1})]e^{-sx_{n,-2}} - T_n\alpha_3\sin(r_3x_{n,0})e^{-sx_{n,-3}} \\ &= \sum_{i=0}^{n} F_i e^{-s(x-ih)} \\ &= \sum_{i=0}^{n} F_i e^{-(x-ih)}\Big[1 - \alpha_1\sin(r_1(x-ih))e^{h} - \alpha_2\sin(r_2(x-ih))e^{2h} - \alpha_3\sin(r_3(x-ih))e^{3h}\Big]. \end{aligned} \tag{2.18}$$

Proof. Taking $w(x) = e^{-sx}$ in (2.2) and applying (2.3), we get the equation (2.18). □

An illustration of the verification of Corollary 2.25 is seen in the following example.

Example 2.26. Setting $x = 8$, $\alpha_1 = 0.6$, $n = 3$, $\alpha_2 = 0.5$, h =0.7, $\alpha_3 = 1$, $r_1 = 4$, $r_2 = 2$, and $r_2 = 1$ in (2.18), then we obtain

$$e^{-24} - T_4 e^{-15.6} - [T_3(0.6)\sin(11.8) + T_2\sin(6.6)]e^{-13.5} - T_3\sin(5.9)e^{-11.4}$$
$$= \sum_{i=0}^{3} F_i y(x-(0.7)i) = 26483382788,$$

where $T_0 = 1$, $T_1 = 0.331$, $T_2 = -0.303$, $T_3 = 0.964$, and $T_4 = -0.386$.

3. Conclusion

We derived a summation formula for the $\overline{\varphi(x)}$-Tribonacci sequence by introducing a generalized third-order nabla operator with coefficients of trigonometric functions. We have obtained results on the closed and summation form solutions of the generalized third-order difference equation with coefficients of trigonometric functions, which will be utilized in our future research, to examine the relationship between the Tribonacci ratio and atomic structure. Specific nuclides exhibit the Tribonacci ratio between protons and neutrons, creating a noticeable pattern on the nuclide chart.

The typical Tribonacci sequence and numbers consist of positive integers. Using our definitions and observations enables us to create an infinite variety of real Tribonacci sequences, related polynomials, and real Tribonacci numbers.

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