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A constrained problem of state dependent pantograph functional equation constrained by its conjugate

A. M. A. El-Sayed[a], Eman. M. Al-Barg[b,*], H. R. Ebead[a]

[a]*Faculty of Science, Alexandria University, Alexandria, Egypt.*

[b]*Faculty of Science, Sirt University, Libya.*

Abstract

In this paper, we define the state-dependent pantograph functional equation and study the existence and uniqueness of its solution and prove some data dependence theorems, then we investigate the existence of the solution of a constrained problem of the state-dependent pantograph functional equation constrained by its conjugate equation. Moreover, we demonstrate the continuous dependence of the solution. We also examine the Hyres-Ulam stability of our problem.

Keywords: Constrained problem, pantograph functional equation, state-dependent, continuous dependence, Hyres-Ulam stable.

2020 MSC: 34K43, 34K20.

1. Introduction

Constrained problems are of great significance in accurately representing real-world scenarios mathematically, as they can be converted into models in mathematics, see [6, 19]. The management of control variables or constraints is necessary because of unexpected factors that consistently disrupt biological systems in the actual world, which can result in changes in biological characteristics such as survival rates. Determining if an ecosystem is resistant to these unpredictable and disruptive occurrences is important in ecology. In the context of constraints, we refer to these disruptive events as control variables. Extensive research has been conducted on the investigation of differential equations with constraints. One specific category of such equations involves constraints within a bounded interval, see [1, 7, 12, 17] and while another category involves constraints on an unbounded interval, see [8].

Differential equations with deviating arguments constitute a significant and well-known subfield of nonlinear analysis, finding applications in various fields. Equations with deviating arguments typically have a deviation that depends solely on time. However, self-referential or state-dependent equations arise when the deviation of the arguments depends on both the state variable x and the time variable t, which is crucial in theory and practice [3, 5, 10, 11, 18].

*Corresponding author

Email addresses: amasayed@alexu.edu.eg (A. M. A. El-Sayed), eman.albarq@su.edu.ly (Eman. M. Al-Barg), hanaarezqalla@alexu.edu.eg (H. R. Ebead)

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The Ulam stability problem investigates the effects of tiny disturbances on the solutions to a functional equation. It specifically inquires as to whether functional equation approximation solutions are nearly exact. Many recent papers concerts with the study of the Ulam stability problem, see for example [21–23]. Specialists are interested in differential equations with state-dependent delays as they frequently originate from application models, such as the classical electrodynamics two-body problem. These equations have many applications, particularly in problems involving memories, such as hereditary phenomena (see [2, 14, 16, 20]). Differential equations with delays come in various forms. The differential equation

$$\frac{d\xi(t)}{dt} = \mathcal{F}(t, \xi(t), \xi(\gamma t)),$$

where $\gamma \in (0, 1)$ is called " pantograph differential equations " which is a fundamental mathematical model in the field of a delay differential equation, pantograph differential equation form of equation that belongs to the proportional delay differential equations category. The significance of these equations is due to their capacity to simulate various economic, chemical and chemistry, biology, medicine, infectious diseases, pharmacological, and physiological kinetics, see [4, 15, 24].

In [9], the authors define the pantograph functional equation

$$\xi(t) = \mathcal{F}(t, \xi(t), \xi(\gamma t)), \ t \in [0, T], \ \gamma \in (0, 1),$$

and they study the existence of its solution in $C[0, T]$ and $L_1[0, T]$. Here we investigate the existence of a unique solution to the constrained problem of the state-dependent pantograph functional equation

$$\xi(t) = \mathcal{F}_1(t, \xi(t), \xi(\gamma_1 \eta(t))), \ t \in [0, T], \ \gamma_1 \in (0, 1). \tag{1.1}$$

Constrained by it's conjugate

$$\eta(t) = \mathcal{F}_2(t, \eta(t), \eta(\gamma_2 \xi(t))), \ t \in [0, T], \ \gamma_2 \in (0, 1), \tag{1.2}$$

where $\mathcal{F}_i : [0, T] \times [0, T] \times [0, T] \to [0, T], \ i = 1, 2$ are continuous. Furthermore, we will demonstrate that the solution $\xi \in C[0, T]$ is continuously dependent on the functions $\mathcal{F}_i$ and γ_i. Also the continuous dependence of ξ on η and η on ξ will be studied. Moreover, the Hyres-Ulam stability of the problem will be established. Firstly, we begin with the state-dependent pantograph equation

$$\xi(t) = \mathcal{F}(t, \xi(t), \xi(\gamma \xi(t))), \ \ t \in [0, T], \ \gamma \in (0, 1), \tag{1.3}$$

where $\mathcal{F} : [0, T] \times [0, T] \times [0, T] \to [0, T]$ is a continuous. The existence of the unique solution of (1.3) will be studied. Additionally, we will show that the solution $\xi(t)$ continuously depends on both the function $\mathcal{F}$ and the parameter γ. Furthermore, we will establish the Hyres-Ulam stability of the problem.

2. Solution of (1.3)

Consider the class of continuous functions defined on $[0, T]$ denoted by $C[0, T]$ with the norm

$$\|\xi\|_C = \sup_{t \in [0,T]} |\xi(t)|.$$

Consider the following assumptions

(i) $\mathcal{F} : [0, T] \times [0, T] \times [0, T] \to [0, T]$ is continuous and there exist positive constants K_1, K_2 such that

$$|\mathcal{F}(t, x, u) - \mathcal{F}(s, y, v)| \leqslant K_1|t - s| + K_2(|x - y| + |u - v|);$$

(ii) the next algebraic equation has a real positive root $L \in (0,1)$,

$$K_2\gamma L^2 + (K_2 - 1)L + K_1 = 0.$$

Define the subset Q_L of $C[0,T]$, through

$$Q_L = \{\xi \in C : |\xi(t) - \xi(s)| \leqslant L|t-s|, \ \forall \ t,s \in [0,T]\},$$

and the operator G by

$$G\xi(t) = \mathcal{F}(t, \xi(t), \xi(\gamma\xi(t))).$$

It is evident that the subset Q_L of $C[0,T]$ is bounded, closed, convex, and nonempty. For $\xi \in Q_L$, we have

$$|\xi(t) - \xi(0)| \leqslant L|t-0| \ \Rightarrow \ |\xi(t)| \leqslant |\xi(0)| + L|t| \leqslant |\xi(0)| + LT.$$

Theorem 2.1. *Assume that* (i)-(ii) *are valid. If* $(2K_2 + K_2L\gamma) < 1$, *then the equation* (1.3) *has a unique solution* $\xi \in C[0,T]$.

Proof. Let $\xi \in Q_L, \ t_1, t_2 \in [0,T], \ t_1 < t_2$, then

$$\begin{aligned} |G\xi(t_2) - G\xi(t_1)| &= |\mathcal{F}(t_2, \xi(t_2), \xi(\gamma\xi(t_2))) - \mathcal{F}(t_1, \xi(t_1), \xi(\gamma\xi(t_1)))| \\ &= K_1|t_2 - t_1| + K_2|\xi(t_2) - \xi(t_1)| + K_2|\xi(\gamma\xi(t_2)) + \xi(\gamma\xi(t_1))) \\ &\leqslant K_1|t_2 - t_1| + K_2L|t_2 - t_1| + K_2L\gamma|\xi(t_2) - \xi(t_1)| \\ &\leqslant K_1|t_2 - t_1| + K_2L|t_2 - t_1| + K_2L^2\gamma|t_2 - t_1| \\ &\leqslant (K_1 + K_2L + K_2L^2\gamma)|t_2 - t_1| \leqslant L|t_2 - t_1|, \end{aligned}$$

then

$$G : Q_L \to Q_L.$$

Now, let $\xi, \bar{\xi} \in Q_L$, then

$$\begin{aligned} |G\xi(t) - G\bar{\xi}(t)| &= |\mathcal{F}(t, \xi(t), \xi(\gamma\xi(t)) - \mathcal{F}(t, \bar{\xi}(t), \bar{\xi}(\gamma\bar{\xi}(t)))| \\ &\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma\xi(t)) - \bar{\xi}(\gamma\bar{\xi}(t))| \\ &\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma\xi(t) - \xi(\gamma\bar{\xi}(t)) + \xi(\gamma\bar{\xi}(t)) + \bar{\xi}(\gamma\bar{\xi}(t))| \\ &\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma\xi(t) - \xi(\gamma\bar{\xi}(t))| + K_2|\xi(\gamma\bar{\xi}(t)) + \bar{\xi}(\gamma\bar{\xi}(t))| \\ &\leqslant K_2|\xi - \bar{\xi}\| + K_2L|\gamma\xi(t) - \gamma\bar{\xi}(t)| + K_2\|\xi - \bar{\xi}\| \\ &\leqslant K_2\|\xi - \bar{\xi}\| + K_2L\gamma\|\xi - \bar{\xi}\| + K_2\|\xi - \bar{\xi}\| \\ &\leqslant 2K_2\|\xi - \bar{\xi}\| + K_2L\gamma\|\xi - \bar{\xi}\|. \end{aligned}$$

Hence

$$\|G\xi - G\bar{\xi}\| \leqslant (2K_2 + K_2L\gamma)\|\xi - \bar{\xi}\|.$$

□

Then G is a contraction mapping and by the Banach fixed point Theorem [13], G has a unique fixed point. Consequently, the equation (1.3) has a unique solution $\xi \in C[0,T]$.

2.1. Hyres-Ulam stability

Definition 2.2 ([8, 12]). Let the solution $\xi \in C[0,T]$ of the equation (1.3) exists, then the equation (1.3) is Hyers-Ulam stable if $\forall \epsilon > 0, \ \exists \ \delta(\epsilon) > 0$ such that for any δ-approximate solution $\xi_s \in C$ of (1.3) satisfies

$$|\xi_s(t) - \mathcal{F}(t, \xi_s(t), \xi_s(\gamma\xi_s(t)))| < \delta,$$

then $\|\xi - \xi_s\| < \epsilon$.

Theorem 2.3. *If the hypothesis of Theorem 2.1 are met, then the equation (1.3) is Hyers-Ulam stable.*

Proof. Let

$$|\xi_s(t) - F(t, \xi_s(t), \xi_s(\gamma\xi_s(t)))| < \delta,$$

then

$$\begin{aligned}
|\xi(t) - \xi_s(t)| &= |F(t, \xi(t), \xi(\gamma\xi(t))) - \xi_s(t)| \\
&= |F(t, \xi_s(t), \xi_s(\gamma\xi_s(t))) - \xi_s(t) + F(t, \xi_s(t), \xi_s(\gamma\xi_s(t))) - F(t, \xi(t), \xi(\gamma\xi(t)))| \\
&= |\xi_s(t) - F(t, \xi_s(t), \xi_s(\gamma(\xi_s(t))))| + |F(t, \xi_s(t), \xi_s(\gamma\xi_s(t))) - F(t, \xi(t), \xi(\gamma\xi(t)))| \\
&\leqslant \delta + |F(t, \xi(t), \xi(\gamma\xi(t))) - F(t, \xi_s(t), \xi_s(\gamma\xi_s(t)))| \\
&\leqslant \delta + K_2|\xi(t) - \xi_s(t)| + K_2|\xi_s(\gamma\xi_s(t)) - \xi(\gamma\xi(t))| \\
&\leqslant \delta + K_2\|\xi - \xi_s\| + K_2|\xi_s(\gamma\xi_s(t)) - \xi(\gamma\xi_s(t)) + \xi(\gamma\xi_s(t)) - \xi(\gamma\xi(t))| \\
&\leqslant \delta + K_2\|\xi - \xi_s\| + K_2|\xi_s(\gamma\xi_s(t)) - \xi(\gamma\xi_s(t))| + K_2|\xi(\gamma\xi_s(t)) - \xi(\gamma\xi(t))| \\
&\leqslant \delta + K_2\|\xi - \xi_s\| + K_2\|\xi - \xi_s\| + K_2L\gamma|\xi(t) - \xi_s(t)| \\
&\leqslant \delta + 2K_2\|\xi - \xi_s\| + K_2L\gamma\|\xi - \xi_s\|,
\end{aligned}$$

and $(1 - (2K_2 + K_2L\gamma))\|\xi - \xi_s\| \leqslant \delta$. Hence

$$\|\xi - \xi_s\| \leqslant \frac{\delta}{1 - (2K_2 + K_2L\gamma)}.$$

□

2.2. Continuous dependence

Definition 2.4. The solution $\xi \in C[0, T]$ of (1.3) depends continuously on the function F and the parameter γ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that

$$\max\{|F - F^*|, |\gamma - \gamma^*|\} < \delta \;\Rightarrow\; \|\xi - \xi^*\| < \epsilon,$$

where

$$\xi^*(t) = F^*(t, \xi^*(t), \xi^*(\gamma^*(\xi^*(t)))). \tag{2.1}$$

Theorem 2.5. *If the conditions of Theorem 2.1 are met, then the solution $\xi \in C[0, T]$ of (1.3) depends continuously on the function F and the parameter γ.*

Proof. Considering the two solutions ξ and ξ^* of (1.3) and (2.1) respectively, we obtain

$$\begin{aligned}
|\xi(t) - \xi^*(t)| &= |F(t, \xi(t), \xi(\gamma(\xi(t)))) - F^*(t, \xi^*(t), \xi^*(\gamma^*(\xi^*(t))))| \\
&= |F(t, \xi(t), \xi(\gamma\xi(t))) - F(t, \xi^*(\gamma\xi^*(t))) + F(t, \xi^*(t), \xi^*(\gamma^*\xi^*(t))) - F^*(t, \xi^*(t), \xi^*(\gamma^*\xi^*(t)))| \\
&\leqslant |F(t, \xi(t), \gamma\xi(t)) - F(t, \xi^*(t), \xi^*(\gamma^*\xi^*(t)))| + |F(t, \xi^*(t), \xi^*(\gamma^*\xi^*(t))) \\
&\quad - F^*(t, \xi^*(t), \xi^*(\gamma^*\xi^*(t)))| \\
&\leqslant K_2|\xi(t) - \xi^*(t)| + k_2|\xi(\gamma\xi(t)) - \xi^*(\gamma^*\xi^*(t))| + \delta \\
&\leqslant K_2\|\xi - \xi^*\| + K_2|\xi(\gamma\xi(t)) - \xi(\gamma^*\xi^*(t)) + \xi(\gamma^*\xi^*(t)) - \xi^*(\gamma^*\xi^*(t))| + \delta \\
&\leqslant K_2\|\xi - \xi^*\| + K_2|\xi(\gamma\xi(t)) - \xi(\gamma^*\xi^*(t))| + k_2|\xi(\gamma^*\xi^*(t)) - \xi^*(\gamma^*\xi^*(t))| + \delta
\end{aligned}$$

$$
\begin{aligned}
&\leqslant K_2\|\xi-\xi^*\| + K_2L|\gamma\xi(t)-\gamma^*\xi^*(t)| + K_2\|\xi-\xi^*\| + \delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L|\gamma\xi(t)-\gamma\xi^*(t)+\gamma\xi^*(t)-\gamma^*\xi^*(t)| + K_2\|\xi-\xi^*\| + \delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L|\gamma\xi(t)-\gamma\xi^*(t)| + K_2L|\gamma\xi^*(t)-\gamma^*\xi^*(t)| + K_2\|\xi-\xi^*\| + \delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma\|\xi-\xi^*\| + K_2L\|\xi^*\||\gamma-\gamma^*| + K_2\|\xi-\xi^*\| + \delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma\|\xi-\xi^*\| + K_2L\|\xi^*\|\delta + K_2\|\xi-\xi^*\| + \delta\\
&\leqslant (2K_2+K_2L\gamma)\|\xi-\xi^*\| + (1+K_2L\|\xi^*\|)\delta.
\end{aligned}
$$

Thus

$$(1-(2K_2+K_2L\gamma))\|\xi-\xi^*\| \leqslant (1+K_2L\|\xi^*\|)\delta,$$

and

$$\|\xi-\xi^*\| \leqslant \frac{(1+K_2L\|\xi^*\|)}{(1-(2K_2+K_2L\gamma))}\delta = \epsilon,$$

then the solution $\xi \in C[0,T]$ depends continuously on $\mathcal{F}$ and γ. □

3. Solution of constrained problem (1.1)-(1.2)

Consider the Banach space $X = C[0,T] \times C[0,T]$ with the norm $\|(u,v)\|_X = \|u\|_C + \|v\|_C$, with

$$\|u\|_C = \sup_{t\in[0,T]} |u(t)|.$$

Consider the following hypothesis:

(i) $\mathcal{F}_i : [0,T]\times[0,T]\times[0,T] \to [0,T]$ are continuous and there exist positive constants K_1, K_2 such that

$$|\mathcal{F}_i(t,x,u) - \mathcal{F}_i(s,y,v)| \leqslant K_1|t-s| + K_2(|x-y|+|u-v|),\ i=1,2;$$

(ii) the algebraic equation $K_2\gamma L^2 + (K_2-1)L + K_1 = 0$ has a positive solution $L \in (0,1)$, where $\gamma = \max\{\gamma_1,\gamma_2\}$.

Define the subset S_L of $C[0,T]$ as

$$S_L = \{U=(\xi,\eta)\in X : |\xi(t)-\xi(s)| \leqslant L|t-s|,\ |\eta(t)-\eta(s)| \leqslant L|t-s|,\ \forall t,s\in[0,T]\},$$

and the operator F by $F(\xi,\eta) = (\mathcal{F}_1\eta, \mathcal{F}_2\xi)$, where

$$F_1\eta(t) = \mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))), \qquad F_2\xi(t) = \mathcal{F}_2(t,\eta(t),\eta(\gamma_2\xi(t))).$$

Lemma 3.1. *Let $X = C[0,T]\times C[0,T]$ be a Banach space, then S_L is convex and compact in the Banach space $(X, \|\ .\ \|)$.*

Proof. Let $U = (\xi,\eta) \in S_L$, $\bar{U} = (\bar{\xi},\bar{\eta}) \in S_L$, $t\in[0,T]$,

$$
\begin{aligned}
(\delta U + (1-\delta)\bar{U})(t) &= (\delta(\xi,\eta) + (1-\delta)(\bar{\xi},\bar{\eta}))(t)\\
&= (\delta\xi(t),\delta\eta(t)) + ((1-\delta)\bar{\xi}(t),(1-\delta)\bar{\eta}(t))\\
&= (\delta\xi(t)+(1-\delta)\bar{\xi}(t), \delta\eta(t)+(1-\delta)\bar{\eta}(t))\\
&= (\delta\xi(t)+(1-\delta)\bar{\xi}(t), \delta\eta(t)+(1-\delta)\bar{\eta}(t)).
\end{aligned}
$$

Now

$$|\delta\xi(t)+(1-\delta)\bar{\xi}(t) - \delta\xi(s) + (1-\delta)\bar{\xi}(s)| \leqslant \delta|\xi(t)-\xi(s)| + (1-\delta)|\bar{\xi}(t)-\bar{\xi}(s)|$$

$$\leqslant \delta L|t-s| + (1-\delta)L|t-s| \leqslant L|t-s|.$$

Similarly,

$$|\delta\eta(t) + (1-\delta)\bar{\eta}(t) - \delta\xi(s) + (1-\delta)\bar{\xi}(s)| \leqslant L|t-s|.$$

Then

$$(\delta(\xi,\eta) + (1-\delta)(\bar{\xi},\bar{\eta}))(t) \in S_L.$$

So S_L is convex. From definition of S_L we deduce that $\xi(t), \eta(t)$ are equi-continuous, then $\forall\ U = (\xi,\eta) \in S_L$, U is equi-continuous, and

$$\|U\|_X = \|\xi\|_C + \|\eta\|_C \leqslant LT + |\xi(0)| + LT + |\eta(0)|,$$

then $\forall U = (\xi,\eta) \in S_L$, U is uniformly bounded. Using the Arzella-Ascoli theorem we deduce that S_L is compact. □

Theorem 3.2. *Let the assumptions* (i)-(ii) *be met. If* $(2K_2 + K_2L\gamma) < 1$, *then the constrained problem* (1.1)-(1.2) *has a unique solution* $(\xi,\eta) \in X$.

Proof. Let $U = (\xi,\eta) \in S_L$ and $t_1, t_2 \in [0,T],\ t_1 < t_2$ such that $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}
|F_1\eta(t_2) - F_1\eta(t_1)| &= |\mathcal{F}_1(t_2, \xi(t_2), \xi(\gamma_1\eta(t_2))) - \mathcal{F}_1(t_1, \xi(t_1), \xi(\gamma_1\eta(t_1)))| \\
&\leqslant K_1|t_2-t_1| + K_2|\xi(t_2) - \xi(t_1)| + K_2|\xi(\gamma_1\eta(t_2)) - \xi(\gamma_1\eta(t_1))| \\
&\leqslant K_1|t_2-t_1| + K_2L|t_2-t_1| + K_2L\gamma_1|y((t_2)) - \eta(t_1)| \\
&\leqslant K_1|t_2-t_1| + K_2L|t_2-t_1| + K_2L^2\gamma_1|t_2-t_1| \\
&\leqslant (K_1 + K_2L + K_2L^2\gamma_1)|t_2-t_1| \leqslant L|t_2-t_1|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\mathcal{F}_2\xi(t_2) - \mathcal{F}_2\xi(t_1)| &\leqslant |\mathcal{F}_2(t_2, \eta(t_2), \eta(\gamma_2\xi(t_2))) - \mathcal{F}_2(t_1, \eta(t_1), \eta(\gamma_2\xi(t_1)))| \\
&\leqslant (K_1 + K_2L + K_2L^2\gamma_2)|t_2-t_1| \leqslant L|t_2-t_1|.
\end{aligned}$$

Then $FU = (\mathcal{F}_1\eta, \mathcal{F}_2\xi) \in S_L$ and this prove that $\mathcal{F}: S_L \to S_L$. Now, let $U = (\xi,\eta) \in X$, $V = (\bar{\xi},\bar{\eta}) \in X$, then

$$\mathcal{F}(\xi\eta) = (\mathcal{F}_1\eta, \mathcal{F}_2\xi), \quad \mathcal{F}(\bar{\xi},\bar{\eta}) = (F_1\bar{\eta}, \mathcal{F}_2\bar{\xi}),$$

and

$$\begin{aligned}
|F_1\eta(t) - F_1\bar{\eta}(t)| &= |\mathcal{F}_1(t, \xi(t), \xi(\gamma_1\eta(t))) - \mathcal{F}_1(t, \bar{\xi}(t), \bar{\xi}(\gamma_1\bar{\eta}(t)))| \\
&\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma_1\eta(t)) - \bar{\xi}(\gamma_1\bar{\eta}(t))| \\
&\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma_1\eta(t)) - \xi(\gamma_1\bar{\eta}(t)) + \xi(\gamma_1\bar{\eta}(t)) + \bar{\xi}(\gamma_1\bar{\eta}(t))| \\
&\leqslant K_2|\xi(t) - \bar{\xi}(t)| + K_2|\xi(\gamma_1\eta(t)) - \xi(\gamma_1\bar{\eta}(t))| + |\xi(\gamma_1\bar{\eta}(t)) + \bar{\xi}(\gamma_1\bar{\eta}(t))| \\
&\leqslant K_2\|\xi - \bar{\xi}\| + K_2L|\gamma_1\eta(t) - \gamma_1\bar{\eta}(t)| + K_2\|\xi - \bar{\xi}\| \\
&\leqslant K_2\|\xi - \bar{\xi}\| + K_2L\gamma_1\|\eta - \bar{\eta}\| + K_2\|\xi - \bar{\xi}\| \\
&\leqslant 2K_2\|\xi - \bar{\xi}\| + K_2L\gamma_1\|\eta - \bar{\eta}\|,
\end{aligned}$$

and

$$\|F_1\eta - F_1\bar{\eta}\| \leqslant 2K_2\|\xi - \bar{\xi}\| + K_2L\gamma_1\|\eta - \bar{\eta}\|.$$

Similarly, we can prove that

$$\|\mathcal{F}_2\xi - \mathcal{F}_2\bar{\xi}\| \leqslant 2K\|\eta - \bar{\eta}\| + K_2L\gamma_2\|\xi - \bar{\xi}\|.$$

Hence

$$\begin{aligned}
\|\mathcal{F}(\xi,\eta) - \mathcal{F}(\bar{\xi},\bar{\eta})\|_X &= \|(\mathcal{F}_1\eta, \mathcal{F}_2\xi) - (F_1\bar{\eta}, \mathcal{F}_2\bar{\xi})\|_X \\
&= \|\mathcal{F}_1(\eta - \mathcal{F}_1\bar{\eta}, \mathcal{F}_2\xi - \mathcal{F}_2\bar{\xi})\|_X \\
&= \|\mathcal{F}_1\eta - \mathcal{F}_1\bar{\eta}\|_C + |\mathcal{F}_2\eta - \mathcal{F}_2\bar{\xi}\|_C \\
&\leqslant 2K_2\|\xi - \bar{\xi}\| + K_2L\gamma_1\|\eta - \bar{\eta}\| + 2K_2\|\eta - \bar{\eta}\| + K_2L\gamma_2\|\xi - \bar{\xi}\| \\
&\leqslant (2K_2 + K_2L\gamma)(\|\xi - \bar{\xi}\| + \|\eta - \bar{\eta}\|) \\
&\leqslant (2K_2 + K_2L\gamma)(\|\xi - \bar{\xi}\| + \|\eta - \bar{\eta}\|) \\
&\leqslant (2K_2 + K_2L\gamma)\|(\xi,\eta) - (\bar{\xi} - \bar{\eta})\|.
\end{aligned}$$

Since $(2K_2 + K_2L\gamma) < 1$, then $\mathcal{F}$ is a contraction mapping and by the Banach fixed point Theorem [13], $\mathcal{F}$ has a unique solution, consequently the constrained problem (1.1)-(1.2) has a unique solution $(\xi,\eta) \in X$. □

Corollary 3.3. *If we put $\xi = \eta$ and $\gamma_1 = \gamma_2$, then we deduce Theorem 2.1.*

3.1. Hyres-Ulam stability

Definition 3.4 ([7, 8])**.** Let the solution $(\xi,\eta) \in X$ of (1.1)-(1.2) exists, then the problem (1.1)-(1.2) is Hyers-Ulam stable if $\forall\epsilon > 0$, $\exists\delta(\epsilon) > 0$ such that for any δ-approximate solution $(\xi_s,\eta_s) \in X$ of (1.1)-(1.2) satisfies

$$\max\{|\xi_s(t) - \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1(\eta_s(t))))|,\ |\eta_s(t) - \mathcal{F}_2(t,\eta_s(t),\eta_s(\gamma\xi_s(t)))|\} < \delta,$$

implies

$$\|(\xi,\eta) - (\xi_s,\eta_s)\|_X < \epsilon.$$

Theorem 3.5. *Assume that the assumptions of Theorem 3.2 be valid, then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof. Let

$$\max\{|\xi_s(t) - \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t)))|,\ |\eta_s(t) - \mathcal{F}_2(t,\eta_s(t),\eta_s(\gamma_2\xi_s(t)))|\} < \delta,$$

then

$$\begin{aligned}
|\xi(t) - \xi_s(t)| &= |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))) - \xi_s(t)| \\
&= |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))) - \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t))) + \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t))) - \xi_s(t)| \\
&\leqslant |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))) - \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t)))| + |\mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t))) - \xi_s(t)| \\
&\leqslant |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))) - \mathcal{F}_1(t,\xi_s(t),\xi_s(\gamma_1\eta_s(t)))| + \delta \\
&\leqslant K_2|\xi(t) - \xi_s(t)| + K_2|\xi(\gamma_1\eta_s(t)) - \xi_s(\gamma_1\eta(t))| + \delta \\
&\leqslant K_2\|\xi - \xi_s\| + K_2|\xi(\gamma_1\eta_s(t)) - \xi_s(\gamma_1\eta_s(t)) + \xi_s(\gamma_1\eta_s(t)) - \xi(\gamma_1\eta(t))| + \delta \\
&\leqslant K_2\|\xi - \xi_s\| + K_2|\xi(\gamma_1\eta_s(t)) - \xi_s(\gamma_1\eta_s(t))| + K_2|\xi_s(\gamma_1\eta_s(t)) - \xi(\gamma_1\eta(t))| + \delta \\
&\leqslant K_2\|\xi - \xi_s\| + K_2\|\xi - \xi_s\| + K_2L\gamma_1|\eta_s(t) - \eta(t)| + \delta \\
&\leqslant 2K_2\|\xi - \xi_s\| + K_2L\gamma_1\|\eta - \eta_s\| + \delta,
\end{aligned}$$

and

$$(1 - 2K_2)\|\xi - \xi_s\| \leqslant \delta + K_2L\gamma_1\|\eta - \eta_s\|.$$

Hence

$$\|\xi-\xi_s\| \leqslant \frac{\delta}{1-2K_2} + \frac{K_2L\gamma_1}{1-2K_2}\|\eta-\eta_s\|. \tag{3.1}$$

Similarly,

$$\|\eta-\eta_s\| \leqslant \frac{\delta}{1-2K_2} + \frac{K_2L\gamma_2}{1-2K_2}\|\xi-\xi_s\|. \tag{3.2}$$

By addition of (3.1) and (3.2), we obtain

$$\begin{aligned}(\|\xi-\xi_s\|+\|\eta-\eta_s\|) &\leqslant \frac{2\delta}{1-2K} + \frac{KL\gamma_1}{1-2K}\|\eta-\eta_s\| + \frac{KL\gamma_2}{1-2K}\|\xi-\xi_s\| \\ &\leqslant \frac{2\delta}{1-2K_2} + \frac{K_2L\gamma}{1-2K_2}(\|\xi-\xi_s\|+\|\eta-\eta_s\|)\end{aligned}$$

and

$$(1-\frac{K_2L\gamma}{1-2K_2})(\|\xi-\xi_s\|+\|\eta-\eta_s\|) \leqslant \frac{2\delta}{1-2K_2}.$$

Hence

$$(\|\xi-\xi_s\|+\|\eta-\eta_s\|) \leqslant \frac{2\delta}{1-(2K_2+K_2L\gamma)} = \epsilon.$$

Then

$$\|(\xi,\eta)-(\xi_s,\eta_s)\|_X = \|(\xi-\xi_s),(\eta-\eta_s)\|_X = \|(\xi-\xi_s)\|_C + \|(\eta-\eta_s)\|_C \leqslant \epsilon.$$

□

Corollary 3.6. *If we put $\xi=\eta$ and $\gamma_1=\gamma_2$, then we deduce Theorem 2.3.*

3.2. Continuous dependence

Definition 3.7. The solution $(\xi,\eta)\in X$ of (1.1)-(1.2) depends continuously on the functions $\mathcal{F}_i$ and γ_i if $\forall\epsilon>0,\ \exists\delta>0$ such that

$$\max\{|\mathcal{F}_i-\mathcal{F}_i^*|,\ |\gamma_i-\gamma_i^*|\}<\delta \quad\Rightarrow\quad \|(\xi,\eta)-(\xi^*,\eta^*)\|_X<\epsilon,$$

where

$$\xi^*(t)=\mathcal{F}_1^*(t,\xi^*(t),\xi^*(\gamma_1^*(\eta^*(t)))),\quad \eta^*(t)=\mathcal{F}_2^*(t,\eta^*(t),\eta^*(\gamma_2^*(\xi^*(t)))).$$

Theorem 3.8. *Let the assumptions of Theorem 3.2 be met, then $(\xi,\eta)\in X$ depends continuously on $\mathcal{F}_i$ and γ_i.*

Proof. Let (ξ,η) and (ξ^*,η^*) be two solutions of the problem (1.1)-(1.2), then

$$\begin{aligned}|\xi(t)-\xi^*(t)| &= |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t)))-\mathcal{F}_1^*(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t)))| \\ &= |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1(\eta(t))))-\mathcal{F}_1(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t)))+\mathcal{F}_1(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t))) \\ &\quad -\mathcal{F}_1^*(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t)))| \\ &\leqslant |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t)))-\mathcal{F}(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t)))|+|\mathcal{F}_1(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t))) \\ &\quad -\mathcal{F}_1^*(t,\xi^*(t),\xi^*(\gamma_1^*\eta^*(t)))| \\ &\leqslant K_2|\xi(t)-\xi^*(t)|+K_2|\xi(\gamma_1\eta(t))-\xi^*(\gamma_1^*\eta^*(t))|+|\mathcal{F}-\mathcal{F}^*| \\ &\leqslant K_2\|\xi-\xi^*\|+K_2|\xi(\gamma_1\eta(t))-\xi(\gamma_1^*\eta^*(t))+\xi(\gamma_1^*\eta^*(t))-\xi^*(\gamma_1^*\eta^*(t))|+\delta \\ &\leqslant K_2\|\xi-\xi^*\|+K_2|\xi(\gamma_1\eta(t))-\xi(\gamma_1^*\eta^*(t))|+K_2|\xi(\gamma_1^*\eta^*(t))-\xi^*(\gamma_1^*\eta^*(t))|+\delta \\ &\leqslant K_2\|\xi-\xi^*\|+K_2L|\gamma_1\eta(t)-\gamma_1^*\eta^*(t)|+K_2\|\xi-\xi^*\|+\delta\end{aligned}$$

$$
\begin{aligned}
&\leqslant K_2\|\xi-\xi^*\| + K_2L|\gamma_1\eta(t)-\gamma_1\eta^*(t)+\gamma_1\eta^*(t)-\gamma_1^*\eta^*(t)| + K_2\|\xi-\xi^*\|+\delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L|\gamma_1\eta(t)-\gamma_1\eta^*(t)| + K_2L|\gamma_1\eta^*(t)-\gamma_1^*\eta^*(t)| + K_2\|\xi-\xi^*\|+\delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma_1\|\eta-\eta^*\| + K_2L\|\eta^*\||\gamma_1-\gamma_1^*| + K_2\|\xi-\xi^*\|+\delta\\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma_1\|\eta-\eta^*\| + K_2L\|\eta^*\|\delta + K_2\|\xi-\xi^*\|+\delta\\
&\leqslant 2K_2\|\xi-\xi^*\| + K_2L\gamma_1\|\eta-\eta^*\| + (1+K_2L\|\eta^*\|)\delta.
\end{aligned}
$$

Thus

$$(1-2K_2)\|\xi-\xi^*\| \leqslant K_2L\gamma_1\|\eta-\eta^*\| + (1+K_2L\|\eta^*\|)\delta,$$

and

$$\|\xi-\xi^*\| \leqslant \frac{K_2L\gamma_1}{1-2K_2}\|\eta-\eta^*\| + \frac{(1+K_2L\|\eta^*\|)}{1-2K_2}\delta. \tag{3.3}$$

Also

$$
\begin{aligned}
|\eta(t)-\eta^*(t)| &= |\mathcal{F}_2(t,\eta(t),\eta(\gamma_2\xi(t))) - \mathcal{F}_2^*(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))|\\
&= |\mathcal{F}_2(t,\eta(t),\eta(\gamma_2\xi(t))) - \mathcal{F}_2(t,\eta^*(t),\eta^*(\gamma_2\xi^*(t)) + \mathcal{F}_2(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))\\
&\quad - \mathcal{F}_2^*(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))|\\
&\leqslant |\mathcal{F}_2(t,\eta(t),\eta(\gamma_2\xi(t))) - \mathcal{F}_2(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))| + |\mathcal{F}_2(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))\\
&\quad - \mathcal{F}_2^*(t,\eta^*(t),\eta^*(\gamma_2^*\xi^*(t)))|\\
&\leqslant K_2|\eta(t)-\eta^*(t)| + K_2|\eta(\gamma_2\xi(t)) - \eta^*(\gamma_2^*\xi^*(t))| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2|\eta(\gamma_2\xi(t)) - \eta(\gamma_2^*\xi^*(t)) + \eta(\gamma_2^*\xi^*(t)) - \eta^*(\gamma_2^*\xi^*(t))| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2|\eta(\gamma_2\xi(t)) - \eta(\gamma_2^*\xi^*(t))| + K_2|\eta(\gamma_2^*\xi^*(t)) - \eta^*(\gamma_2^*\xi^*(t))| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2L|\gamma_2\xi(t) - \gamma_2^*\xi^*(t)| + K_2\|\eta-\eta^*\| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2L|\gamma_2\xi(t) - \gamma_2\xi^*(t) + \gamma_2\xi^*(t) - \gamma_2^*\xi^*(t)| + K\|\eta-\eta^*\| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2L|\gamma_2\xi(t) - \gamma_2\xi^*(t) + K_2L|\gamma_2\xi^*(t) - \gamma_2^*\xi^*(t)| + K_2\|\eta-\eta^*\| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2L\gamma_2\|\xi-\xi^*\| + K_2L\|\xi^*\||\gamma_2-\gamma_2^*| + K_2\|\eta-\eta^*\| + \delta\\
&\leqslant K_2\|\eta-\eta^*\| + K_2L\gamma_2\|\xi-\xi^*\| + K_2L\|\xi^*\|\delta + K_2\|\eta-\eta^*\| + \delta\\
&\leqslant 2K_2\|\eta-\eta^*\| + K_2L\gamma_2\|\xi-\xi^*\| + (1+K_2L\|\xi^*\|)\delta,
\end{aligned}
$$

thus

$$(1-2K_2)\|\eta-\eta^*\| \leqslant K_2L\gamma_2\|\xi-\xi^*\| + (1+K_2L\|\xi^*\|)\delta,$$

and

$$\|\eta-\eta^*\| \leqslant \frac{K_2L\gamma_2}{1-2K_2}\|\xi-\xi^*\| + \frac{(1+K_2L\|\xi^*\|)}{1-2K_2}\delta. \tag{3.4}$$

By addition of (3.3) and (3.4), we obtain

$$
\begin{aligned}
(\|\xi-\xi^*\| + \|\eta-\eta^*\|) &\leqslant \frac{K_2L\gamma_1}{1-2K_2}\|\xi-\xi^*\| + \frac{K_2L\gamma_2}{1-2K_2}\|\eta-\eta^*\| + \frac{1+K_2L\|\eta^*\|}{1-2K_2}\delta + \frac{(1+K_2L\|\xi^*\|)}{1-2K_2}\delta\\
&\leqslant \frac{K_2L\gamma}{1-2K_2}(\|\xi-\xi^*\| + \|\eta-\eta^*\|) + \frac{2+K_2L(\|\xi^*\|+\|\eta^*\|)}{1-2K_2}\delta,
\end{aligned}
$$

and

$$(1-\frac{K_2L\gamma}{1-2K_2})(\|\xi-\xi^*\| + \|\eta-\eta^*\|) \leqslant \frac{2+K_2L(\|\xi^*\|+\|\eta^*\|)}{1-2K_2}\delta.$$

Hence

$$(\|\xi-\xi^*\| + \|\eta-\eta^*\|) \leqslant \frac{2+K_2L(\|\xi^*\|+\|\eta^*\|)}{1-(2K_2+K_2L\gamma)}\delta = \epsilon.$$

Then

$$\|(\xi,\eta)-(\xi^*,\eta^*)\|_X = \|(\xi-\xi^*),(\eta-\eta^*)\|_X = \|(\xi-\xi^*)\|_C + \|\eta-\eta^*)\|_C \leqslant \epsilon.$$

□

Corollary 3.9. *If we put* $\xi = \eta$ *and* $\gamma_1 = \gamma_2$, *then we deduce Theorem 2.5.*

Definition 3.10. The solution $\xi \in C[0,T]$ of (1.1) depends continuously on η if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$|\eta - \eta^*| < \delta \ \Rightarrow \ \|\xi - \xi^*\| < \epsilon,$$

where

$$\xi^*(t) = \mathcal{F}_1(t, \xi^*(t), \xi^*(\gamma_1\eta^*(t))), \quad \eta^*(t) = \mathcal{F}_2(t, \eta^*(t), \eta^*(\gamma_2\xi^*(t))).$$

Theorem 3.11. *Assume that the assumptions of Theorem 3.2 be valid, then* $\xi \in C[0,T]$ *depends continuously on* η.

Proof. Let ξ and ξ^* be two solutions of (1.1), then

$$\begin{aligned}
|\xi(t)-\xi^*(t)| &= |\mathcal{F}_1(t,\xi(t),\xi(\gamma_1\eta(t))) - \mathcal{F}_1(t,\xi^*(t),\xi^*(\gamma_2\eta^*(t)))| \\
&\leqslant K_2|\xi(t)-\xi^*(t)| + K_2|\xi(\gamma_1\eta(t)) - \xi^*(\gamma_1\eta^*(t))| \\
&\leqslant K_2\|\xi-\xi^*\| + K_2|\xi(\gamma_1\eta(t)) - \xi(\gamma_1\eta^*(t)) + \xi(\gamma_1\eta^*(t)) - \xi^*(\gamma_1\eta^*(t))| \\
&\leqslant K_2\|\xi-\xi^*\| + K_2|\xi(\gamma_1\eta(t)) - \xi(\gamma_1\eta^*(t))| + K|\xi(\gamma_1\eta^*(t)) - \xi^*(\gamma_1\eta^*(t))| \\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma_1|\eta(t)-\eta^*(t)| + K_2|\xi(\gamma_1\eta^*(t)) - \xi^*(\gamma_1\eta^*(t))| \\
&\leqslant K_2\|\xi-\xi^*\| + K_2L\gamma_1\delta + K_2\|\xi-\xi^*\| \leqslant 2K_2\|\xi-\xi^*\| + K_2L\gamma_1\delta,
\end{aligned}$$

thus

$$(1-2K_2)\|\xi-\xi^*\| \leqslant K_2L\gamma_1\delta,$$

then

$$\|\xi-\xi^*\| \leqslant \frac{K_2L\gamma_1}{1-2K_2}\delta = \epsilon.$$

□

Definition 3.12. The solution $\eta \in C[0,T]$ of (1.2) depends continuously on ξ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$|\xi - \xi^*| < \delta \Rightarrow \|\eta - \eta^*\| < \epsilon,$$

where

$$\xi^*(t) = \mathcal{F}_1(t, \xi^*(t), \xi^*(\gamma_1\eta^*(t))), \quad \eta^*(t) = \mathcal{F}_2(t, \eta^*(t), \eta^*(\gamma_2\xi^*(t))).$$

Theorem 3.13. *Let the assumptions of Theorem 3.2 be satisfied, then* $\eta \in C[0,T]$ *depends continuously on* ξ.

Proof. Let η and η^* be two solutions of (1.2), then

$$\begin{aligned}
|\eta(t)-\eta^*(t)| &= \mathcal{F}_2(t,\eta(t),\eta(\gamma_2\xi(t))) - \mathcal{F}_2(t,\eta^*(t),\eta^*\gamma_2(\xi^*(t))) \\
&\leqslant K_2|\eta(t)-\eta^*(t)| + K_2|\eta(\gamma_2\xi(t)) - \eta^*(\gamma_2\xi^*(t))| \\
&\leqslant K_2\|\eta-\eta^*\| + K_2|\eta(\gamma_2\xi(t)) - \eta(\gamma_2\xi^*(t)) + \eta(\gamma_2\xi^*(t)) - \eta^*(\gamma_2\xi^*(t))| \\
&\leqslant K_2\|\eta-\eta^*\| + K_2|\eta(\gamma_2\xi(t)) - \eta(\gamma_2\xi^*(t))| + K_2|\eta(\gamma_2\xi^*(t)) - \eta^*(\gamma_2\xi^*(t))| \\
&\leqslant K_2\|\eta-\eta^*\| + K_2L\gamma_2|\xi(t)-\xi^*(t)| + K_2|\eta(\gamma_2\xi^*(t) - \eta^*(\gamma_2\xi^*(t)| \\
&\leqslant K_2\|\eta-\eta^*\| + K_2L\gamma_2\delta + K_2\|\eta-\eta^*\| \leqslant 2K_2\|\eta-\eta^*\| + K_2L\gamma_2\delta,
\end{aligned}$$

thus

$$(1-2K_2)\|\eta-\eta^*\| \leqslant K_2L\gamma_2\delta,$$

then

$$\|\eta-\eta^*\| \leqslant \frac{K_2 L\gamma_2}{1-2K_2}\delta = \epsilon.$$

□

4. Examples

Example 4.1. Consider the problem

$$\xi(t) = \frac{1}{5}t + \frac{1}{10}(\xi(t) + \xi(0.8\xi(t))),$$

where $\gamma = 0.8$, $K_1 = \frac{1}{5}$, $K_2 = \frac{1}{10}$. Thus we have

$$L = \frac{(1-k_1) \pm \sqrt{(1-K_2)^2 - 4K_1K_2\gamma}}{2K_2\gamma} = 0.02225 < 1,$$

and $2K_2 + K_2L\gamma = 0.20178 < 1$. It is evident that the conditions of Theorem 2.1 are satisfied. Therefore, the equation (1.3) has a unique solution $\xi \in C[0, T]$.

Example 4.2. Consider the problem

$$\xi(t) = \frac{1}{9}t + \frac{1}{6}(\xi(t) + \xi(0.3\eta(t))), \quad \eta(t) = \frac{1}{8}t + \frac{1}{7}(\eta(t) + \eta(0.6\xi(t))), \tag{4.1}$$

where $\gamma = \max\{\gamma_1, \gamma_2\} = 0.3$, and $K_1 = \max\{\frac{1}{9}, \frac{1}{8}\} = \frac{1}{8}$ and $K_2 = \max\{\frac{1}{6}, \frac{1}{7}\} = \frac{1}{6}$. Thus we have

$$L = \frac{(1-k_1) \pm \sqrt{(1-K_2)^2 - 4K_1K_2\gamma}}{2K_2\gamma} = 0.145 < 1,$$

and $2K_2 + K_2L\gamma = 0.3406 < 1$. It is evident that the conditions of Theorem 3.2 are satisfied. Hence there exist unique solution $\xi \in C[0, T]$ of the problem (4.1).

5. Conclusion

In this study, we defined the state-dependent pantograph functional equation (1.3), and we examined the existence of a unique solution of (1.3). Furthermore, we proved the problem's Hyers-Ulam stability. We demonstrated that the unique solution of (1.3) is continuously dependent on the function $\mathcal{F}$, and γ. Then we studied a constrained problem of the state-dependent pantograph functional equation (1.1) constrained by its conjegate equation (1.2). Moreover, we analyzed the sufficient criteria for the solution's existence and uniqueness. We studied The Hyres-Ulam stability of the problem. Furthermore, we proved the continuous dependence of the solution on the function $\mathcal{F}_i$ and γ_i and the continuous dependence of ξ on η and η on ξ are studied. Finally, we introduced some examples to illustrate our results.

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