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Some important results for the conformable fractional stochastic pantograph differential equations in the L^p space

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Abstract

Important mathematical topics include existence, uniqueness, continuous dependency, regularity, and the averaging principle. In this research work, we establish these results for the conformable fractional stochastic pantograph differential equations (CFSPDEs) in $\mathbf{L}^p$ space. The situation of $p = 2$ is generalized by the obtained findings. First, we establish the existence and uniqueness results by applying the contraction mapping principle under a suitably weighted norm and demonstrating the continuous dependency of solutions on both the initial values and fractional exponent ϕ. The second section is devoted to examining the regularity of time. As a result, we find that, for each $\Phi \in (0, \phi - \frac{1}{2})$, the solution to the considered problem has a Φ-Hölder continuous version. Next, we study the averaging principle by using Jensen's, Grönwall-Bellman's, Hölder's, and Burkholder-Davis-Gundy's inequalities. To help with the understanding of the theoretical results, we provide three applied examples at the end.

Keywords: Pantograph problem, existence and uniqueness, continuous dependency, regularity, averaging principle, conformable fractional derivative.

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1. Introduction

Fractional calculus (FC) is a branch of mathematics that studies the extension of derivatives and integrals to any arbitrary order, real or complex. In 1695, not too long after the development of classical calculus, FC emerged. FC has long been thought of as a field of pure mathematics with no practical uses. But things have altered in this regard in the last few decades. FC is especially helpful in explaining the behavior of intricate physical systems. Numerous natural phenomena have nonlocal properties, which means that events from both their recent and distant pasts influence them in the present. Fractional operators are a more precise way to describe these nonlocal relationships than standard integer-order derivatives. This

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is due to two core reasons: first, we have the choice to select any order for the fractional order derivative (FrOD) and not be bound to an integer order only. Secondly, non-integer-order derivatives do not rely only on local conditions but also on the past for support when the system has a long-term memory. The FrOD is a superior way to represent many real-world phenomena than integer-order calculus.

Different from integer-order derivatives, there are several kinds of definitions for FrOD [3, 25, 26]; some of them are Grunuwald Letnikov, Riemann-Liouville (R-L), Caputo-Fabrizio, Atangana-Baleanu, Caputo, and conformable. These definitions are generally not equivalent to each other. Khalil et al. [21] developed the conformable fractional derivative (CFrD), which is a distinctive description of the FrOD compared to earlier versions. For a mapping $\aleph(\tau) : [0, \infty[\to \mathbb{R}$, the CFrD of order ϕ is specified via:

$$\mathcal{T}_{\tau}^{\phi}\aleph(\tau) = \lim_{\epsilon\to 0} \frac{\aleph^{\lceil\phi\rceil-1}(\tau + \epsilon\tau^{\lceil\phi\rceil-\phi}) - \aleph^{\lceil\phi\rceil-1}(\tau)}{\epsilon}, \tag{1.1}$$

$u - 1 < \phi \leqslant u$, $\tau > 0$, $u \in \mathbb{N}$, and $\lceil\phi\rceil$, the lowest integer larger than or equivalent to ϕ. In a particular case, if $0 < \phi \leqslant 1$, next, we get

$$\mathcal{T}_{\tau}^{\phi}\aleph(\tau) = \lim_{\epsilon\to 0} \frac{\aleph(\tau + \epsilon\tau^{1-\phi}) - \aleph(\tau)}{\epsilon},\ \tau > 0.$$

If $\aleph(\tau)$ is ϕ-differentiable in some $(0, \mathbb{T})$, $\mathbb{T} > 0$ and $\lim_{\tau\to 0^+} \aleph^{\phi}(\tau)$ exists, then define $\aleph^{(\phi)}(0) = \lim_{\tau\to 0^+} \aleph^{(\phi)}(\tau)$. The conformable fractional integral of a function $\aleph(\tau)$ starting from $\tilde{\alpha} \geqslant 0$ is defined as:

$$\mathcal{I}_{\phi}^{\tilde{\alpha}}(\aleph)(\tau) = \int_{\tilde{\alpha}}^{\tau} \frac{\aleph(\ell)}{(\ell - \tilde{\alpha})^{1-\phi}} d\ell,\ \phi \in (0,\ 1].$$

Variables and their interrelationships are represented mathematically to forecast future events or explain the observed system. In mathematical terms, it offers details on the intricate phenomenon. It facilitates comprehension of the complex dynamical system's behavior. In addition, a variety of scientific, technical, and social science fields employ mathematical models. There are occurrences across various disciplines that, when modeled mathematically, are found to be differential equations (DEs). The DEs with non-integer-order derivatives are called fractional-order differential equations (FODEs). The FODEs can model and analyze complex structures with complex non-linear processes and higher-order behaviors, making them sometimes a better choice for modeling than integer-order DEs. FODEs have been the subject of much interest and attention in recent decades because of their applications in science and engineering, including viscoelasticity, control, electrochemistry, star clusters, and stellar interiors [11, 22, 38].

On the other hand, the behavior of a system that is affected by random fluctuations is represented by mathematical models called fractional stochastic differential equations (FSDEs). The FSDEs combine the concepts of FC and stochastic processes. The FSDEs are a particular kind of DEs in which there are stochastic and deterministic factors that affect the dynamics of the system. Wiener processes, also known as Brownian motions, are continuous-time stochastic processes that move unpredictably and randomly. These processes are typically used to simulate the stochastic component. Kiyoshi Ito first presented FSDEs in 1940 as a method of simulating particle diffusion in a fluid; since then, they have been widely applied in this and numerous other fields [30]. The ability of FSDEs to capture the impacts of noise and unpredictability on a system, which is frequently crucial in real-world applications, is one of the factors contributing to its popularity. Numerous disciplines, including physics, chemistry, biology, finance, and engineering, can benefit from their use [2, 5, 27, 43].

Several writers have recently been actively researching the FSDEs. Among these, exponential stability in the mean square of delay FSDEs is established by Li and Xu [24]. The new standards are derived for the exponential stability of the mean square of the examined problems in this article. A few instances are examined to clarify the theory. Li and Peng [23] used the fixed point theory of Sadovskii$'$ to obtain the controllability of a class of FSDEs. Cui and Yan [9] used the same fixed-point theorem to draw some findings about the existence of mild solutions for fractional stochastic integrodifferential equations with delay in Hilbert spaces. In [33], Niu and Xie looked at the regularity, uniqueness, and existence of the solutions for a certain class of one-dimensional FSDEs with white noise. Using the Schauder fixed point theorem, Chen and Li [7] demonstrated the existence of global mild solutions as well as saturated mild solutions. The authors of the works [6, 8] demonstrated the existence of FSDEs solutions under various hypotheses. In [20], A. Karczewska and C. Lizama presented several findings on the perturbation of the stochastic Volterra equations in addition to discussing the presence of mild, weak, and strong solutions of FSDEs. In [36], R. Schnaubelt and M. Veraar demonstrate the path-wise continuity features of solutions to a stochastic Volterra equation with a local martingale-provided additive noise factor. Xiaoa and Wang [40] use the stopping time technique to investigate the stability of FSDEs of the Caputo type.

Dynamic systems may rely on previous states in addition to their current ones. The CFSPDEs are widely employed to simulate these events, whose systems depend on the state $\omega(\zeta\tau)$, where $0 < \zeta < 1$. In particular, the CFSPDEs have far more practical applications in biology, economics, the sciences, engineering, control, and electrodynamics.

One of the key topics in mathematics is the existence and uniqueness of solutions to FSDEs. It is fundamental to understand whether a certain FSDE has a unique solution and if it does. The regularity of a solution to a FSDEs refers to the smoothness of the solution. Alternatively, it characterizes the degree of behavior of the solution. The averaging principle is a helpful method for simplifying both stochastic and deterministic systems. The basis for averaging techniques in mathematics, engineering mechanics, control, and other complicated issues is the averaging principle, a vital and fundamental approximation theory. It is an approximation principle that has some ability to balance both complex and basic systems. The averaging principle is based on the proof of an approximation theorem for FSDEs, which functions as a kind of substitute for the original system, and the subsequent optimal order convergence theorem. Some scholars have recently become interested in the FSDEs' averaging concept. For example, Luo et al. [31] established an averaging principle for a specific set of FSDEs with time delays in the $\mathbf{L}^2$ space, based on innovative assumptions. Similarly, Xu and collaborators demonstrated the averaging principle within the $\mathbf{L}^2$ space for FSDEs featuring Caputo derivatives driven by Brownian motion [42]. In another study [37], researchers investigated the averaging principle for SDEs with Poisson noises, while Xu examined the concept for SDEs driven by L'evy noise [41]. Furthermore, in [1], the authors explored the approximation theorem as an averaging approach for the solutions of Itô-Doob-type FSDEs characterized by non-Lipschitz coefficients in terms of probability and mean square. For more details about the averaging approach, see [12, 13].

Inequalities are fundamental tools for analyzing various important concepts in stochastic differential problems. We utilize Grönwall-Bellman's inequality (Grön-Bell-Ineq), Jensen's inequality (Jen-Ineq), Burkholder-Davis-Gundy's (BHDG-Ineq), and Hölder's inequality (Höld-Ineq). Each of the mentioned inequalities plays a crucial role in understanding different aspects of FSDEs. We explore the importance of each inequality in the context of FSDEs.

Grön-Bell-Ineq is a fundamental tool in the analysis of DEs, providing bounds on the solutions and aiding in establishing existence and uniqueness. In the context of FSDEs, which involve fractional operators and stochastic processes, the application of Grön-Bell-Ineq enhances our understanding in several ways [29, 35].

i. Existence of Solutions: Grön-Bell-Ineq helps establish conditions under which solutions to FSDEs exist. By providing bounds on the solutions, it ensures that solutions remain within certain limits, even in the presence of FrOD and stochastic noise. This is crucial for ensuring that solutions do not diverge or become undefined.

ii. Uniqueness of Solutions: Grön-Bell-Ineq aids in proving the uniqueness of solutions to FSDEs. It provides a tool for comparing different solutions and showing that they cannot deviate too much from each other. This is essential for demonstrating that a given FSDE has a unique solution under certain conditions, despite the presence of randomness and nonlinearity.

Incorporating Jen-Ineq into the study of continuous dependency in FSDEs can lead to several advancements.

i. Stability Analysis: Jen-Ineq can be used to establish stability criteria for solutions to FSDEs. By providing bounds on convex functions of stochastic processes, Jen-Ineq helps in controlling the growth of solutions. This can be crucial for understanding the continuous dependence of solutions on initial conditions and parameters. Advancements in stability analysis can lead to better predictions of the long-term behavior of systems described by FSDEs.
ii. Quantitative Estimates: Jen-Ineq provides quantitative estimates on the growth of solutions to FSDEs. By bounding convex functions of stochastic processes, it allows researchers to quantify how solutions evolve over time and how they are influenced by various parameters and initial conditions. Incorporating Jen-Ineq into the study of continuous dependency can lead to more precise estimates of how solutions change with perturbations in the system.
iii. Nonlinear Dynamics: Many FSDEs exhibit nonlinear dynamics, where the drift and diffusion coefficients depend nonlinearly on the state variables and stochastic processes. Jen-Ineq can help in analyzing the impact of nonlinearity on the continuous dependency of solutions. By providing bounds on convex functions, it aids in understanding how nonlinearities affect the stability and behavior of solutions over time.

Overall, incorporating Jen-Ineq into the study of continuous dependency in FSDEs can lead to advancements in stability analysis, quantitative estimation, and understanding of nonlinear dynamics. These advancements can deepen our understanding of how solutions to FSDEs depend continuously on initial conditions and parameters, leading to improved predictions and control of complex systems described by FSDEs.

The BHDG-Ineq can offer valuable insights into the behavior of solutions to FSDEs under varying conditions. Here are several ways in which it can provide new insights [4, 18].

i. Estimation of Stochastic Integrals: BHDG-Ineq provides bounds on stochastic integrals involving fractional Brownian motion. These integrals often appear in the drift and diffusion terms of FSDEs. By estimating these integrals, researchers can gain insights into the overall behavior of the solutions to FSDEs, particularly how they are affected by the underlying stochastic processes.
ii. Control of Solution Growth: In FSDEs, the driving noise is typically represented by a fractional Brownian motion, which can exhibit long-range dependence and roughness. BHDG-Ineq helps in controlling the growth of solutions by providing bounds on the stochastic integrals involving fractional Brownian motion. This control of solution growth is crucial for understanding how solutions evolve over time and under varying conditions.
iii. Stability Analysis: The BHDG-Ineq can be used to analyze the stability of solutions to FSDEs. By bounding the growth of stochastic integrals, BHDG-Ineq aids in determining whether solutions remain bounded or converge to certain equilibrium states over time. This is essential for understanding the long-term behavior of systems described by FSDEs.
iv. Quantitative Estimates: BHDG-Ineq provides quantitative estimates of the growth of stochastic integrals, which are essential for predicting the behavior of solutions to FSDEs. These estimates help in understanding how solutions evolve over time and how they are influenced by various parameters and initial conditions.
v. Comparison with Classical SDEs: BHDG-Ineq can also be used to compare solutions to FSDEs with solutions to classical SDEs. By bounding the growth of stochastic integrals, researchers can assess how the presence of fractional Brownian motion in FSDEs affects the behavior of solutions compared to traditional SDEs. This comparison can lead to new insights into the unique characteristics of FSDEs.

The application of Höld-Ineq is instrumental in studying regularity properties of solutions to FSDEs. Here's how Höld-Ineq facilitates this study [10, 39].

i. Estimation of Moments: Höld-Ineq provides a bound on the $\mathbf{L}^p$ norm of a product of functions in terms of the individual $\mathbf{L}^p$ norms of the functions and their Hölder exponents. In the context of FSDEs, where solutions may exhibit irregular behavior due to the presence of fractional derivatives and stochastic noise, Höld-Ineq allows researchers to estimate moments of the solutions. By controlling the growth of these moments, Höld-Ineq aids in understanding the regularity properties of solutions and their behavior under various conditions.
ii. Control of Solution Growth: Höld-Ineq helps in controlling the growth of solutions to FSDEs by providing bounds on the $\mathbf{L}^p$ norms of the solutions. This is particularly useful for establishing conditions under which solutions remain bounded or converge to certain equilibrium states. By bounding the growth of solutions in terms of their Hölder exponents, Höld-Ineq facilitates the study of regularity properties and stability of solutions over time.
iii. Analysis of Smoothness: Höld-Ineq characterizes the smoothness of functions in terms of their Hölder exponents. In the context of FSDEs, where solutions may exhibit fractal-like behavior or irregularities, Höld-Ineq helps in quantifying the degree of smoothness or regularity of solutions. By estimating Hölder exponents and bounding the growth of solutions, Höld-Ineq provides insights into the regularity properties of solutions and how they evolve over time.
iv. Stability Analysis: Höld-Ineq aids in stability analysis of solutions to FSDEs by providing bounds on the differences between solutions at different points in time or space. By controlling the growth of these differences in terms of Hölder exponents, Höld-Ineq helps in understanding how solutions behave under perturbations and whether they exhibit stable or chaotic behavior over time.

Inspired by these findings, we established the regularity of the solutions as well as the existence, uniqueness, and continuous dependence of solutions on the initial values and on the fractional exponent ϕ of the CFSPDEs in the $\mathbf{L}^p$ space. We also established the averaging principle result for the CFSPDEs in the sense of pth moment by utilizing the Grönwall-Bellman's inequality (Grön-Bell-Ineq), Jensen's inequality (Jen-Ineq), Burkholder-Davis-Gundy's (BHDG-Ineq), Hölder's inequality (Höld-Ineq) and the interval translation approach. To demonstrate that the mathematical approach is valid, three numerical examples are also built. We examined the following CFSPDEs of order $\frac{1}{2} < \phi < 1$:

$$\mathcal{T}_\tau^\phi \omega(\tau) = \Lambda\big(\tau, \omega(\tau), \omega(\tau\zeta)\big) + \delta\big(\tau, \omega(\tau), \omega(\tau\zeta)\big)\frac{d\mathbb{W}_\tau}{d\tau}, \tag{1.2}$$

where $\tau\zeta$ represents the past state and $\zeta \in (0,1)$ and ϕ represent the CFrD within the range, $\Lambda : [0,\ \mathbb{T}] \times \mathcal{R}^{\mathbf{b}} \times \mathcal{R}^{\mathbf{b}} \to \mathcal{R}^{\mathbf{b}}$, $\delta : [0,\ \mathbb{T}] \times \mathcal{R}^{\mathbf{b}} \times \mathcal{R}^{\mathbf{b}} \to \mathcal{R}^{\mathbf{b}\times\mathbf{m}}$ are measurable and on an underlying complete filtered probability space $(\mho, \widetilde{\mathbb{F}}_\tau, \mathfrak{P})$, $(\mathbb{W}_\tau)_{\tau\in[0,\ \infty)}$ is a scalar Brownian motion, with the filtration $\{\widetilde{\mathscr{F}_\tau}\}_{\text{fi}\geqslant 0}$.

Applying numerical techniques to solve CFSPDEs can present several computational complexities, including [34, 44]:

i. Non-locality of Fractional Operators: Fractional operators in CFSPDEs are non-local operators, meaning they depend on the entire history of the process. This non-locality complicates numerical discretization, as traditional finite difference or finite element methods designed for local operators may not be directly applicable. Mitigation strategies include using specialized numerical schemes tailored for fractional operators, such as the Grünwald-Letnikov or Caputo discretizations, or approximating FrOD with local approximations in specific cases.
ii. Stochasticity and Randomness: CFSPDEs involve stochastic processes, such as fractional Brownian motion, introducing randomness into the equations. Numerically simulating these stochastic processes accurately can be computationally demanding, especially for long simulation times or high-dimensional problems. Monte Carlo methods, such as the Euler-Maruyama or Milstein methods, are commonly used but can require many samples to achieve convergence. Variance reduction techniques, like control variates or importance sampling, can help improve the efficiency of Monte Carlo simulations.

iii. Time-stepping and Stability: Numerical integration methods for CFSPDEs must be carefully chosen to ensure stability and accuracy. These equations often exhibit stiff behavior due to the presence of both fractional operators and pantograph terms, requiring implicit or semi-implicit time-stepping methods. However, such methods can increase computational complexity and may require solving nonlinear systems of equations at each time step. Adaptive time-stepping strategies and advanced numerical solvers can help mitigate stability issues and improve efficiency.

iv. Dimensionality: CFSPDEs with multiple variables or dimensions can result in high-dimensional state spaces, leading to increased computational complexity. Direct numerical methods for high-dimensional problems may be impractical due to memory and computational constraints. Dimensionality reduction techniques, such as model order reduction or proper orthogonal decomposition, can help mitigate this complexity by approximating the dynamics in a lower-dimensional subspace while preserving essential features of the system.

v. Accuracy vs. Efficiency Trade-off: Achieving high accuracy in numerical solutions of CFSPDEs often requires fine discretizations or high-order numerical methods, which can be computationally expensive. Balancing accuracy and efficiency is crucial, particularly for large-scale problems or real-time applications. Adaptive mesh refinement, adaptive time-stepping, and error estimation techniques can help optimize computational resources and improve efficiency without sacrificing accuracy.

The format of the study is as follows. In Section 2, we present some important definitions, some key results, and assumptions that will serve as foundations to support the results regarding CFSPDEs. In the first subsection of Section 3, we first prove the well-posedness of the solution of CFSPDEs, and in the second subsection, we will prove the regularity. In Section 4, we established the averaging principle theorem and included examples to support our findings in Section 5. Section 6 then presents the conclusion.

2. Preliminaries

In this section, we go over definitions, some assumptions that are the pillars of our results, and a lemma and a corollary that will be useful in this paper.

Definition 2.1. When $p \geqslant 2$, $\tau \in [0,\infty)$, suppose $\widetilde{\mathbb{S}}^p_\tau = \mathbf{L}^p(\mho, \widetilde{\mathbb{F}}_\tau, \mathfrak{P})$ represents all $\widetilde{\mathbb{F}}_\tau$-measurable, p^{th} functions that are integrable $\omega = (\omega_1, \omega_2, \cdots, \omega_{\mathfrak{b}})^T : \mho \to \mathcal{R}^{\mathfrak{b}}$ with

$$\|\omega\|_p = \left(\sum_{\imath=1}^{\mathfrak{b}} \mathbf{E}(|\omega_\imath|^p)\right)^{\frac{1}{p}}.$$

A measurable procedure $\omega(t) : [0,\mathbb{T}] \to \mathbf{L}^p(\mho, \widetilde{\mathbb{F}}_\tau, \mathfrak{P})$ becomes $\widetilde{\mathscr{F}}_\tau$-adapted process if $\omega(\tau) \in \widetilde{\mathbb{S}}^p_\tau$ for each $\tau \geqslant 0$. For $\forall\theta \in \widetilde{\mathbb{S}}^p_0$, a $\widetilde{\mathscr{F}}$-adapted process $\omega(\tau)$ is solution of Eq. (1.2) with initial condition (In.C) $\omega(0) = \theta$ if $\omega(0) = \theta$ and the subsequent equality satisfies on $\widetilde{\mathbb{S}}^p_\tau$ for $\tau \in [0,\mathbb{T}]$:

$$\omega(\tau) = \theta + \int_0^\tau \ell^{\phi-1}\Lambda\big(\ell, \omega(\ell), \omega(\ell\zeta)\big)\,d\ell + \int_0^\tau \ell^{\phi-1}\delta\big(\ell, \omega(\ell), \omega(\ell\zeta)\big)\,d\mathbb{W}_\tau.$$

Definition 2.2. For the purposes of this article, we make the assumption that coefficients Λ and δ in Eq. (1.2) meet the following requirements.

($\mathbb{A}_1$) Global Lipschitz continuity in $\mathcal{R}^{\mathfrak{b}}$ of the drift term Λ and the diffusion term δ: when $\forall \mathcal{O}_1, \mathcal{O}_2, \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{R}^{\mathfrak{b}}$ there is $\mathbf{L}$ such as

$$\|\delta(\tau, \mathcal{O}_1, \mathcal{O}_2) - \delta(\tau, \mathcal{V}_1, \mathcal{V}_2)\|_p \leqslant \mathbf{L}\big(\|\mathcal{O}_1 - \mathcal{V}_1\|_p + \|\mathcal{O}_2 - \mathcal{V}_2\|_p\big),$$
$$\|\Lambda(\tau, \mathcal{O}_1, \mathcal{O}_2) - \Lambda(\tau, \mathcal{V}_1, \mathcal{V}_2)\|_p \leqslant \mathbf{L}\big(\|\mathcal{O}_1 - \mathcal{V}_1\|_p + \|\mathcal{O}_2 - \mathcal{V}_2\|_p\big).$$

($\mathbb{A}_2$) The drift term $\Lambda(\tau,0,0)$ and the diffusion $\delta(\tau,0,0)$ are essential bounded in time, i.e.,

$$\operatorname*{esssup}_{\tau\in[0,\mathbb{T}]}\|\Lambda(\tau,0,0)\|_p < \mathfrak{U},\ \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]}\|\delta(\tau,0,0)\|_p < \mathfrak{U}.$$

Keep in consideration that the assumptions ($\mathbb{A}_1$) and ($\mathbb{A}_2$) do not depend on the norm selected on $\mathcal{R}^{\mathbf{b}}$. Nonetheless, we provide $\mathcal{R}^{\mathbf{b}}$ with the p norm for convenience in our subsequent estimates: for any vector $\omega = (\omega_1, \omega_2, \omega_3, \cdots, \omega_m)^T \in \mathcal{R}^{\mathbf{b}}$, $\|\omega\|_p = \left(\sum_{\iota=1}^{m} |\omega_\iota|^p\right)^{\frac{1}{p}}$ provides the p norm $\|\omega\|_p$ of ω.

Now we propose some conditions that are pillars for the results of averaging principle.

($\mathfrak{C}_1$) We make the condition that coefficient Λ in Eq. (1.1) when $\forall \mathcal{O}_1, \mathcal{O}_2, \mathcal{V}_1, \mathcal{V}_2, \mathcal{O}, \mathcal{V} \in \mathcal{R}^{\mathbf{b}}$, $\tau \in [0,\mathbb{T}]$ there is $\mathscr{U}_1 > 0$ such that meets the following:

$$\|\Lambda(\tau,\mathcal{O}_1,\mathcal{O}_2) - \Lambda(\tau,\mathcal{V}_1,\mathcal{V}_2)\| \vee \|\delta(\tau,\mathcal{O}_1,\mathcal{O}_2) - \delta(\tau,\mathcal{V}_1,\mathcal{V}_2)\| \leqslant \mathscr{U}_1\big(\|\mathcal{O}_1 - \mathcal{V}_1\| + \|\mathcal{O}_2 - \mathcal{V}_2\|\big),$$

where, $\Lambda \vee \delta = \max(\Lambda, \delta)$.

($\mathfrak{C}_1$) Now we make the condition that coefficient δ in Eq. (1.1) when $\forall \mathcal{O}_1, \mathcal{O}_2, \mathcal{V}_1, \mathcal{V}_2, \mathcal{O}, \mathcal{V} \in \mathcal{R}^{\mathbf{b}}$, $\tau \in [0,\mathbb{T}]$ there is $\mathscr{U}_2 > 0$ that satisfies the following:

$$\|\Lambda(\tau,\mathcal{O},\mathcal{V})\| \vee \|\delta(\tau,\mathcal{O},\mathcal{V})\| \leqslant \mathscr{U}_2\big(1 + \|\mathcal{O}\| + \|\mathcal{V}\|\big).$$

($\mathfrak{C}_1$) Functions $\widetilde{\Lambda}$ and $\widetilde{\delta}$ exist and for $\mathbb{T}_1 \in [0,\mathbb{T}]$, $\tau \in [0,\mathbb{T}]$, and $p \geqslant 2$, we are able to identify positively bound functions $\mathscr{Y}_1(\mathbb{T}_1)$ and $\mathscr{Y}_2(\mathbb{T}_1)$ that fulfill

$$\frac{1}{\mathbb{T}_1}\int_0^{\mathbb{T}_1} \|\Lambda(\tau,\mathcal{O},\mathcal{V}) - \widetilde{\Lambda}(\mathcal{O},\mathcal{V})\|^p d\tau \leqslant \mathscr{Y}_1(\mathbb{T}_1)\big(1 + \|\mathcal{O}\|^p + \|\mathcal{V}\|^p\big),$$
$$\frac{1}{\mathbb{T}_1}\int_0^{\mathbb{T}_1} \|\delta(\tau,\mathcal{O},\mathcal{V}) - \widetilde{\delta}(\mathcal{O},\mathcal{V})\|^p d\tau \leqslant \mathscr{Y}_2(\mathbb{T}_1)\big(1 + \|\mathcal{O}\|^p + \|\mathcal{V}\|^p\big),$$

where $\lim_{\mathbb{T}_1\to\infty} \mathscr{Y}_1(\mathbb{T}_1) = 0$ and $\lim_{\mathbb{T}_1\to\infty} \mathscr{Y}_2(\mathbb{T}_1) = 0$.

Corollary 2.3. *For every $\Upsilon \in (0, \phi - \frac{1}{2})$, there occurred a modification $\omega_2(\tau)$ of $\omega_1(\tau)$ with Φ-Hölder continuous paths, i.e.,*

$$\mathfrak{P}\big(\omega_1(\tau) = \omega_2(\tau)\big) = 1,\ \forall \tau \in [0,\mathbb{T}].$$

Proof. By utilzing Kolmogorov test [19], $\omega(\tau)$ has Υ-Hölder continuous modification for all $\Upsilon \in (0, \phi - \frac{1}{2})$. □

Lemma 2.4 ([14])**.** *Assume that there are real numbers $\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\upsilon (\upsilon \in \mathbb{N})$ and meet $\mathfrak{w}_\iota \geqslant 0 (\iota = 1,2,\ldots,\upsilon)$. Then*

$$\left(\sum_{\iota=1}^{\upsilon} \mathfrak{w}_\iota\right)^p \leqslant \upsilon^{p-1} \sum_{\iota=1}^{\upsilon} \mathfrak{w}_\iota^p,\ \forall p > 1.$$

3. The main results

In this part, we demonstrated the well-posedness and regularity of the solutions to CFSPDEs.

3.1. *Well-posedness of solutions of CFSPDEs under the standard Lipschitz condition of coefficients*

To prove the well-posedness of solutions, we must demonstrate the solution's existence, uniqueness, and continuous dependency on ϕ and the starting data in order to achieve this goal. Suppose $\widetilde{\mathcal{H}}^p(0,\mathbb{T})$

is the space of all processes $\omega(\tau)$ that are measurable $\widetilde{\mathscr{F}}_{\mathbb{T}}$-adapted, with $\widetilde{\mathscr{F}}_{\mathbb{T}} = (\widetilde{\mathbb{F}}_{\tau})_{\tau\in[0,\mathbb{T}]}$ and satisfy the following:

$$\|\omega(\tau)\|_{\widetilde{\mathcal{H}}^p} = \underset{\tau\in[0,\mathbb{T}]}{\text{esssup}}\|\omega(\tau)\|_p < \infty.$$

$(\widetilde{\mathcal{H}}^p(0,\mathbb{T}), \|\cdot\|_{\widetilde{\mathcal{H}}^p})$ is surely a Banach space. We construct an operator $\mathfrak{I}_\theta : \widetilde{\mathcal{H}}^p(0,\mathbb{T}) \to \widetilde{\mathcal{H}}^p(0,\mathbb{T})$ by $\mathfrak{I}_\theta(\omega(0)) = \theta$ for any $\theta \in \widetilde{\mathcal{S}}_0^p$ and for $\tau \in [0,\mathbb{T}]$, the subsequent equality is valid:

$$\mathfrak{I}_\theta(\omega(\tau)) = \theta + \int_0^\tau \ell^{\phi-1}\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\,d\ell + \int_0^\tau \ell^{\phi-1}\delta(\ell,\omega(\ell),\omega(\ell\zeta))\,d\mathbb{W}_\ell. \tag{3.1}$$

The well-defined property of this operator is demonstrated in the ensuing lemma. The elementary inequality below is employed in the proof of this result as well as multiple others that follow,

$$\|\omega_1+\omega_2\|_p^p \leqslant 2^{p-1}\Big(\|\omega_1\|_p^p + (\|\omega_2\|_p^p\Big),\ \forall\omega_1,\omega_2 \in \mathcal{R}^{\mathbf{b}}. \tag{3.2}$$

Lemma 3.1. *Assume that* $(\mathbb{A}_1)$ *and* $(\mathbb{A}_2)$ *are valid. The operator* $\mathfrak{I}_\theta$ *is then well defined for any* $\theta \in \widetilde{\mathcal{S}}_0^p$.

Proof. Suppose $\omega(\tau) \in \widetilde{\mathcal{H}}^p[0,\mathbb{T}]$ and here $\omega(\tau)$ is arbitrary. We have the following $\forall\tau \in [0,\mathbb{T}]$ from the description of $\mathfrak{I}_\theta(\omega(\tau))$ as in Eq. (3.1) and the inequality (3.2):

$$\begin{aligned}\|\mathfrak{I}_\theta(\omega(\tau))\|_p^p \leqslant 2^{p-1}\|\theta\|_p^p &+ 2^{2p-2}\Big\|\int_0^\tau \ell^{\phi-1}\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\,d\ell\Big\|_p^p\\ &+ 2^{2p-2}\Big\|\int_0^\tau \ell^{\phi-1}\delta(\ell,\omega(\ell),\omega(\ell\zeta))\,d\mathbb{W}_\tau\Big\|_p^p.\end{aligned} \tag{3.3}$$

The Höld-Ineq gives us the result that

$$\begin{aligned}\Big\|\int_0^\tau \ell^{\phi-1}\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\,d\ell\Big\|_p^p &\leqslant \sum\nolimits_{\imath=1}^m \mathbf{E}\Big(\int_0^\tau \ell^{\phi-1}|\Lambda_\imath(\ell,\omega(\ell),\omega(\ell\zeta))|\,d\ell\Big)^p\\ &\leqslant \sum\nolimits_{\imath=1}^m \mathbf{E}\bigg(\Big(\int_0^\tau \ell^{\frac{(\phi-1)p}{(p-1)}}\,d\ell\Big)^{p-1}\int_0^\tau |\Lambda_\imath(\ell,\omega(\ell),\omega(\ell\zeta))|^p d\ell\bigg)\\ &\leqslant \frac{\mathbb{T}^{(p\phi-1)}(p-1)^{(p-1)}}{(p\phi-1)^{(p-1)}}\int_0^\tau \|\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\|_p^p d\ell.\end{aligned} \tag{3.4}$$

According to $(\mathbb{A}_1)$, we acquire

$$\begin{aligned}\|\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\|_p^p &\leqslant 2^{p-1}\Big(\|\Lambda(\ell,\omega(\ell),\omega(\ell\zeta)) + \Lambda(\ell,0,0)\|_p^p - \|\Lambda(\ell,0,0)\|_p^p\Big)\\ &\leqslant 2^{p-1}\Big(\mathbf{L}^p\Big(\|\omega(\ell)\|_p^p + \|\omega(\ell\zeta)\|_p^p\Big) + \|\Lambda(\ell,0,0)\|_p^p\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\tau \|\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\|_p^p d\ell\\ &\leqslant 2^{p-1}\mathbf{L}^p\bigg(\Big(\underset{\ell\in[0,\mathbb{T}]}{\text{esssup}}\|\omega(\ell)\|_p\Big)^p + \Big(\underset{\ell\in[0,\mathbb{T}]}{\text{esssup}}\|\omega(s'')\|_p\Big)^p\bigg)\int_0^\tau 1\,d\ell + 2^{p-1}\int_0^\tau \|\Lambda(\ell,0,0)\|_p^p d\ell\\ &\leqslant 2^{p-1}\mathbf{L}^p\mathbb{T}\Big(\|\omega(\ell)\|_{\widetilde{\mathcal{H}}^p}^p + \|\omega(\ell'')\|_{\widetilde{\mathcal{H}}^p}^p\Big)2^{p-1}\int_0^\tau \|\Lambda(\ell,0,0)\|_p^p d\ell.\end{aligned} \tag{3.5}$$

By Eqs. (3.4) and (3.5), we get

$$\left\| \int_0^\tau \ell^{\phi-1}\Lambda(\ell,\omega(\ell),\omega(\ell\zeta))\,d\ell \right\|_p^p \leqslant \frac{\mathbb{T}^{(p\phi-1)}(2p-2)^{(p-2)}}{(p\phi-1)^{(p-1)}}\left(\mathbf{L}^p\mathbb{T}\left(\|\omega(\ell)\|^p_{\widetilde{\mathcal{H}}^p}+\|\omega(\ell\zeta)\|^p_{\widetilde{\mathcal{H}}^p}\right)+\int_0^\tau \|\Lambda(\ell,0,0)\|_p^p d\ell\right).$$

Now, applying the BHDG-Ineq and Höld-Ineq, we get

$$\begin{aligned}
\left\| \int_0^\tau \ell^{\phi-1}\delta(\ell,\omega(\ell),\omega(\ell\zeta))\,d\mathbb{W}_\tau \right\|_p^p &\leqslant \sum\nolimits_{\imath=1}^{m} \mathbf{E}\left| \int_0^\tau \ell^{\phi-1}(\delta_\imath(\ell,\omega(\ell),\omega(\ell\zeta))\,d\mathbb{W}_\ell \right|^p \\
&\leqslant \sum\nolimits_{\imath=1}^{m} \mathcal{C}_p\mathbf{E}\left| \int_0^\tau \ell^{2\phi-2}\left|\delta_\imath(\ell,\omega(\ell),\omega(\ell\zeta))\right|^2 d\ell\right|^{\frac{p}{2}} \\
&\leqslant \sum\nolimits_{\imath=1}^{m} \mathcal{C}_p\mathbf{E}\int_0^\tau \ell^{2\phi-2}\left|\delta_\imath(\ell,\omega(\ell),\omega(\ell\zeta))\right|^p d\ell\left(\int_0^\tau \ell^{2\phi-2}d\ell\right)^{\frac{p-2}{2}} \\
&\leqslant \mathcal{C}_p\left(\frac{\mathbb{T}^{2\phi-1}}{2\phi-1}\right)^{\frac{p-2}{2}}\int_0^\tau \ell^{2\phi-2}\left\|\delta(\ell,\omega(\ell,\omega(\ell\zeta))\right\|_p^p d\ell,
\end{aligned} \tag{3.6}$$

where $\mathcal{C}_p = \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}$. By utilizing $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$, we get as follows:

$$\begin{aligned}
\left\|\delta(\ell,\omega(\ell),\omega(\ell\zeta))\right\|_p^p &\leqslant 2^{p-1}\mathbf{L}^p\left(\|\omega(\ell)\|_p^p+\|\omega(\ell\zeta)\|_p^p\right)+2^{p-1}\|\delta(\ell,0,0)\|_p^p \\
&\leqslant 2^{p-1}\mathbf{L}^p\left(\|\omega(\ell)\|_p^p+\|\omega(\ell\zeta)\|_p^p\right)+2^{p-1}\mathcal{U}^p.
\end{aligned} \tag{3.7}$$

Thus, $\forall\tau\in[0,\mathbb{T}]$, we get the following:

$$\begin{aligned}
&\int_0^\tau \ell^{2\phi-2}\left\|\delta(\ell,\omega(\ell),\omega(\ell\zeta))\right\|_p^p d\ell \\
&\leqslant 2^{p-1}\mathbf{L}^p\int_0^\tau \ell^{2\phi-2}\left(\left(\operatorname*{esssup}_{\ell\in[0,\mathbb{T}]}\|\omega(\ell)\|_p\right)^p+\left(\operatorname*{esssup}_{\ell\in[0,\mathbb{T}]}\|\omega(\ell\zeta)\|_p\right)^p\right)d\ell+2^{p-1}\mathcal{U}^p\int_0^\tau \ell^{2\phi-2}d\ell \\
&\leqslant \frac{2^{p-1}\mathbb{T}^{2\phi-1}}{2\phi-1}\left(\mathbf{L}^p\left(\|\omega(\ell)\|^p_{\widetilde{\mathcal{H}}_p}+\|\omega(\ell\zeta)\|^p_{\widetilde{\mathcal{H}}_p}\right)+\mathcal{U}^p\right).
\end{aligned}$$

With Eqs. (3.4), (3.6), and $(\mathbb{A}_2)$, we obtain that $\|\mathfrak{I}(\omega(\tau))\|_{\widetilde{\mathcal{H}}_p}<\infty$. As a consequence, the map $\mathfrak{I}_\theta$ is well-defined. □

We must demonstrate the following lemma in order to establish existence and uniqueness.

Lemma 3.2. *For any $\phi>\frac{1}{2}$ and $\tau>0$, the following inequality holds:*

$$\hbar\int_0^\tau \ell^{2\phi-2}\mathbb{E}_{2\phi-1}(\hbar\ell^{2\phi-1})d\ell \leqslant \mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1}),$$

where Mittag-Leffler function $\mathbb{E}_{2\phi-1}(.)$ is defined as

$$\mathbb{E}_{(2\phi-1)}(\tau)=\sum\nolimits_{\imath=0}^{\infty}\frac{\tau^\imath}{\Gamma((2\phi-1)\imath+1)}. \tag{3.8}$$

Proof. Let $\hbar > 0$ be arbitrary. We utilize the following identity after first substituting integral and sum:

$$\int_0^\tau \ell^{2\phi-2}\ell^{\iota(2\phi-1)}d\ell = \tau^{(\iota+1)(2\phi-1)}\mathbb{B}\big(2\phi-1, \iota(2\phi-1)+1\big),\ \iota = 0,1,2,\ldots.$$

So, we get

$$\begin{aligned}\frac{\hbar}{\Gamma(2\phi-1)}\int_0^\tau \ell^{2\phi-2}\mathbb{E}_{2\phi-1}(\hbar\ell^{2\phi-1})d\ell =&\hbar\sum_{\iota=0}^{\infty}\frac{\hbar^\iota}{\Gamma(\iota(2\phi-1)+1)}\int_0^\tau \ell^{2\phi-2}\ell^{\iota(2\phi-1)}d\ell\\ =&\sum_{\iota=0}^{\infty}\frac{\hbar^{\iota+1}\tau^{(\iota+1)(2\phi-1)}}{\Gamma(2\phi-1)\Gamma(\iota(2\phi-1)+1)}\\ =&\sum_{\iota=1}^{\infty}\frac{\hbar^{\iota}\tau^{\iota(2\phi-1)}}{\Gamma(\iota(2\phi-1)+1)}\\ =&\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})-1 \leqslant \mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1}),\end{aligned}$$

here, $\mathbb{B}$ is a beta function. Hence, the proof is completed. □

We shall demonstrate that the operator $\mathfrak{I}_\theta$ is contractive under an appropriate weighted norm ([16, Remark 2.1]) in order to establish the existence and uniqueness of solutions. The weight function in this case is the Mittag-Leffler function $\mathbb{E}_{(2\phi-1)}(\tau)$, which is defined as Eq. (3.8).

Theorem 3.3. *If* $(\mathbb{A}_1)$ *and* $(\mathbb{A}_2)$ *are valid, then the problem* (1.2) *with* $\omega(0) = \theta$ *has unique solution on* $[0, \mathbb{T}]$ *for any* $\theta \in \widetilde{\mathbb{S}}_0^p$.

Proof. First of all, choose a fix positive constant $\hbar$ as follows:

$$\hbar > \Psi 2^{p-1}\Gamma(2\phi-1), \tag{3.9}$$

where

$$\Psi = 2^{p-1}\mathbf{L}^p\left(\left(\mathbb{T}^{(p-2)\phi+1}\right)\frac{1}{\left(\frac{(p-2)\phi+1}{p-1}\right)^{p-1}} + \left(\frac{\mathbb{T}^{2\phi-1}}{2\phi-1}\right)^{\frac{p-2}{2}}\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\right). \tag{3.10}$$

We establish a weighted norm $\|\cdot\|_\hbar$ over the space $\widetilde{\mathcal{H}}^p([0,\mathbb{T}])$ as:

$$\|\omega(\tau)\|_\hbar = \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]}\left(\frac{\|\omega(\tau)\|_p^p}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}\right)^{\frac{1}{p}},\ \forall\omega(\tau)\in\widetilde{\mathcal{H}}^p([0,\mathbb{T}]). \tag{3.11}$$

Two norms, $\|\cdot\|_{\widetilde{\mathcal{H}}^p}$ and $\|\cdot\|_\hbar$, are equivalent. $\left(\widetilde{\mathcal{H}}^p([0,\mathbb{T}]), \|\cdot\|_\hbar\right)$ is a Banach space as a result. Choose and fix $\theta \in \widetilde{\mathbb{S}}_0^p$. By virtue of Lemma 3.1, the operator $\mathfrak{I}_\theta$ is well-defined. Now, we will prove that the map $\mathfrak{I}_\theta$ is contractive with respect to the norm $\|\cdot\|_\hbar$. For this purpose, let ω, $\widetilde{\omega}$ be arbitrary. We obtain the following $\forall\tau\in[0,\mathbb{T}]$ from Eqs. (3.1) and (3.2):

$$\begin{aligned}\|\mathfrak{I}_\theta(\omega(\tau)) - \mathfrak{I}_\theta(\widetilde{\omega}(\tau))\|_p^p \leqslant& 2^{p-1}\left\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda(\ell,\omega(\ell),\omega(\ell\zeta)) - \Lambda(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)d\ell\right\|_p^p\\ &+2^{p-1}\left\|\int_0^\tau \ell^{\phi-1}\Big(\delta(\ell,\omega(\ell),\omega(\ell\zeta)) - \delta(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)dW_\ell\right\|_p^p.\end{aligned}$$

Using the Höld-Ineq and $(\mathbb{A}_1)$, we obtain

$$\left\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda(s,\omega(\ell),\omega(\ell\zeta)) - \Lambda(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)d\ell\right\|_p^p$$

$$\leqslant \sum_{\iota=1}^{m} \mathbf{E}\Big(\int_0^{\tau} \ell^{\phi-1}\Big(\Lambda_\iota(\ell,\omega(\ell),\omega(\ell\zeta)) - \Lambda_\iota(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)d\ell\Big)^p$$
$$\leqslant \sum_{\iota=1}^{m} \mathbf{E}\bigg(\Big(\int_0^{\tau} \ell^{\frac{(\phi-1)(p-2)}{p-1}} d\ell\Big)^{p-1}\Big(\int_0^{\tau} \ell^{2\phi-2}\big|\Lambda_\iota(\ell,\omega(\ell),\omega(\ell\zeta)) - \Lambda_\iota(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\big|\Big)\bigg)$$
$$\leqslant \frac{\mathbf{L}^p\mathbb{T}^{p\phi-2\phi+1}(p-1)^{p-1}}{(p\phi-2\phi+1)^{p-1}}\int_0^{\tau} \ell^{2\phi-2}\Big(\|\omega(\ell)-\widetilde{\omega}(\ell))\|_p^p + \|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta))\|_p^p\Big)d\ell.$$

However, using $(\mathbb{A}_1)$ and the BHDK-Ineq, we have

$$\Big\|\int_0^{\tau} \ell^{\phi-1}\Big(\delta(\ell,\omega(\ell),\omega(\ell\zeta)) - \delta(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)d\mathbb{W}_\ell\Big\|_p^p$$
$$= \sum_{\iota=1}^{m} \mathbf{E}\Big|\int_0^{\tau} \ell^{\phi-1}\Big(\delta_\iota(\ell,\omega(\ell),\omega(\ell\zeta)) - \delta_\iota(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\Big)d\mathbb{W}_\ell\Big|^p$$
$$\leqslant \sum_{\iota=1}^{m} \mathcal{C}_p\mathbf{E}\Big|\int_0^{\tau} \ell^{2\phi-2}\big|\delta_\iota(\ell,\omega(\ell),\omega(\ell\zeta)) - \delta_\iota(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\big|^2 d\ell\Big|^{\frac{p}{2}}$$
$$\leqslant \sum_{\iota=1}^{m} \mathcal{C}_p\mathbf{E}\int_0^{\tau} \ell^{2\phi-2}\big|\delta_\iota(\ell,\omega(\ell),\omega(\ell\zeta)) - \delta_\iota(\ell,\widetilde{\omega}(\ell),\widetilde{\omega}(\ell\zeta))\big|^p d\ell\Big(\int_0^{\tau} \ell^{2\phi-2}d\ell\Big)^{\frac{p-2}{2}}$$
$$\leqslant \Big(\frac{\mathbb{T}^{2\phi-1}}{2\phi-1}\Big)^{\frac{p-2}{2}}\mathbf{L}^p\mathcal{C}_p\int_0^{\tau} \ell^{2\phi-2}\Big(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_p^p + \|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_p^p\Big)d\ell.$$

Thus, $\forall\tau \in [0,\mathbb{T}]$, we have

$$\|\mathfrak{I}_\theta(\omega(\tau)) - \mathfrak{I}_\theta(\widetilde{\omega}(\tau))\|_p^p \leqslant \Psi\int_0^{\tau}\Big(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_p^p + \|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_p^p\Big)\ell^{2\phi-2}d\ell,$$

here Ψ is specified in Eq. (3.10). The result suggests that using the definition of $\|\cdot\|_\hbar$ from Eq. (3.11),

$$\frac{\|\mathfrak{I}_\theta\omega(\tau) - \mathfrak{I}_\theta\widetilde{\omega}(\tau)\|_p^p}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}$$
$$\leqslant \frac{\Psi\int_0^{\tau}\ell^{2\phi-2}\frac{\left(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_p^p+\|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_p^p\right)}{\mathbb{E}_{2\phi-1}(\hbar\ell^{2\phi-1})}\mathbb{E}_{2\phi-1}(\hbar\ell^{2\phi-1})d\ell}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}$$
$$\leqslant \Psi\left(\operatorname*{esssup}_{\ell\in[0,\mathbb{T}]}\left(\frac{\left(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_p^p+\|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_p^p\right)}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}\right)^{\frac{1}{p}}\right)^p\frac{\int_0^{\tau}\ell^{2\phi-2}\mathbb{E}_{2\phi-1}(\hbar\ell^{2\phi-1})d\ell}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}$$
$$\leqslant \frac{\Psi\Gamma(2\phi-1)}{\hbar}\Big(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_\hbar^p + \|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_\hbar^p\Big).$$

By utilizing Lemma 3.2, we get the required result,

$$\|\mathfrak{I}_\theta(\omega(\tau)) - \mathfrak{I}_\theta(\widetilde{\omega}(\tau))\|_\hbar \leqslant \Big(\frac{\Psi\Gamma(2\phi-1)}{\hbar}\Big)^{\frac{1}{p}}\big(\|\omega(\ell)-\widetilde{\omega}(\ell)\|_\hbar + \|\omega(\ell\zeta)-\widetilde{\omega}(\ell\zeta)\|_\hbar\big).$$

From Eqs. (3.9), we get $\frac{\Psi\Gamma(2\phi-1)}{\hbar} < 1$, the operator $\mathfrak{I}_\theta$ on $\big(\widetilde{\mathcal{H}}^p([0,\mathbb{T}]),\|\cdot\|_\hbar\big)$ is a contractive map. There is a single fixed point of this map in $\widetilde{\mathcal{H}}^p([0,\mathbb{T}])$, according to the Banach fixed point thorem. The unique solution of Eq. (1.2) with the In.C $\omega(0) = \theta$ is also this fixed point. This theorem is proved. □

In the following theorem, we will demonstrate that the solution continuously depends on ϕ.

Theorem 3.4. *The solution $\wp_{\phi}(\tau,\theta)$ depends continuously on ϕ, i.e.,*

$$\lim_{\phi\to\tilde{\phi}} \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\phi}(\tau,\theta)-\wp_{\tilde{\phi}}(\tau,\theta)\|_{p}=0.$$

Proof. Suppose $\phi,\tilde{\phi}\in(\frac{1}{2},1)$ further take $\theta\in\widetilde{\mathbb{S}}_{0}^{p}$. As $\wp_{\phi}(\theta,\tau)$ and $\wp_{\tilde{\phi}}(\theta,\tau)$ are solutions to Eq. (1.2), we obtain the following:

$$\begin{aligned}\wp_{\phi}(\theta,\tau)-\wp_{\tilde{\phi}}(\theta,\tau)=&\int_0^{\tau}\ell^{\phi-1}\big(\Lambda(\ell,\wp_{\phi}(\ell),\wp_{\phi}(\ell\zeta))-\Lambda(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))\big)d\ell\\&+\int_0^{\tau}(\ell^{\phi-1}-\ell^{\tilde{\phi}-1})\Lambda(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))d\ell\\&+\int_0^{\tau}\ell^{\phi-1}\big(\delta(\ell,\wp_{\phi}(\ell),\wp_{\phi}(\ell\zeta))-\delta(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))\big)d\mathbb{W}_{\ell}\\&+\int_0^{\tau}(\ell^{\phi-1}-\ell^{\tilde{\phi}-1})\delta(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell))d\mathbb{W}_{\ell}.\end{aligned}\tag{3.12}$$

Using Eq. (3.2), we get the following result from Eq. (3.12):

$$\begin{aligned}\|\wp_{\phi}(\theta,\tau)-\wp_{\tilde{\phi}}(\theta,\tau)\|_p^p\leqslant&\, 2^{p-1}\Psi\int_0^{\tau}\ell^{2\phi-2}\|\wp_{\phi}(\theta,\tau)-\wp_{\tilde{\phi}}(\theta,\tau)\|_p^p d\ell\\&+2^{2p-2}\left\|\int_0^{\tau}(\ell^{\phi-1}-\ell^{\tilde{\phi}-1})\Lambda(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))d\ell\right\|_p^p\\&+\left\|\int_0^{\tau}(\ell^{\phi-1}-\ell^{\tilde{\phi}-1})\delta(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))d\mathbb{W}_{\ell}\right\|_p^p.\end{aligned}\tag{3.13}$$

Suppose the following:

$$\mathcal{A}(\tau,\ell,\phi,\tilde{\phi})=\left|\ell^{\phi-1}-\ell^{\tilde{\phi}-1}\right|.\tag{3.14}$$

Now we will simplify Eq. (3.13) one by one. First, using Eq. (3.2), the Höld-Ineq, $(\mathbb{A}_1)$, and $(\mathbb{A}_2)$, we get the following result:

$$\begin{aligned}&\left\|\int_0^{\tau}(\ell^{\phi-1}-\ell^{\tilde{\phi}-1})\Lambda(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))d\ell\right\|_p^p\\&\leqslant\sum_1^m\mathbf{E}\left(\int_0^{\tau}\mathcal{A}(\tau,\ell,\phi,\tilde{\phi})\left|\Lambda_{\imath}(\ell,\wp_{\tilde{\phi}})\right|d\ell\right)^p\\&\leqslant\sum_1^m\mathbf{E}\left(\left(\int_0^{\tau}(\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^{\frac{p}{p-1}}\right)^{p-1}\int_0^{\tau}\left|\Lambda_{\imath}(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))\right|^p d\ell\right)\\&\leqslant\left(\int_0^{\tau}(\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2\right)^{\frac{p}{2}}\left(\int_0^{\tau}1d\ell\right)^{\frac{p-2}{2}}\int_0^{\tau}\|\Lambda(\ell,\wp_{\tilde{\phi}}(\ell),\wp_{\tilde{\phi}}(\ell\zeta))\|_p^p d\ell\\&\leqslant\left(\int_0^{\tau}(\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2\right)^{\frac{p}{2}}\mathbb{T}^{\frac{p-2}{2}}\int_0^{\tau}2^{p-1}\left(\mathbf{L}^p\|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p)+\|\Lambda(\ell,0)\|_p^p\right)\\&\leqslant\left(\int_0^{\tau}(\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2\right)^{\frac{p}{2}}\mathbb{T}^{\frac{p}{2}}2^{p-1}\left(\mathbf{L}^p\operatorname*{esssup}_{\tau\in[0,\mathbb{T}]}\|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p)+\mathcal{U}^p\right).\end{aligned}$$

Now we will simplify the second part of Eq. (3.13). For this, using the BHDG-Ineq, Eq. (3.14), $(\mathbb{A}_1)$, $(\mathbb{A}_2)$, we have as a result,

$$\begin{aligned}
&\Big\| \int_0^\tau (\ell^{\phi-1} - \ell^{\tilde{\phi}-1})\delta(\ell, \wp_{\tilde{\phi}}(\ell), \wp_{\tilde{\phi}}(\ell\zeta))d\mathbb{W}_\ell \Big\|_p^p \\
&\quad \leqslant \sum\nolimits_1^m \mathbf{E}\Big| \int_0^\tau \mathcal{A}(\tau,\ell,\phi,\tilde{\phi})\delta_\iota(\ell, \wp_{\tilde{\phi}}(\ell), \wp_{\tilde{\phi}}(\ell\zeta))d\mathbb{W}_\ell \Big|^p \\
&\quad \leqslant \sum\nolimits_1^m \mathcal{C}_p \mathbf{E}\Big| \int_0^\tau \mathcal{A}(\tau,\ell,\phi,\tilde{\phi})^2 \big|\delta_\iota(\ell, \wp_{\tilde{\phi}}(\ell), \wp_{\tilde{\phi}}(\ell\zeta))\big|^2 d\mathbb{W}_\ell \Big|^{\frac{p}{2}} \\
&\quad \leqslant \sum\nolimits_1^m \mathcal{C}_p \mathbf{E}\Big[\Big(\int_0^\tau \mathcal{A}(\tau,\ell,\phi,\tilde{\phi})^2 \big|\delta_\iota(\ell, \wp_{\tilde{\phi}}(\ell), \wp_{\tilde{\phi}}(\ell\zeta))\big|^p d\ell \Big)^{\frac{2}{p}} \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p-2}{p}} \Big]^{\frac{p}{2}} \\
&\quad = \mathcal{C}_p \int_0^\tau \mathcal{A}(\tau,\ell,\phi,\tilde{\phi})^2 \|\delta(\ell, \wp_{\tilde{\phi}}(\ell), \wp_{\tilde{\phi}}(\ell\zeta))\|_p^p d\ell \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p-2}{p}} \\
&\quad \leqslant \mathcal{C}_p \Big(\int_0^\tau \mathcal{A}(\tau,\ell,\phi,\tilde{\phi})^2 d\ell \Big)^{\frac{p}{2}} 2^{p-1} \Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big).
\end{aligned}$$

By utilizing the above results and definition $\|\cdot\|_\hbar$, we get the following:

$$\begin{aligned}
\frac{\|\wp_\phi(\theta,\tau) - \wp_{\tilde{\phi}}(\theta,\tau)\|_p^p}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})} &\leqslant \frac{\Psi 2^{p-1} \int_0^\tau \ell^{2\phi-2} \frac{\|\wp_\phi(\ell,\theta) - \wp_{\tilde{\phi}}(\ell,\theta)\|_p^p}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})} \mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})}{\mathbb{E}_{2\phi-1}(\hbar\tau^{2\phi-1})} \\
&\quad + 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}} \mathbb{T}^{\frac{p}{2}} \\
&\quad + 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \mathcal{C}_p \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}} \\
&\leqslant \frac{\Psi 2^{p-1}\Gamma(2\phi-1)}{\hbar} \|\wp_\phi(\tau,\theta) - \wp_{\tilde{\phi}(\tau,\theta)}\|_\hbar^p \\
&\quad + 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}} \mathbb{T}^{\frac{p}{2}} \\
&\quad + 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \mathcal{C}_p \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}}.
\end{aligned}$$

Finally, utilizing Lemma 3.2, we get

$$\begin{aligned}
&\Big(1 - \frac{\Psi 2^{p-1}\Gamma(2\phi-1)}{\hbar}\Big) \|\wp_\phi(\tau,\theta) - \wp_{\tilde{\phi}}(\tau,\theta)\|_\hbar^p \\
&\quad \leqslant 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}} \mathbb{T}^{\frac{p}{2}} \\
&\quad + 2^{3p-3}\Big(\mathbf{L}^p \operatorname*{esssup}_{\tau\in[0,\mathbb{T}]} \|\wp_{\tilde{\phi}}(\ell,\theta)\|_p^p + \mathcal{U}^p \Big) \mathcal{C}_p \Big(\int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell \Big)^{\frac{p}{2}}.
\end{aligned}$$

Thus, by Eq. (3.9) and $p \geqslant 2$, it is required to demonstrate the following in order to complete the proof:

$$\lim_{\tilde{\phi}\to\phi} \sup_{\tau\in[0,\mathbb{T}]} \int_0^\tau (\mathcal{A}(\tau,\ell,\phi,\tilde{\phi}))^2 d\ell = 0.$$

We have the following:

$$\int_0^{\tau}\left(\mathcal{A}(\tau,\ell,\phi,\tilde{\phi})\right)^2 d\ell=\int_0^{\tau}\ell^{2\phi-2}d\ell+\int_0^{\tau}\ell^{2\tilde{\phi}-2}d\ell-\int_0^{\tau}\ell^{\phi+\tilde{\phi}-2}d\ell=\frac{\ell^{2\phi-1}}{2\phi-1}+\frac{\ell^{2\tilde{\phi}-1}}{2\tilde{\phi}-1}-\frac{2\tau^{\phi+\tilde{\phi}-1}}{\phi+\tilde{\phi}-1}.$$

Hence, it proved the required result. □

The evaluation of the difference between two distinct solutions yields the following result. Consequently, we derive the Lipschitz continuity dependency solutions with respect to the initial values.

Theorem 3.5. *For any $\theta,\gamma\in\widetilde{\mathbb{S}}_0^{p}$ the solution $\wp_\phi(\tau,\theta)$ depends Lipschitz continuously on θ, i.e., there exists $\mathbf{L}>0$ such that*

$$\|\wp_\phi(\tau,\theta)-\wp_\phi(\tau,\gamma)|_p\leqslant \mathbf{L}\|\theta-\gamma\|_p,\ \textit{for all}\ \tau\in[0,\ \mathbb{T}].$$

Proof. Choose and fix $\theta\in\widetilde{\mathbb{S}}_0^{p}$. Let $\theta\in\widetilde{\mathbb{S}}_0^{p}$ arbitrarily. Since $\wp_\phi(\tau,\theta)$ and $\wp_\phi(\tau,\gamma)$ are solutions of Eq. (1.2) it follows that

$$\begin{aligned}\wp_\phi(\tau,\theta)-\wp_\phi(\tau,\gamma)=\theta-\gamma+\int_0^{\tau}\ell^{\phi-1}\big(\Lambda(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta))-\Lambda(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma))\big)d\ell\\ +\int_0^{\tau}\ell^{\phi-1}\big(\delta(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta))-\delta(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)d\mathbb{W}_\ell.\end{aligned}$$

Hence, using Eq. (3.2),

$$\begin{aligned}&\|\wp_\phi(\tau,\theta)-\wp_\phi(\tau,\gamma)\|_p^p\\ &\leqslant 2^{p-1}\left\|\int_0^{\tau}\ell^{\phi-1}\Big(\Lambda\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta),)\big)-\Lambda\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\ell\right\|_p^p\\ &\quad+2^{p-1}\left\|\int_0^{\tau}\ell^{\phi-1}\Big(\delta\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\delta\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\mathbb{W}_\ell\right\|_p^p.\end{aligned}\tag{3.15}$$

Now we simplify Eq. (3.15) one by one. So, by using Eq. (3.2), Höld-Ineq and ($\mathbb{A}_1$), we get the following reuslt:

$$\begin{aligned}&\left\|\int_0^{\tau}\ell^{\phi-1}\Big(\Lambda\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\Lambda\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\ell\right\|_p^p\\ &\leqslant\sum_{\imath=1}^{m}\mathbf{E}\left(\int_0^{\tau}\ell^{\phi-1}\Big(\Lambda_\imath\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\Lambda_\imath\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\ell\right)^p\\ &\leqslant\sum_{\imath=1}^{m}\mathbf{E}\left(\left(\int_0^{\tau}\ell^{\frac{(\phi-1)(p-2)}{p-1}}d\ell\right)^{p-1}\left(\int_0^{\tau}\ell^{2\phi-2}\big|\Lambda_\imath\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)\right.\right.\\ &\quad\left.\left.-\Lambda_\imath\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\big|\right)\right)\\ &\leqslant\left(\frac{\mathbf{L}^p\mathbb{T}^{p\phi-2\phi+1}(p-1)^{p-1}}{(p\phi-2\phi+1)^{p-1}}\right)\int_0^{\tau}\ell^{2\phi-2}\Big(\|\wp_\phi(\ell,\theta)-\wp_\phi(\ell,\gamma)\|_p^p+\|\wp_\phi(\ell\zeta,\theta)-\wp_\phi(\ell\zeta,\gamma)\|_p^p\Big)ds.\end{aligned}\tag{3.16}$$

Now using Eq. (3.2), the Höld-Ineq, BHDG-Ineq and $(\mathbb{A}_1)$, we get

$$\begin{aligned}
&\left\| \int_0^\tau \ell^{\phi-1}\Big(\delta\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\delta\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\mathbb{W}_\ell \right\|_p^p \\
&= \sum_{\iota=1}^m \mathbf{E}\left| \int_0^\tau \ell^{\phi-1}\Big(\delta_\iota\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\delta_\iota\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\Big)d\mathbb{W}_\ell \right|^p \\
&\leqslant \sum_{\iota=1}^m \mathcal{C}_p\mathbf{E}\left| \int_0^\tau \ell^{2\phi-2}\big|\delta_\iota\big(\ell,\wp_\phi(\ell,\theta)\wp_\phi(\ell\zeta,\theta)\big)-\delta_\iota\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\big|^2 d\ell \right|^{\frac{p}{2}} \qquad (3.17)\\
&\leqslant \sum_{\iota=1}^m \mathcal{C}_p\mathbf{E}\int_0^\tau \ell^{2\phi-2}\big|\delta_\iota\big(\ell,\wp_\phi(\ell,\theta),\wp_\phi(\ell\zeta,\theta)\big)-\delta_\iota\big(\ell,\wp_\phi(\ell,\gamma),\wp_\phi(\ell\zeta,\gamma)\big)\big|^p d\ell\left(\int_0^\tau \ell^{2\phi-2}d\ell\right)^{\frac{p-2}{2}} \\
&\leqslant \mathbf{L}^p\mathcal{C}_p\left(\frac{\mathbb{T}^{2\phi-1}}{2\phi-1}\right)^{\frac{p-2}{2}}\int_0^\tau \ell^{2\phi-2}\Big(\|\wp_\phi(\ell,\theta)-\wp_\phi(\ell,\gamma)\|_p^p+\|\wp_\phi(\ell\zeta,\theta)-\wp_\phi(\ell\zeta,\gamma)\|_p^p\Big)d\ell.
\end{aligned}$$

Utilizing Eqs. (3.16) and (3.17), we can therefore extract the following from Eq. (3.15),

$$\begin{aligned}
&\|\wp_\phi(\tau,\theta)-\wp_\phi(\tau,\gamma)\|_p^p \\
&\leqslant 2^{p-1}\|\theta-\gamma\|_p^p+2^{p-1}\Psi\int_0^\tau \ell^{2\phi-2}\Big(\|\wp_\phi(\ell,\theta)-\wp_\phi(\ell,\gamma)\|_p^p+\|\wp_\phi(\ell\zeta,\theta)-\wp_\phi(\ell\zeta,\gamma)\|_p^p\Big)d\ell.
\end{aligned}$$

Considering the Grön-Bell-Ineq, we obtain the following ([17, Lemma 7.1.1]):

$$\|\wp_\phi(\tau-\zeta,\theta)-\wp_\phi(\tau-\zeta,\gamma)\|_p^p\leqslant 2^{p-1}\mathbb{E}_{2\phi-1}\big(2^{p-1}\Psi\Gamma(2\phi-1)\tau^{2\phi-1}\big)\|\theta-\gamma\|_p^p.$$

Hence, the proof is complete. □

3.2. *The regularity of solutions to CFSPDEs*

In this subsection, we will prove the regularity of solutions to CFSPDEs.

Theorem 3.6. *Consider that* $(\mathbb{A}_1)$ *and* $(\mathbb{A}_2)$ *are valid. After that, a constant* $\mathcal{J}>0$ *that depends on* $\phi,\mathbf{L},\mathcal{J},\mathcal{U},\mathbb{T}$ *exists, so*

$$\|\wp_\phi(\theta,\tau)-\wp_\phi(\theta,f)\|_p\leqslant \mathcal{J}|\tau-f|^{\phi-\frac{1}{2}},\ \forall\tau,f\in[0,\mathbb{T}].$$

Proof. Take $\tau,u\in[0,\mathbb{T}]$, such that $\tau>u$. By applying inequality (3.2), we get the following:

$$2^{2-2p}\|\wp_\phi(\tau,\theta)-\wp_{\tilde\phi}(\tau,\theta)\|_p^p\leqslant\left\|\int_u^\tau \ell^{\phi-1}\Lambda(\ell,\wp_\phi(\ell,\theta))d\ell\right\|_p^p+\left\|\int_u^\tau \ell^{\phi-1}\delta(\ell,\wp_\phi(\ell,\theta))d\mathbb{W}_\ell\right\|_p^p.$$

Now utilizing Höld-Ineq, BHDG-Ineq, as a result, we get the following from above:

$$\begin{aligned}
2^{2-2p}\|\wp_\phi(\tau,\theta)-\wp_{\tilde\phi}(\tau,\theta)\|_p^p&\leqslant\frac{(p-1)^{p-1}}{(p\phi-1)^{p-1}}\big(\tau^{\frac{p\phi-1}{p-1}}-u^{\frac{p\phi-1}{p-1}}\big)^{p-1}\int_u^\tau\|\Lambda(\ell,\wp_\phi(\ell,\theta))\|_p^p d\ell \\
&\quad+\mathcal{C}_p\int_u^\tau\frac{\|\delta(\ell,\wp_\phi(\ell,\theta))\|_p^p}{\ell^{2-2\phi}}d\ell\left(\int_u^\tau\frac{1}{\ell^{2-2\phi}}d\ell\right)^{\frac{p-2}{2}}.
\end{aligned}$$

However, $\mathcal{U}_1>0$ also exists, as $\operatorname*{esssup}_{\tau\in[0,\mathbb{T}]}\|\wp_\phi(\tau,\theta)\|_p^p\leqslant\mathcal{U}_1$ because $\wp_\phi(\ell,\theta)\in\widetilde{\mathcal{H}}^p([0,\mathbb{T}])$. Along with $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$, this implies

$$\|\Lambda(\ell,\wp_\phi(\ell,\theta))\|_p^p\leqslant 2^{p-1}\big(\mathbf{L}^p\|\wp_\phi(\ell,\theta))\|_p^p+\|\Lambda(\ell,0)\|_p^p\big)\leqslant 2^{p-1}\big(\mathbf{L}^p\mathcal{U}_1+\mathcal{U}^p\big),$$

$$\left\|\delta(\ell,\wp_{\Phi}(\ell,\theta))\right\|_p^p \leqslant 2^{p-1}\left(\mathbf{L}^p\left\|\wp_{\Phi}(\ell,\theta)\right\|_p^p + \|\delta(\ell,0)\|_p^p\right) \leqslant 2^{p-1}\left(\mathbf{L}^p\mathcal{U}_1 + \mathcal{U}^p\right).$$

The estimate that results from combining the calculations above is as follows:

$$\begin{aligned}2^{2-2p}\left\|\wp_{\Phi}(\tau,\theta)-\wp_{\tilde{\Phi}}(\tau,\theta)\right\|_p^p \leqslant& \frac{(2p-2)^{p-1}}{(p\Phi-1)^{p-1}}(\tau-f)^{\frac{(2\Phi-1)p}{2}}\left(\mathbf{L}^p\mathcal{U}_1+\mathcal{U}^p\right)\mathbb{T}^{\frac{p}{2}}\\&+\frac{1}{(2\Phi-1)^{\frac{p}{2}}}(\tau-f)^{\frac{(2\Phi-1)p}{2}}\left(\mathbf{L}^p\mathcal{U}_1+\mathcal{U}^p\right)2^{p-1}\mathcal{C}_p.\end{aligned}$$

Hence, we get

$$\left\|\wp_{\Phi}(\tau,\theta)-\wp_{\tilde{\Phi}}(\tau,\theta)\right\|_p \leqslant \mathcal{J}(\tau- f)^{\Phi-\frac{1}{2}},$$

where

$$\mathcal{J}^p = 2^{2p-2}\left(\frac{(2p-2)^{p-1}}{(p\Phi-1)^{p-1}}\left(\mathbf{L}^p\mathcal{U}_1+\mathcal{U}^p\right)\mathbb{T}^{\frac{p}{2}} + \frac{1}{(2\Phi-1)^{\frac{p}{2}}}\left(\mathbf{L}^p\mathcal{U}_1+\mathcal{U}^p\right)2^{p-1}\mathcal{C}_p\right).$$

□

4. Averaging principle result

In this section, we establish the growth requirements for $\widetilde{\delta}$ and the averaging principle result of CF-SPDEs in the sense of $\mathbf{L}^p$. The averaging principle for FSDEs in the $\mathbf{L}^p$ space provides a powerful approach for solving real-world challenges in various fields, such as finance, physics, biology, and engineering. This principle involves taking averages of solutions over time or space to obtain smoother and more stable representations of the underlying processes. Here's how it contributes and its limitations.

The contributions of the averaging principle are as follows [32, 34, 44].

i. Noise Reduction: Averaging over time or space helps reduce the impact of random fluctuations or noise present in the system. In real-world applications, noise can obscure the underlying dynamics of a process. By averaging, one can obtain a clearer signal that better represents the true behavior of the system.
ii. Improved Predictions: Averaging allows for more accurate predictions of future states or behaviors of the system. By obtaining smoother trajectories or representations of the system dynamics, one can make better-informed decisions or forecasts, which is crucial in fields like finance for risk management or in biology for predicting population dynamics.
iii. Enhanced Stability: Averaging can improve the stability of the numerical methods used to solve FSDEs. It can help mitigate numerical instabilities or oscillations that may arise due to the complex interplay between fractional operators and stochastic noise, leading to more reliable numerical solutions.
iv. Regularization: Averaging acts as a form of regularization, smoothing out irregularities in the solutions of FSDEs. This regularization can help in obtaining well-behaved solutions, especially in cases where the equations are ill-posed or have singularities.

The limitations of the averaging principle are as follows.

i. Loss of Information: Averaging over time or space may lead to a loss of detailed information about the system dynamics. In some cases, important features or transient behaviors may be smoothed out or obscured, making it difficult to capture the full complexity of the underlying processes.
ii. Assumption of Stationarity: The averaging principle often assumes stationarity of the underlying processes, which may not hold true in all real-world scenarios. If the processes exhibit non-stationary behavior or significant changes over time, simple averaging techniques may not be appropriate or effective.

iii. Computational Complexity: Averaging can significantly increase computational complexity, especially for high-dimensional systems or when dealing with large datasets. This can pose challenges in terms of computational resources and efficiency, particularly for real-time applications or when working with big data.
iv. Limited Applicability: The averaging principle may not be applicable to all types of FSDEs or all real-world problems. Its effectiveness depends on the specific characteristics of the system under consideration and the goals of the analysis. In some cases, alternative approaches may be more suitable or necessary.

In summary, while the averaging principle for FSDEs in the $\mathbf{L}^p$ space offers valuable benefits in terms of noise reduction, improved predictions, stability enhancement, and regularization, it also has limitations related to information loss, stationarity assumptions, computational complexity, and limited applicability. Careful consideration of these factors is essential when applying the averaging principle to real-world challenges.

The interval translation approach is a powerful method for demonstrating the averaging principle for FSDEs. This approach involves dividing the time interval into smaller subintervals and analyzing the behavior of the solutions within each subinterval. Here are some implications of the interval translation approach in demonstrating the averaging principle.

i. Smoothing Effect: By dividing the time interval into smaller subintervals, the interval translation approach allows researchers to analyze the behavior of solutions over shorter time scales. This approach effectively "smoothes out" the effects of stochastic noise and random fluctuations, leading to more stable and predictable solutions within each subinterval.
ii. Averaging Over Subintervals: Within each subinterval, the interval translation approach involves averaging the solutions over multiple realizations or trajectories of the stochastic process. This averaging process helps in reducing the impact of random fluctuations and noise, leading to a clearer and more robust representation of the underlying dynamics of the system.
iii. Consistency Across Subintervals: The interval translation approach ensures consistency of solutions across different subintervals. By averaging over multiple realizations within each subinterval, the approach provides a coherent and consistent representation of the system dynamics, despite the presence of stochastic noise and variability.
iv. Quantitative Estimates: The interval translation approach allows for quantitative estimates of the averaging effect. By analyzing the behavior of solutions within each subinterval and comparing the averaged solutions across different subintervals, researchers can quantify the extent to which averaging reduces the impact of stochastic noise and improves the stability of the solutions.
v. Validity Across Different Conditions: The interval translation approach demonstrates the robustness and validity of the averaging principle across different conditions and scenarios. By analyzing the behavior of solutions within multiple subintervals under varying initial conditions, parameters, or external influences, the approach confirms the general applicability and effectiveness of the averaging principle for FSDEs.

Lemma 4.1. *For every $\mathbb{T}_1 \in [0, \mathbb{T}]$, we can derive the following growth requirements for $\widetilde{\delta}$ by utilizing conditions* $(\mathfrak{C}_2)$ *and* $(\mathfrak{C}_3)$*:*

$$\|\widetilde{\delta}(\mathcal{O}, \mathcal{V})\|^p \leqslant \mathscr{U}_3 \left(1 + \|\mathcal{O}\|^p + \|\mathcal{V}\|^p\right),$$

where $\mathscr{U}_3 = \left(2^{p-1}\mathscr{Y}_2\left(\mathbb{T}_1\right) + 6^{p-1}\mathscr{U}_2^p\right)$.

Proof. Considering Jen-Ineq and conditions $(\mathfrak{C}_2)$ and $(\mathfrak{C}_3)$, we derive the following result:

$$\begin{aligned}
\|\widetilde{\delta}(\mathcal{O}, \mathcal{V})\|^p &\leqslant 2^{p-1}\|\delta(\tau, \mathcal{O}, \mathcal{V}) - \widetilde{\delta}(\mathcal{O}, \mathcal{V})\|^p + 2^{p-1}\|\delta(\tau, \mathcal{O}, \mathcal{V})\|^p \\
&\leqslant 2^{p-1}\mathscr{Y}_2\left(\mathbb{T}_1\right)\left(1 + \|\mathcal{O}\|^p + \|\mathcal{V}\|^p\right) + 2^{p-1}\mathscr{U}_2^p(1 + \|\mathcal{O}\| + \|\mathcal{V}\|)^p \\
&\leqslant \left(2^{p-1}\mathscr{Y}_2\left(\mathbb{T}_1\right) + 6^{p-1}\mathscr{U}_2^p\right)\left(1 + \|\mathcal{O}\|^p + \|\mathcal{V}\|^p\right).
\end{aligned}$$

□

Now we examin the averaging principle of CFSPDEs in the sense of $\mathbf{L}^p$. Initially, the standard form of Eq. (3.1) will be examined,

$$\omega_\varepsilon(\tau) = \theta + \varepsilon^\varphi \int_0^\tau \ell^{\varphi-1}\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big)\,d\ell + \varepsilon^{\varphi-\frac{1}{2}} \int_0^\tau \ell^{\varphi-1}\delta\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big)\,dW_\ell, \tag{4.1}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a positive small parameter, ε_0 is a fixed point, and Λ and δ satisfy the conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$. The averaged representation of Eq. (4.1) is thus depicted below:

$$\omega^*_\varepsilon(\tau) = \theta + \varepsilon^\varphi \int_0^\tau \ell^{\varphi-1}\widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\,d\ell + \varepsilon^{\varphi-\frac{1}{2}} \int_0^\tau \ell^{\varphi-1}\widetilde{\delta}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\,dW_\ell, \tag{4.2}$$

where $\widetilde{\Lambda} : \mathcal{R}^{\mathbf{b}} \times \mathcal{R}^{\mathbf{b}} \to \mathcal{R}^{\mathbf{b}}, \widetilde{\delta} : \mathcal{R}^{\mathbf{b}} \times \mathcal{R}^{\mathbf{b}} \to \mathcal{R}^{\mathbf{b}\times\mathbf{m}}$.

Theorem 4.2. *Consider that conditions $(\mathfrak{C}_1)$-$(\mathfrak{C}_3)$ are met. We can determine the corresponding $\varepsilon_1 \in (0, \varepsilon_0]$, $\varphi > 0, \chi \in (0,1)$ satisfies for all $\varepsilon \in (0, \varepsilon_1]$ when $p \in \big[2, (1-\phi)^{-1}\big)$ and for $\digamma > 0$, which is an arbitrarily small number. The formula for this is obtained as follows:*

$$\mathbf{E}\Big[\sup_{\tau\in[0,\varphi\varepsilon^{-\chi}]} \big\|\omega_\varepsilon(\tau) - \omega^*_\varepsilon(\tau)\big\|^p\Big] \leqslant \digamma. \tag{4.3}$$

Proof. We achieve the following outcome for any $\forall\tau \in [0, \mathfrak{a}] \subset [0, \mathbb{T}]$ via Eqs. (4.1) and (4.2):

$$\begin{aligned}\omega_\varepsilon(\tau) - \omega^*_\varepsilon(\tau) = (\theta-\theta) &+ \varepsilon^\phi \int_0^\tau \ell^{\phi-1}\big(\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\big)\,d\ell \\ &+ \varepsilon^{\phi-\frac{1}{2}} \int_0^\tau \ell^{\phi-1}\big(\delta\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\delta}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\big)\,dW_\ell.\end{aligned} \tag{4.4}$$

By using Jen-Ineq, we get the following from Eq. (4.4) as a result:

$$\begin{aligned}\big\|\omega_\varepsilon(\tau) - \omega^*_\varepsilon(\tau)\big\|^p &\leqslant 2^{p-1}\bigg\|\varepsilon^\phi \int_0^\tau \ell^{\phi-1}\Big(\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,d\ell\bigg\|^p \\ &\quad + 2^{p-1}\bigg\|\varepsilon^{\phi-\frac{1}{2}} \int_0^\tau \ell^{\phi-1}\Big(\delta\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\delta}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,dW_\ell\bigg\|^p \\ &\leqslant 2^{p-1}\varepsilon^{\phi p}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,d\ell\bigg\|^p \\ &\quad + 2^{p-1}\varepsilon^{(\phi-\frac{1}{2})p}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\delta\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\delta}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,dW_\ell\bigg\|^p.\end{aligned} \tag{4.5}$$

Utilizing Eq. (4.5) in Eq. (4.3),

$$\begin{aligned}&\mathbf{E}\Big[\sup_{0\leqslant\tau\leqslant\mathfrak{a}} \big\|\omega_\varepsilon(\tau) - \omega^*_\varepsilon(\tau)\big\|^p\Big] \\ &\leqslant 2^{p-1}\varepsilon^{\phi p}\mathbf{E}\bigg[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,d\ell\bigg\|^p\bigg] \\ &\quad + 2^{p-1}\varepsilon^{(\phi-\frac{1}{2})p}\mathbf{E}\bigg[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\delta\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \widetilde{\delta}\big(\mathscr{U}^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,dW_\ell\bigg\|^p\bigg] = \mathbb{Y}_1 + \mathbb{Y}_2.\end{aligned} \tag{4.6}$$

From $\mathbb{Y}_1$,

$$\begin{aligned}\mathbb{Y}_1 &\leqslant 2^{2p-2}\varepsilon^{\phi p}\mathbf{E}\bigg[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda\big(\ell, \omega_\varepsilon(\ell), \omega_\varepsilon(\ell\zeta)\big) - \Lambda\big(\ell, \omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,d\ell\bigg\|^p\bigg] \\ &\quad + 2^{2p-2}\varepsilon^{\phi p}\mathbf{E}\bigg[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\bigg\|\int_0^\tau \ell^{\phi-1}\Big(\Lambda\big(\ell, \omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big) - \widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell), \omega^*_\varepsilon(\ell\zeta)\big)\Big)\,d\ell\bigg\|^p\bigg] = \mathbb{Y}_{11} + \mathbb{Y}_{12}.\end{aligned} \tag{4.7}$$

Using Höld-Ineq, Jen-Ineq, and ($\mathfrak{C}_1$) on $\mathbb{Y}_{11}$, we get the following result:

$$\begin{aligned}\mathbb{Y}_{11} &\leqslant 2^{2p-2}\varepsilon^{\phi p}\left(\int_0^{\mathfrak{a}} \ell^{\frac{(\phi-1)p}{p-1}} d\ell\right)^{p-1} \mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\int_0^{\tau} \|\Lambda(\ell,\omega_\varepsilon(\ell),\omega_\varepsilon(\ell\zeta)) - \Lambda(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta))\|^p d\ell\right]\\ &\leqslant 2^{3p-3}\varepsilon^{\phi p}\mathfrak{a}^{\phi p-1}\mathscr{U}_2^p\left(\frac{p-1}{\phi p-1}\right)^{p-1}\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\int_0^{\tau}\|\omega_\varepsilon(\ell)-\omega^*_\varepsilon(\ell)\|^p d\ell\right]\\ &\quad+\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\int_0^{\tau}\|\omega_\varepsilon(\ell\zeta)-\omega^*_\varepsilon(\ell\zeta)\|^p d\ell\right]\\ &= \mathscr{D}_{11}\varepsilon^{\phi p}\mathfrak{a}^{\phi p-1}\left(\int_0^{\mathfrak{a}}\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho)-\omega^*_\varepsilon(\rho)\|^p\right]d\ell + \int_0^{\mathfrak{a}}\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho\zeta)-\omega^*_\varepsilon(\rho\zeta)\|^p\right]d\ell\right),\end{aligned} \tag{4.8}$$

where $\mathscr{D}_{11} = 2^{3p-3}\mathscr{U}_2^p\left(\frac{p-1}{\phi p-1}\right)^{p-1}$. Using Höld-Ineq, Jen-Ineq, and ($\mathfrak{C}_3$) on $\mathbb{Y}_{12}$, we get the following result:

$$\begin{aligned}\mathbb{Y}_{12} &\leqslant 2^{2p-2}\varepsilon^{\phi p}\Big(\int_0^{\mathfrak{a}}\ell^{\frac{(\phi-1)p}{p-1}}d\ell\Big)^{p-1}\mathbf{E}\Big[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\int_0^{\tau}\Big\|\Lambda\big(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta)\big)-\widetilde{\Lambda}\big(\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta)\big)\Big\|^p d\ell\Big]\\ &\leqslant 2^{2p-2}\varepsilon^{\phi p}\left(\frac{p-1}{\phi p-1}\right)^{p-1}\mathfrak{a}^{\phi p}\mathscr{Y}_1(\mathfrak{a})\big(1+\mathbf{E}\big\|\omega^*_\varepsilon(\ell)\big\|^p+\mathbf{E}\big\|\omega^*_\varepsilon(\ell\zeta)\big\|^p\big) = \mathscr{D}_{12}\varepsilon^{\phi p}\mathfrak{a}^{\phi p},\end{aligned} \tag{4.9}$$

where $\mathscr{D}_{12} = 2^{2p-2}\mathscr{Y}_1(\mathfrak{a})\left(1+\mathbf{E}\|\omega^*_\varepsilon(\ell)\|^p+\mathbf{E}\big\|\omega^*_\varepsilon(\ell\zeta)\big\|^p\right)\left(\frac{p-1}{\phi p-1}\right)^{p-1}$. Through the use of Jen-Ineq, $\mathbb{Y}_2$ provides the following:

$$\begin{aligned}\mathbb{Y}_2 &\leqslant 2^{2p-2}\varepsilon^{(\phi-\frac{1}{2})p}\left(\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\left\|\int_0^{\tau}\ell^{\phi-1}\Big[\delta\big(\ell,\omega_\varepsilon(\ell),\omega_\varepsilon(\ell\zeta)\big)-\delta\big(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta)\big)\Big]d\mathbb{W}_\ell\right\|^p\right]\right)\\ &\quad+2^{2p-2}\varepsilon^{(\phi-\frac{1}{2})p}\left(\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\left\|\int_0^{\tau}\ell^{\phi-1}\Big[\delta\big(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta)\big)-\widetilde{\delta}\big(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta)\big)\Big]d\mathbb{W}_\ell\right\|^p\right]\right)\\ &= \mathbb{Y}_{21}+\mathbb{Y}_{22}.\end{aligned}$$

Using ($\mathfrak{C}_1$), Höld-Ineq, and BHDG-Ineq on $\mathbb{Y}_{21}$, we achieve the following outcomes:

$$\begin{aligned}\mathbb{Y}_{21} &\leqslant 2^{2p-2}\varepsilon^{(\phi-\frac{1}{2})p}\big(2^{-1}(p-1)^{1-p}p^{p+1}\big)^{\frac{p}{2}}\\ &\quad\times\mathbf{E}\left[\int_0^{\mathfrak{a}}\ell^{2\phi-2}\|\delta(\ell,\omega_\varepsilon(\ell),\omega_\varepsilon(\ell\zeta))-\delta(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta))\|^2 d\ell\right]^{\frac{p}{2}}\\ &\leqslant 2^{2p-2}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\left(p^{p+1}2^{-1}(p-1)^{1-p}\right)^{\frac{p}{2}}\\ &\quad\times\mathbf{E}\left[\int_0^{\mathfrak{a}}\ell^{(\phi-1)p}\|\delta(\ell,\omega_\varepsilon(\ell),\omega_\varepsilon(\ell\zeta))-\delta(\ell,\omega^*_\varepsilon(\ell),\omega^*_\varepsilon(\ell\zeta))\|^p d\ell\right]\\ &\leqslant 2^{3p-3}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\mathscr{U}_2^p\big(p^{p+1}2^{-1}(p-1)^{1-p}\big)^{\frac{p}{2}}\\ &\quad\times\int_0^{\mathfrak{a}}\ell^{(\phi-1)p}\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\left[\|\omega_\varepsilon(\rho)-\omega^*_\varepsilon(\rho)\|^p+\|\omega_\varepsilon(\rho\zeta)-\omega^*_\varepsilon(\rho\zeta)\|^p\right]d\ell\right]\\ &= \mathscr{D}_{21}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\left(\int_0^{\mathfrak{a}}\ell^{(\phi-1)p}\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho)-\omega^*_\varepsilon(\rho)\|^p d\ell\right]\right.\\ &\quad\left.+\int_0^{\mathfrak{a}}\ell^{(\phi-1)p}\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho\zeta)-\omega^*_\varepsilon(\rho\zeta)\|^p d\ell\right]\right),\end{aligned} \tag{4.10}$$

where $\mathscr{D}_{21} = 2^{3p-3}\mathscr{U}_2^p\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}$. Again using Höld-Ineq and BHDG-Ineq on $\mathbb{Y}_{22}$, we achieve the following outcomes:

$$\begin{aligned}
\mathbb{Y}_{22} &\leqslant 2^{2p-2}\left(2^{-1}(p-1)^{1-p}p^{p+1}\right)^{\frac{p}{2}}\varepsilon^{(\phi-\frac{1}{2})p} \\
&\quad\times \mathbf{E}\left[\int_0^{\mathfrak{a}}\left\|\Lambda(\ell,\omega_\varepsilon^*(\ell),\omega_\varepsilon^*(\ell\zeta)) - \widetilde{\delta}(\ell,\omega_\varepsilon^*(\ell),\omega_\varepsilon^*(\ell\zeta))\right\|^2 \ell^{2\phi-2}d\ell\right]^{\frac{p}{2}} \\
&\leqslant 2^{2p-2}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\left(2^{-1}(p-1)^{1-p}p^{p+1}\right)^{\frac{p}{2}} \\
&\quad\times \mathbf{E}\left[\int_0^{\mathfrak{a}}\ell^{(\phi-1)p}\left(\|\delta(\ell,\omega_\varepsilon^*(\ell),\omega_\varepsilon^*(\ell\zeta))\|^p + \left\|\widetilde{\delta}(\omega_\varepsilon^*(\ell),\omega_\varepsilon^*(\ell\zeta))\right\|^p\right)d\ell\right] \\
&\leqslant \frac{2^{3p-3}3^{p-1}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\phi p-\frac{p}{2}}\mathscr{U}_2^p\left(\mathscr{U}_2^p+\mathscr{U}_3\right)^p}{((\phi-1)p+1)}\left(2^{-1}(p-1)^{1-p}p^{p+1}\right)^{\frac{p}{2}} \\
&\quad\times\left(1+\mathbf{E}[\|\omega_\varepsilon^*(\ell)\|^p]+\mathbf{E}[\|\omega_\varepsilon^*(\ell\zeta)\|^p]\right) = \mathscr{D}_{22}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\phi p-\frac{p}{2}},
\end{aligned} \tag{4.11}$$

where $\mathscr{D}_{22} = 2^{3p-3}3^{p-1}\mathscr{U}_2^p\left(\mathscr{U}_2^p+\mathscr{U}_3\right)^p\left(2^{-1}(p-1)^{1-p}p^{p+1}\right)^{\frac{p}{2}}\left(1+\mathbf{E}[\|\omega_\varepsilon^*(\ell)\|^p]+\mathbf{E}[\|\omega_\varepsilon^*(\ell\zeta)\|^p]\right)$. By utilizing Eqs. (4.7)-(4.11) in (4.6), as a result, we get the following outcomes:

$$\begin{aligned}
&\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\|^p\right] \\
&\leqslant \mathscr{D}_{12}\varepsilon^{\phi p}\mathfrak{a}^{\phi p} + \mathscr{D}_{22}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\phi p-\frac{p}{2}} \\
&\quad+\int_0^{\mathfrak{a}}\left[\mathscr{D}_{11}\varepsilon^{\phi p}\mathfrak{a}^{\phi p-1} + \mathscr{D}_{21}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\ell^{(\phi-1)p}\right]\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho)-\omega_\varepsilon^*(\rho)\|^p\,d\ell\right] \\
&\quad+\int_0^{\mathfrak{a}}\left[\mathscr{D}_{11}\varepsilon^{\phi p}\mathfrak{a}^{\phi p-1} + \mathscr{D}_{21}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\frac{p}{2}-1}\ell^{(\phi-1)p}\right]\mathbf{E}\left[\sup_{0\leqslant\rho\leqslant\ell}\|\omega_\varepsilon(\rho\zeta)-\omega_\varepsilon^*(\rho\zeta)\|^p\,d\ell\right].
\end{aligned} \tag{4.12}$$

Consequently, we obtain the subsequent outcome from Eq. (4.12):

$$\begin{aligned}
&\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\mathfrak{a}}\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\|^p\right] \\
&\leqslant\left(\mathscr{D}_{12}\varepsilon^{\phi p}\mathfrak{a}^{\phi p} + \mathscr{D}_{22}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\phi p-\frac{p}{2}}\right)\exp\left(2\mathscr{D}_{11}\varepsilon^{\phi p}\mathfrak{a}^{\phi p} + \frac{2\mathscr{D}_{21}}{(\phi-1)p+1}\varepsilon^{(\phi-\frac{1}{2})p}\mathfrak{a}^{\phi p-\frac{p}{2}}\right).
\end{aligned}$$

This implies that for any $\forall\tau\in[0,\varphi\varepsilon^{-\chi}]\subseteq[0,\mathbb{T}]$, there are $\varphi>0$ and $\chi\in(0,1)$ as well

$$\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\varphi\varepsilon^{-\chi}}\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\|^p\right]\leqslant\mathcal{Z}\varepsilon^{1-\chi},$$

where

$$\begin{aligned}
\mathcal{Z} = &\left(\mathscr{D}_{12}\varphi^{\phi p}\varepsilon^{\phi p+\chi-\phi\chi p-1} + \mathscr{D}_{22}\varphi^{\phi p-\frac{p}{2}}\varepsilon^{\frac{p}{2}(\chi-1)+\phi p+\chi-\phi\chi p-1}\right) \\
&\times\exp\left[2\mathscr{D}_{11}\varphi^{\phi p}\varepsilon^{\phi p-p\phi\chi} + \frac{2\mathscr{D}_{21}}{(\phi-1)p+1}\varphi^{\phi p-\frac{p}{2}}\varepsilon^{\frac{p}{2}(\chi-1)+\phi p-\phi\chi p}\right]
\end{aligned}$$

is a constant. As a result, when $\forall\digamma>0$, finding $\varepsilon_1\in(0,\varepsilon_0]$ that satisfies $\forall\varepsilon\in(0,\varepsilon_1]$ and $\tau\in[0,\varphi\varepsilon^{-\chi}]$

allows us to deduce

$$\mathbf{E}\left[\sup_{0\leqslant\tau\leqslant\varphi\varepsilon^{-\chi}}\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\|^p\right]\leqslant \digamma.$$

□

Corollary 4.3. *Assume that the conditions* ($\mathfrak{C}_1$) *and* ($\mathfrak{C}_2$) *are valid. Considering any arbitrary number* $\digamma_1>0$*, the subsequent criteria are established:* $\chi\in(0,1)$*,* $\varphi>0$*, and* $\varepsilon_1\in(0,\varepsilon_0)$ *occur for* $\forall\varepsilon\in(0,\varepsilon_1]$*, we possess*

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(\sup_{\tau\in[0,\varphi\varepsilon^{-\chi}]}\big\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\big\|>\digamma_1\Big)=0.$$

Proof. Using the Chebyshev-Markov inequality and Theorem 4.2, one can deduce the following for any number $\digamma_1>0$:

$$\mathbb{P}\Big[\sup_{\tau\in[0,\varphi\varepsilon^{-\chi}]}\big\|\omega_\varepsilon(\tau)-\omega_\varepsilon^*(\tau)\big\|>\digamma_1\Big]\leqslant\frac{1}{\digamma_1^2}\mathbf{E}\Big[\sup_{\tau\in[0,\varphi\varepsilon^{-\chi}]}\big\|\omega_\varepsilon(\ell)-\omega_\varepsilon^*(\ell)\big\|^2\Big]\leqslant\frac{\Psi\varepsilon^{1-\chi}}{\digamma_1^2}\leqslant 0\ \text{ as }\ \varepsilon\to 0,$$

where

$$\Psi=\left(\mathscr{D}_{12}\varphi^{\phi p}\varepsilon^{\phi p+\chi-\phi\chi p-1}+\mathscr{D}_{22}\varphi^{\phi p-\frac{p}{2}}\varepsilon^{\frac{p}{2}(\chi-1)+\phi p+\chi-\phi\chi p-1}\right)\times\exp\left[2\mathscr{D}_{11}\varphi^{\phi p}\varepsilon^{\phi p-p\phi\chi}+\frac{2\mathscr{D}_{21}}{(\phi-1)p+1}\varphi^{\phi p-\frac{p}{2}}\varepsilon^{\frac{p}{2}(\chi-1)+\phi p-\phi\chi p}\right].$$

It ends the proof. □

The results developed in our research work to analyze CFSPDE can be extended and adapted to analyze more complex systems beyond CFSPDE. Here are some potential extensions or adaptations.

i. Higher-order Fractional Derivatives: While CFSPDEs involve fractional derivatives, the methods developed can be extended to analyze systems with higher-order fractional derivatives.

ii. Multiscale and Multiphysics Systems: Many real-world systems exhibit multiscale or multiphysics behavior, where multiple processes operate simultaneously at different scales or interact through complex couplings. The methods developed for CFSPDE can be extended to analyze such systems by incorporating multiple fractional processes, coupling terms, or additional physics. This may involve developing hybrid numerical methods or multiscale modeling approaches to capture the system's behavior accurately.

iii. Nonlinear Dynamics and Control: The methods developed for analyzing CFSPDE can be extended to study nonlinear dynamical systems and control problems. This includes systems with nonlinearities, bifurcations, chaos, or complex feedback control mechanisms. Numerical techniques such as bifurcation analysis, stability analysis, or optimal control methods can be applied to study the behavior of nonlinear systems and design control strategies.

iv. Networked and Distributed Systems: Many systems in science and engineering are networked or distributed, involving interactions between multiple interconnected components or agents. The methods developed for CFSPDEs can be adapted to analyze such systems by modeling the interactions between components using fractional operators or stochastic processes. This may include studying synchronization phenomena, collective behavior, or emergent properties in networked systems.

v. Data-driven and Machine Learning Approaches: Data-driven and machine learning approaches can complement traditional analytical and numerical methods for analyzing complex systems. The methods developed for CFSPDE can be combined with data-driven techniques, such as deep learning or reinforcement learning, to analyze large-scale or high-dimensional systems, infer underlying dynamics from observational data, or optimize system performance.

vi. Applications in Biology, Finance, and Engineering: The methods developed for analyzing CFSPDE can be applied to study complex systems in various fields, including biology, finance, and engineering. This may involve adapting the methods to model specific phenomena or processes in these domains, such as biological signaling networks, financial markets, or dynamical systems in engineering.

In the section that follows, we give three examples to show how our presented outcome is valuable.

5. Examples

The average system of a complicated system can be obtained using the average principle result, as shown by the three numerical examples. Numerical techniques play a crucial role in complementing theoretical findings in solving real-world challenges modeled with CFSPDEs. Here's how they complement each other.

i. Validation of Theoretical Results: Numerical techniques provide a means to validate theoretical findings obtained from analytical or semi-analytical approaches. By implementing numerical simulations of CFSPDEs, researchers can verify the behavior of solutions predicted by theoretical models under various conditions and parameter settings. Consistency between theoretical predictions and numerical results enhances confidence in the validity of theoretical findings.
ii. Exploration of Complex Scenarios: Real-world challenges often involve complex scenarios that may not be amenable to analytical solutions. Numerical techniques enable researchers to explore the behavior of CFSPDE in such scenarios by simulating the dynamics of the system over time. This allows for the investigation of how different factors and uncertainties affect the solutions and provides insights that may not be obtainable through purely theoretical analysis.
iii. Parameter Estimation and Sensitivity Analysis: Numerical techniques facilitate parameter estimation and sensitivity analysis for CFSPDE in real-world applications. By fitting numerical simulations to observed data, researchers can estimate unknown parameters in the model and assess their sensitivity to variations in these parameters. This information is valuable for understanding the robustness of theoretical predictions and calibrating models to real-world observations.
iv. Prediction and Forecasting: Numerical simulations enable the prediction and forecasting of future behavior in real-world systems modeled with CFSPDE. By extrapolating solutions obtained from numerical simulations, researchers can anticipate how the system will evolve over time and make informed decisions or interventions accordingly. This predictive capability is essential for addressing real-world challenges and for developing strategies to mitigate risks or optimize performance.
v. Handling Nonlinearity and Complexity: Numerical techniques excel at handling the nonlinearity and complexity inherent in real-world systems modeled with CFSPDE. Through numerical simulations, researchers can explore the behavior of highly nonlinear and complex systems, including those with discontinuities, singularities, or multi-scale dynamics. This allows for a more comprehensive understanding of the system's behavior and its response to various external factors and uncertainties.

Example 5.1. Consider the subsequent CFSPDE:

$$\begin{cases} \mathcal{T}_{\tau}^{0.8}\omega_{\varepsilon}(\tau) = 2\varepsilon^{\phi}\cos^2(\tau)\omega_{\varepsilon}(\tau) - \varepsilon^{\phi}\omega_{\varepsilon}\sin^2(\frac{1}{2}\tau) + \varepsilon^{\phi-\frac{1}{2}}\frac{d\mathbb{W}_{\tau}}{d\tau},\ \tau \in [0,\pi], \\ \omega(0) = \boldsymbol{\mathfrak{X}}_0, \end{cases}$$

where $\phi = 0.8$,

$$\Lambda(\tau,\omega(\tau),\omega(\tau\zeta)) = 2\varepsilon\cos^2(\tau)\omega_{\varepsilon}(\tau) - \varepsilon\omega_{\varepsilon}\sin^2\Big(\frac{1}{2}\tau\Big), \qquad \delta(\tau,\omega(\tau),\omega(\tau\zeta)) = 1.$$

Then,

$$\widetilde{\Lambda}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\int_0^{\pi}\left(2\varepsilon\cos^2(\tau)\omega_\varepsilon(\tau) - \varepsilon\omega_\varepsilon\sin^2\left(\frac{1}{2}\tau\right)\right)d\tau = \frac{1}{2}\omega_\varepsilon(\tau),$$
$$\widetilde{\delta}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\int_0^{\pi} 1 d\tau = 1.$$

Thus, we have the corresponding averaged CFSPDE

$$\begin{cases} \mathcal{T}_\tau^{0.8}\omega_\varepsilon^*(\tau) = \frac{1}{2}\varepsilon^{\phi}\omega_\varepsilon(\tau) + \varepsilon^{\phi-\frac{1}{2}}\frac{d\mathbb{W}_\tau}{d\tau}, \\ \omega(0) = \boldsymbol{\varkappa}_0. \end{cases}$$

Example 5.2. Consider the subsequent CFSPDE:

$$\begin{cases} \mathcal{T}_\tau^{0.8}\omega_\varepsilon(\tau) = \varepsilon^{\phi}\sin^2(\tau)\omega_\varepsilon(\tau\zeta) + \varepsilon^{\phi-\frac{1}{2}}\sin\left(\omega_\varepsilon(\tau)\right)\frac{d\mathbb{W}_\tau}{d\tau}, \ \tau \in [0, \pi], \\ \omega(0) = \boldsymbol{\varkappa}_0, \end{cases}$$

where $\phi = 0.8$,

$$\Lambda(\tau, \omega(\tau), \omega(\tau\zeta)) = \sin^2(\tau)\omega_\varepsilon(\tau\zeta), \quad \delta(\tau, \omega(\tau), \omega(\tau\zeta)) = \sin(\omega_\varepsilon(\tau)).$$

Then,

$$\widetilde{\Lambda}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\int_0^{\pi}\sin^2(\lambda)\omega_\varepsilon(\lambda\zeta)d\lambda = \frac{1}{2}\omega_\varepsilon^*(\tau\zeta),$$
$$\widetilde{\delta}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\int_0^{\pi}\sin(\omega_\varepsilon(\tau))d\lambda = \sin(\omega_\varepsilon(\tau)).$$

Thus, we have the corresponding averaged CFSPDE

$$\begin{cases} \mathcal{T}_\tau^{0.8}\omega_\varepsilon^*(\tau) = \frac{1}{2}\varepsilon^{\phi}\omega_\varepsilon^*(\tau\zeta) + \varepsilon^{\phi-\frac{1}{2}}\sin\left(\omega_\varepsilon^*(\tau)\right)\frac{d\mathbb{W}_\tau}{d\tau}, \ \tau \in [0, \pi], \\ \omega(0) = \boldsymbol{\varkappa}_0. \end{cases}$$

Example 5.3. Consider the subsequent CFSPDE:

$$\begin{cases} \mathcal{T}_\tau^{0.9}\omega_\varepsilon(\tau) = \frac{1}{2}\varepsilon^{\phi}\omega_\varepsilon(\tau\zeta) + \frac{3\pi}{4}\varepsilon^{\phi-\frac{1}{2}}\sin^3\tau.\omega_\varepsilon(\tau)\frac{d\mathbb{W}_\tau}{d\tau}, \ \tau \in [0, \pi], \\ \omega(0) = \boldsymbol{\varkappa}_0, \end{cases}$$

where $\phi = 0.9$,

$$\Lambda(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{2}\omega_\varepsilon(\tau\zeta), \quad \delta(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{3\pi}{4}\sin^3\tau.\omega_\varepsilon(\tau).$$

Then,

$$\widetilde{\Lambda}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\omega_\varepsilon(\lambda\zeta)d\lambda = \frac{1}{2}\omega_\varepsilon^*(\tau\zeta),$$
$$\widetilde{\delta}(\tau, \omega(\tau), \omega(\tau\zeta)) = \frac{1}{\pi}\frac{3\pi}{4}\int_0^{\pi}\sin^3\tau.\omega_\varepsilon(\tau)d\lambda = \omega_\varepsilon^*(\tau).$$

Thus, we have the corresponding averaged CFSPDE

$$\begin{cases} \mathcal{T}_\tau^{0.9}\omega_\varepsilon^*(\tau) = \frac{1}{2}\varepsilon^{\phi}\omega_\varepsilon^*(\tau\zeta) + \varepsilon^{\phi-\frac{1}{2}}\omega_\varepsilon^*(\tau)\frac{d\mathbb{W}_\tau}{d\tau}, \ \tau \in [0, \pi], \\ \omega(0) = \boldsymbol{\varkappa}_0. \end{cases}$$

6. Conclusions

In this study, we extended the findings regarding the existence, uniqueness, continuous dependency, regularity of solutions to CFSPDEs, and the averaging principle within the $\mathbf{L}^p$ realm. We employed the notion of a contraction map to explore the existence and uniqueness of the aforementioned problem. This work uses multiple inequalities and an interval translation approach to illustrate the averaging principle for CFSPDEs in the $\mathbf{L}^p$ space. Finally, three instances are executed in order to understand the determined results and illustrate the effectiveness of our findings.

In our upcoming work, we will apply numerical methodologies to solve several types of real-world problems depicted by CFSPDEs.

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Authors' contributions

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Competing interests

The authors declare that they have no competing interests.

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