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Analyzing solution of the fractional Lorenz system and fractional Riccati equation via the modified Laplace power series residual method

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Abstract

The series approach is commonly used to obtain approximate analytic solutions for differential equations, but it often converges for a short time. To address this limitation, a new algorithm has been developed that enables the solution to be carried out over a longer period. The Laplace residual power series method (LRPSM) is a technique that generates a solution for fractional differential equations in terms of FOPS via simulation generalized Taylors' series in the Laplace space. To apply the LRPSM for a long time space, a new Modified LRPSM (MLRPSM) algorithm is introduced which divides the time into shorter intervals and applies the LRPSM to each interval. The algorithm investigates the continuity of the solution to ensure that the obtained solution for each interval is smoothly connected to the solution for the previous interval. The effectiveness of the proposed algorithm is demonstrated through its application to the Riccati equation and the Lorenz chaotic system. To comprehend the physical features of studied models, the 2D, and 3D graphical representations of the acquired results for some parameters had been drawn. Especially, at the critical value of the fractional derivative, which marks the transition of the solution behavior for the Lorenz system from a chaotic to a non-chaotic attractor. The efficacy, accuracy, and feasibility of this technique are verified numerically. From this viewpoint, the simulations of gained results indicate that the future iterative technique is indeed robust, effective, and convenient in gaining the approximation solutions over a longer period of a wide range of linear and nonlinear fractional physical problems.

Keywords: Fractional calculus, Laplace transform, residual power series, multi-stage method, fractional Riccati equation, fractional chaotic system.

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1. Introduction

The fractional calculus topic (FC) has witnessed fast growth in the last three decades at theoretical and practical levels, where the theory of FC is considered a significant instrument of applied mathematics for studying the integration and differentiation operators of arbitrary order and generalization of the traditional calculus operators. An enormous number of worthy efforts have been carried out on the theoretical fields of FC as well as its practical prominence [15, 21, 24–27]. The merit of utilizing the

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fractional derivative (FD) in modeling dynamical systems against the ordinary derivative (OD) is that the FD is globally in nature, while the OD is locally in nature. This behavior of FD is convenient and reasonable in modeling physical problems which provides an ideal support to describe the memory and hereditary features of diverse processes and materials. Due to the non-local features of dynamical systems of fractional problems, great attention has been paid to describing their hereditary properties, and memories as well as finding and developing instruments that would control the dynamic behavior of resulting solutions for FDs [4, 22, 36, 41, 47]. In this orientation, numerous attractive notions are introduced in terms of FDs by many analysts, including Atangana-Baleanu, Weyl, Grunwald-Letnikov, Riesz, Caputo-Fabrizio, Riemann–Liouville, and Caputo derivatives, which allow accurate characterize dynamical systems with memory relying on the generalized Mittag-Leffer function, singular kernel, non-local kernel, and exponential kernel [8, 14, 17, 29, 31].

Lastly, both linear and non-linear fractional differential equations (FDEs) have gained considerable attention in modeling and simulating dynamic systems due to their importance in providing accurate descriptions and predictions of various dynamic fractional systems and their sake of better understanding of the complex situations of these systems, such as viscoelasticity, signal processing, rheology, entropy generations, fluid flow, nonlinear optics [5, 16, 33, 37, 46, 48], and so on. The solutions of FDEs are essential and profitable for the physical understanding of the applications as mentioned earlier and others [2, 9, 19, 40, 42, 43, 49]. However, closed-form solutions to these equations are not always available, so approximate solutions are often necessary. Several numerical methods have been proposed for solving FDEs, including the Generalized Adam Bashforth method [35] and the Grunwald-Letnikov approximation method [38], which is a finite difference method that approximates the FD using a truncated series expansion. Additionally, many methods for generating approximate analytic solutions to FDEs rely on series expansion techniques. For instance, the power series method represents the solution in the form $\sum_{i=0}^{M} c_i (t-t_0)^i$, while the Adomian decomposition method (ADM) [18] and the homotopy analysis method (HAM) [1] assume the solution can be expressed as $\sum_{i=0}^{M} u_i(t)$.

In general, the series solution will converge for a short domain that is near the initial point. This is one of the limitation points of the methods. So finding the approximate series solution which converges for a long time is one of the biggest challenges in this topic. Several algorithms were proposed to overcome this issue such as the Pade approximation method [13] which assumes facilitate the convergence of the method, and the multistage method [45] which proposes that the time can be divided into several subintervals and then applying the algorithm for each one by observing the continuity condition.

Finding an accurate approximate continuous solution for a non-linear FDE has had great attention in the last century. So, the researchers suggested several methods using different approaches such as pure numerical or approximate analytic types. One of the effective methods for solving FDE is the traditional residual power series method (RPSM) which presents the solution in terms of fractional-order power series (FOPS) after collecting for the unknown coefficients of the suggested series by applying $(n-1)\alpha$-th Caputo-FD of residual-error function. To overcome this limitation, the Laplace RPSM (LRPSM) has been introduced and proved in [23]. LRPSM presents the solution in terms of fractional-order power series (FOPS) with an easy recurrence formula for the coefficients based on the limit idea notion at infinity. This method is a suitable alternative scheme that relies on the simulation of the generalized Taylor's formula of the posed model in Laplace space. The method explores the exact and accurate approximate continuous solutions for a wide range of complex non-linear FDEs arising in numerous systems in engineering and natural science, see [3, 6, 9, 10, 12, 39] for more details.

Most real-world problems are influenced by a variety of external factors that change their behavior and make it more complex and unpredictable, FDEs provide a theoretical framework for adequately describing the nature of these problems. Therefore, the numerical approximation methods are considering a suitable manner to handle these situations to develop a model that provides a workable solution, since this approximate analytic solution in most cases is valued for a short domain, we intend in the current analysis to build a modified LRPSM (MLRPSM) based on the multistage method to provide a convenient methodology that generates a piecewise continuous solution for the FDE for a long time with high accuracy. The modified algorithm is valued for a long domain and the approximation is continuous along its domain. Moreover, we discuss the solution of the fractional Riccati equation and the chaotic fractional Lorenz system with unequal fractional order and compute the residual error.

The non-linear Riccati equation has several applications in science and engineering, such as financial mathematics, stochastic realization theory, and optimal control [28]. Simplified models of lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions, forward osmosis, and electric circuits can all give birth to the Lorenz equations [32]. Thus, finding accurate approximate analytic solutions for general derivatives for those models will highly contribute to the fields that use those equations.

The current analysis is arranged as a quick overview of FD in the Caputo framework notion and the main results relating to LT and LRPSM are presented in Section 2. The principle of implementing the modified algorithm for solving non-linear FDE is presented in Section 3. In Section 4, two attractive non-linear applications of FDEs are stated to demonstrate the accuracy and applicability of our scheme. Finally, the conclusion is presented in Section 5.

2. Preliminary and notations

The Caputo fractional differentiation (FD) operator, introduced by Michel Caputo in the late 1960s [10], is a mathematical tool designed for fractional order differentiation. It is widely employed in diverse fields such as optics, mechanics, control theory, biology, and physics. The Caputo FD operator is a potent instrument for providing precise descriptions and analyzing phenomena exhibiting memory or non-local behaviors. This operator is recognized as one of the most commonly used FD operators and distinguishes itself from others found in the literature by addressing a broad spectrum of fractional differential equations (FDEs), particularly when considering the FD in the traditional sense for the initial point [16, 19, 33]. This section aims to emphasize the fundamental definition of the Caputo FD operator and additionally provides an overview of the foundational principles of the fractional LRPSM sense.

The Riemann-Liouville integral operator of order $\alpha > 0$ for a continuous function $f(t)$ is defined by

$$(I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x)}{(t-x)^{1-\alpha}}dx.$$

While the α order fractional derivative in the Caputo sense is given by

$$({}_aD^{\alpha}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{n-\alpha-1}f^{(n)}(x)dx.$$

provided the integration exists. The Laplace transform of the piecewise continuous function $f(t)$ on $[0,\infty)$ is given by

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt,$$

and the inverse LT of the transform function $F(s)$ is defined by

$$L^{-1}\{F(s)\} = f(t) = \int_{\vartheta-i\infty}^{\vartheta+i\infty} F(s)e^{st}ds.$$

The LT of the piecewise continuous functions $L\{x(t)\} = X(s)$ and $L\{y(t) = Y(s)\}$ for $t \in [0,\infty)$ have the following properties.

- $\lim\limits_{s\to\infty} sY(s) = y(0)$.
- $L\{({}_0D^{\alpha}y)(t)\} = s^{\alpha}Y(s) - \sum\limits_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0)$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$.

- $L\left\{({}_0D^{k\alpha}y)(t)\right\} = s^{k\alpha}Y(s) - \sum_{j=0}^{k-1} s^{(k-j)\alpha-1}({}_0D^{j\alpha}y)(0)$, for $\alpha \in (0,1]$.

Assuming that $(D^{n\alpha}f)(0)$ exist for $n = 1,2,\dots$, then $F(s) = L\{f(t)\}$ can be written in the following series expansion:

$$F(s) = \sum_{n=0}^{\infty} \frac{d_n}{s^{n\alpha+1}}, \quad s > 0, \alpha \in (0,1], \tag{2.1}$$

where $d_n = ({}_0D^{n\alpha}f)(0)$.

Theorem 2.1 ([10]). *Whenever the function* $F(s) = L\{f(t)\}$ *can be expressed in Eq.* (2.1) *with* $\left|sL\left[D_t^{(n+1)\alpha}f(t)\right]\right| \leqslant \pounds$, *where* $0 < \alpha \leqslant 1$, *then the remaining part of the new expansion shape in* (2.1) *satisfies the following inequality:*

$$|R_n(s)| \leqslant \frac{\pounds}{s^{1+(n+1)\alpha}}, \quad 0 < s \leqslant \xi.$$

3. Principle of multi-stage fractional LRPSM

The residual power series (RPS) approach is an efficient and convenient numeric-analytic technique to create approximate fractional-order power series (FOPS) solutions of differential, integral, and integro-differential equations involving FDs. It has shown its accuracy and applicability in handling numerous kinds of FDEs, which provides a straightforward mechanism to find out the unknown terms of the suggested FOPS solution [7, 20, 34]. For some cases, especially non-linear fractional problems, the procedures of obtaining the unknown terms of FOPS and closed-form solutions is very hard work. So, to overcome this defect of RPS approach, the authors in [23] have compiled the Laplace transform (LT) approach with the simulation of RPS approach via implementing the infinite limit notion to determine the expansion of unknown terms instead of the idea of FD of fractional-error function as in RPS approach. The new recommended method, LRPSM, confirmed the speed and simplicity of its methodology to create the FOPS approximate solutions, and that makes an alternative suitable technique to treat a wide range of non-linear fractional physical applications. This segment introduces a modified algorithm to acquire continuous approximate solutions over a longer domain for certain fractional models subject to proper initial points.

Remark 3.1. Because $({}_a\mathbb{D}^{\alpha}y)(t) = ({}_0\mathbb{D}^{\alpha}y)(t-a)$, then the solution of

$$({}_a\mathbb{D}^{\alpha}y)(t) = f(t,y),\ t > a,\ y(a) = y_0,$$

is given by $y(t) = v(u-a)$, where $v(u)$ is a solution to

$$({}_0\mathbb{D}^{\alpha}v)(u) = f(u+a,v),\ u > 0,\ v(0) = y_0.$$

We consider the nonlinear fractional initial value problem (F.I.V.P)

$$({}_0\mathbb{D}^{\alpha}y)(t) = f(t,y),\ 0 < t \leqslant T,\ y(0) = y_0. \tag{3.1}$$

In the following, we present a new algorithm to solve the F.I.V.P (3.1).

Step 1. Divide the interval $[0,T]$ into n sub-intervals $[0,t_1],\dots,[t_{i-1},t_t],\dots,[t_{n-1},t_n = T]$.

Step 2. On the sub-interval $[t_{i-1},t_i]$, we consider $y_i(t)$ to be the solution of

$$({}_{t_{i-1}}\mathbb{D}^{\alpha}y_i)(t) = f(t,y_i(t)), \quad y_i(t_{i-1}) = c_{i-1}, \quad t_{i-1} \leqslant t \leqslant t_i, \tag{3.2}$$

and $y_i(t) = c_{i-1}$ for $t \in [0,t_{i-1}]$, where $c_0 = y_0$, and $c_i = y_i(t_{i-1}), 1 \leqslant i \leqslant n$.

Step 3. Based on the short note in Remark 3.1 the solution of Eq. (3.2) is equivalent to the solution of

$$({}_0\mathbb{D}^{\alpha}\nu_i)(u) = f(u+t_{i-1},\nu_i(u)), \quad \nu_i(0) = c_{i-1}, \tag{3.3}$$

where $y_i(t) = \nu_i(u-t_{i-1})$.

Step 4. Applying the Laplace transform to Eq. (3.3), we have

$$V_i(s) = \frac{\nu_i(0)}{s} + \frac{1}{s^{\alpha}} L[f(u+t_{i-1},\nu_i)(u)],$$

where $V_i(s) = L(\nu_i)(u)$.

Step 5. Substitute $V_i(s) = \sum_{i=0}^{j} \frac{\nu_i}{s^{\alpha i+1}}$ into the residual equation

$$\text{Res}_j = \sum_{i=1}^{j} \frac{\nu_i}{s^{\alpha i+1}} - \frac{1}{s^{\alpha}} L\left[f\left(u+t_{i-1}, L^{-1}\left[\sum_{i=0}^{j} \frac{\nu_i}{s^{\alpha i+1}}\right]\right)\right].$$

Step 6. Assume $L\left[f\left(u+t_{i-1}, L^{-1}\left[\sum_{i=0}^{j} \frac{\nu_i}{s^{\alpha i+1}}\right]\right)\right]$ can be written as

$$L\left[f\left(u+t_{i-1}, L^{-1}\left[\sum_{i=0}^{j} \frac{\nu_i}{s^{\alpha i+1}}\right]\right)\right] = \sum_{i=0}^{M} \frac{g_i}{s^{\alpha i+1}}, \quad M \geqslant j.$$

Step 7. Take the following limit

$$\lim_{s\to\infty} s^{j\alpha+1}\text{Res}_j = \sum_{i=1}^{j} \frac{\nu_i}{s^{\alpha(i-j)}} - \sum_{i=0}^{M} \frac{g_i}{s^{\alpha(i-j+1)}} = 0,$$

for $j = 1, 2, \ldots$, which gives $\nu_j = g_{j-i}$.

Step 8. Write the solution as

$$\nu_i(u) = L^{-1}\left[\sum_{i=0}^{\infty} \frac{\nu_i}{s^{\alpha i+1}}\right] = \sum_{i=0}^{\infty} \frac{\nu_i}{\Gamma[\alpha i+1]} u^{\alpha i+1}.$$

At this line, we can find the solution of $y_i(t) = \nu_i(u-t_{i-1})$.

Remark 3.2. Since $y_i(t) = c_{i-1},\ t \in [0, t_{i-1}]$, then $y_i(t)$ is continuous on $[0, t_i]$. Because $y_i'(t) = 0,\ t \in [0, t_{i-1}]$, then it holds that

$$({}_0\mathbb{D}^{\alpha}y_i)(t) = ({}_{t_{i-1}}\mathbb{D}^{\alpha}y_i)(t).$$

That is $y_i(t)$ is a solution to Eq. (3.1) in the interval $[0, t_i]$.

4. Applications and simulation results

This segment highlights the performance and accuracy of the Multi-stage LRPSM by including some fractional applications in light of Caputo differentiation. The non-linear fractional Riccati equation is investigated in [7, Application 4.1], whereas the fractional Lorenz system is investigated in [11, Application 4.2]. Indeed, replacing the ODs with FDs refines the solution provided by the ordinary Riccati and Loranze models as well as increases their applications in the study of dynamical systems arising in economics, physics, and the science of epidemiology fields. The obtained FOPS approximate solutions in studied applications are compared with each other at varied Caputo-FD values and with the exact solution at OD value for a long time. Further, numerical simulations are carried out to illustrate the accuracy of the recommended algorithm. All symbolic computations and simulations have been done via Mathematica Package 12.

Application 4.1. Consider the following fractional Riccati equation

$$({}_0D^{\alpha}y)(t) = 2y(t) - y^2(t) + 1, \tag{4.1}$$

$t > 0$, $0 < \alpha \leqslant 1$, subject to the initial condition $y(0) = c$. The exact solution of the Riccati equation (4.1), when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left[\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right].$$

By Applying the MLPSM algorithm in Section 4.6, we have following.

- Dividing the domain $[0, T]$ by $h = t_i - t_{i-1}$ into subdomains.
- In the subinterval $[t_{i-1}, t_i]$, Eq. (4.1) becomes

$$({}_{t_{i-1}}D^{\alpha}y_i)(t) = 2y_i(t) - y_i^2(t) + 1,\ y_i(t_{i-1}) = c_{i-1},\ t_{i-1} \leqslant t \leqslant t_i. \tag{4.2}$$

- Equation (4.2) is equivalent to

$$({}_0D^{\alpha}v_i)(u) = 2v_i(u) - v_i^2(u) + 1,\quad v_i(0) = c_{i-1},\quad 0 \leqslant u \leqslant t - t_{i-1}.$$

- Applying Laplace transform to have

$$V_i(s) - \frac{v_i(0)}{s} = \frac{1}{s^{\alpha}}(2V_i(s) - L[L^{-1}[V_i(s)]^2]) + \frac{1}{s^{\alpha+1}}.$$

- Step 5 reads

$$\begin{aligned} L[L^{-1}[V_i(s)]^2]) &= L\left[\sum_{i=0}^{2J} u^{\alpha i} \sum_{n=\max[0,n-J]}^{\min[i,J]} \frac{d_n d_{i-n}}{\Gamma(\alpha n+1)\Gamma(\alpha(i-n)+1)}\right] \\ &= \sum_{i=0}^{2J} \frac{\Gamma(\alpha i+1)}{s^{\alpha i+1}} \sum_{n=\max[0,n-J]}^{\min[i,J]} \frac{d_n d_{i-n}}{\Gamma(\alpha n+1)\Gamma(\alpha(i-n)+1)}, \end{aligned}$$

- Step 6 gives

$$\begin{aligned} \lim_{s\to\infty} s^{\alpha J+1}\mathrm{Res}_J = \lim_{s\to\infty} \Bigg(&\sum_{i=1}^{J} \frac{d_i}{s^{\alpha(i-J)}} - 2\sum_{i=0}^{J} \frac{d_i}{s^{\alpha(i+1-J)}} \\ &+ \sum_{i=0}^{2J} \frac{\Gamma(\alpha i+1)}{s^{\alpha(i+1-J)}} \sum_{n=\max[0,n-J]}^{\min[i,J]} \frac{d_n d_{i-n}}{\Gamma(\alpha n+1)(\Gamma(\alpha(i-n)+1))} - \frac{1}{s^{\alpha(1-J)}}\Bigg) = 0, \end{aligned}$$

This will give us the recurrence relation

$$d_J = 2d_{J-1} + \Gamma(\alpha(J-1)+1) \sum_{n=0}^{J-1} \frac{d_n d_{J-1-n}}{\Gamma(\alpha n+1)(\Gamma(\alpha(J-1-n)+1))} - (1-\chi_J)$$

for $J = 1, 2, \ldots M$, where

$$\chi_J = \begin{cases} 0, & J \leqslant 1, \\ 1, & J > 1. \end{cases}$$

• At this line the approximate solution using M-terms of the series in step 7 is

$$y_i(t) = L^{-1}\left[\sum_{i=0}^{M} \frac{d_i}{s^{\alpha i+1}}\right] = \sum_{i=0}^{M} \frac{(t-t_{i-1})^{\alpha i}}{\Gamma[\alpha i+!]}, \quad t_{i-1} \leqslant t \leqslant t_i. \tag{4.3}$$

By defining the initial condition in the next interval as $y_i(t_i) = c_i = y_{i+1}(t_i)$, the solution will be continuous. If $M = 3$ in Eq. (4.3), we have

$$\begin{aligned} y_i(t) = c_{i-1} &+ \frac{\left(-c_{i-1}^2 + 2c_{i-1} + 1\right)(u - t_{i-1})^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \frac{2\left(c_{i-1}^3 - 3c_{i-1}^2 + c_{i-1} + 1\right)(u - t_{i-1})^{2\alpha}}{\Gamma(2\alpha+1)} - A\,(u - t_{i-1})^{3\alpha}, \end{aligned}$$

where

$$A = \frac{\left((c_{i-1}-2)\,c_{i-1}-1\right)\left(4\Gamma(\alpha+1)^2\,(c_{i-1}-1)^2 + \Gamma(2\alpha+1)\left((c_{i-1}-2)\,c_{i-1}-1\right)\right)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}.$$

Now, we have $c_0 = 0$ in the interval $[0 = t_0, t_1)$ and the solution becomes

$$_0y(t) = t^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\left(4 - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\right)t^{2\alpha}}{\Gamma(3\alpha+1)} + \frac{2t^{\alpha}}{\Gamma(2\alpha+1)}\right), \quad 0 \leqslant t < t_1.$$

In the next interval $[t_1, t_2)$, we start with initial condition $c_1 = {}_1y(t_1) = {}_0y(t_1)$. Following analogous steps, we generate a pairwise continuous solution $y(t)$ along the interval $[0, T]$.

In the following, the geometric behavior of fractional level curves for the gained 7-terms series solution of Application 4.1 at numerous values of Caputo FD α, when $h = 0.002$ are presented in Figure 1 and Table 1, while the residual error for the seventh approximate series solution is computed and displayed in Table 2 when $h = 0.002$ and numerous values of Caputo FD α. From the viewpoint of this simulation, the proposed method is highly accurate in providing a series of approximate solutions that agree with each other and with exact solutions at standard-order of FD.

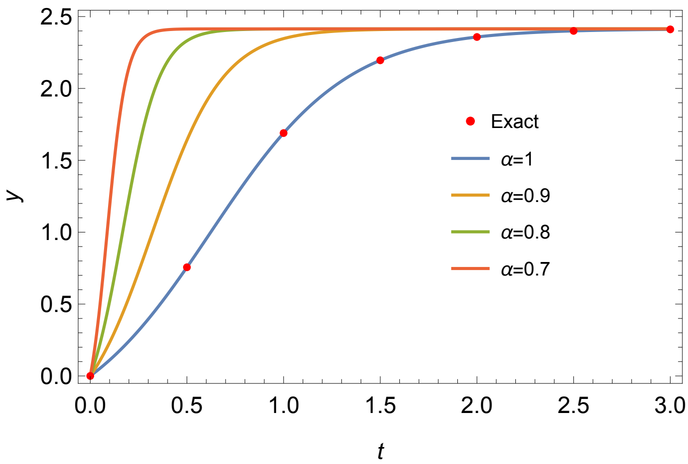

Figure 1: 7-terms series solution of Application 4.1 with different values of α using $h = 0.002$.

Table 1: The solution of fractional Riccati equation using different values of α using MLRPSM.

t	α=1	Exact	α=0.9	α=0.8	α=0.7
0.5	0.756014	0.756014	1.63952	2.33101	2.41347
1.	1.6895	1.6895	2.34671	2.41376	2.41421
1.5	2.19563	2.19563	2.40974	2.41421	2.41421
2.	2.35777	2.35777	2.41392	2.41421	2.41421
2.5	2.40028	2.40028	2.41419	2.41421	2.41421
3.	2.41081	2.41081	2.41421	2.41421	2.41421

Table 2: The residual Error of fractional Riccati equation using different values of α Using MLRPSM.

t	α=1	α=0.9	α=0.8	α=0.7
0.5	1.11022×10^{-16}	1.33227×10^{-15}	1.22125×10^{-15}	1.44329×10^{-15}
1.	3.33067×10^{-16}	1.11022×10^{-16}	9.99201×10^{-16}	6.66134×10^{-16}
1.5	4.44089×10^{-16}	6.66134×10^{-16}	3.33067×10^{-16}	2.22045×10^{-16}
2.	3.33067×10^{-16}	1.55431×10^{-15}	3.33067×10^{-16}	5.55112×10^{-16}
2.5	1.66533×10^{-15}	0.	2.22045×10^{-16}	5.55112×10^{-16}
3.	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	5.55112×10^{-16}

Application 4.2. Consider the fractional Lorenz system

$$({}_0D^{\alpha_1}x)(t) = \sigma(y(t) - x(t)), \tag{4.4}$$
$$({}_0D^{\alpha_2}y)(t) = (24 - 4\rho)x(t) - x(t)z(t) + \rho y(t), \tag{4.5}$$
$$({}_0D^{\alpha_3}z)(t) = x(t)y(t) - \beta z(t). \tag{4.6}$$

Subject to the initial conditions $x(0) = \beta_1, y(0) = \beta_2, z(0) = \beta_3$. The system has chaotic behavior when $\rho = -1, 5, \sigma = 10, \beta = \frac{8}{3}, \alpha_1 = \alpha_2 = \alpha_3 = 1$. To solve the problem in non-commercial cases, we modified the Laplace series solution to be in the form

$$\sum_{i=0}^{j} \frac{d_i}{s^{qi+1}},$$

where $q = \frac{1}{m}, m \in Z^+$, and $\alpha_i = \gamma_i q$, for $i = 1, 2, 3$. Following the same algorithm in Application 4.1, we solve the system (4.4)-(4.6) in the interval $[t_{k-1}, t_k]$ by replacing the fractional derivatives $({}_{t_{k-1}}D^{\alpha_1}x)(t)$, $({}_{t_{k-1}}D^{\alpha_2}y)(t)$, and $({}_{t_{k-1}}D^{\alpha_3}z)(t)$, subject to the initial conditions $x(t_{k-1}) = \beta_{1,k-1}$, $y(t_{k-1}) = \beta_{2,k-1}$, $z(t_{k-1}) = \beta_{3,k-1}$. To take the Laplace transform to the system, we generate the corresponding system

$$({}_0D^{\alpha_1}x)(u) = \sigma(y(u) - x(u)), \tag{4.7}$$
$$({}_0D^{\alpha_2}y)(u) = (24 - 4\rho)x(u) - x(u)z(u) + \rho y(u), \tag{4.8}$$
$$({}_0D^{\alpha_3}z)(u) = x(u)y(u) - \beta z(u), \tag{4.9}$$

subject to the initial conditions $x(0) = \beta_{1,k-1}, y(0) = \beta_{2,k-1}, z(0) = \beta_{3,k-1}$. Now, assume $q = \frac{1}{100}$, $\alpha_i = \gamma_i q$ for $i = 1, 2, 3$, then the Laplace transforms for equations (4.7)-(4.9) become

$$X(s) - \frac{x(0)}{s} = \frac{1}{s^{\gamma_1 q}}\left[\sigma(Y(s) - X(s))\right], \tag{4.10}$$
$$Y(s) - \frac{y(0)}{s} = \frac{1}{s^{\gamma_2 q}}\left[(24 - 4\rho)X(s) - L[L^{-1}[X(s)]L^{-1}[Z(s)]] + \rho Y(s)\right], \tag{4.11}$$
$$Z(s) - \frac{z(0)}{s} = \frac{1}{s^{\gamma_3 q}}\left[L[L^{-1}[X(s)]L^{-1}[Y(s)]] - \beta Z(s)\right]. \tag{4.12}$$

By substituting

$$X(s)=\sum_{i=0}^{j}\frac{x_i}{s^{qi+1}},\quad Y(s)=\sum_{i=0}^{j}\frac{y_i}{s^{qi+1}},\quad Z(s)=\sum_{i=0}^{j}\frac{z_i}{s^{qi+1}},$$

in equations (4.10)-(4.12), and multiplying the residual error Res_j by s^{qj+1}, we get

$$\begin{aligned}
s^{qj+1}Resx_j &= \sum_{i=1}^{j}\frac{x_i}{s^{q(i-j)}}-\frac{1}{s^{\gamma_1 q}}\left[\sigma\Big(\sum_{i=0}^{j}\frac{y_i}{s^{q(i-j)}}-\sum_{i=0}^{j}\frac{x_i}{s^{q(i-j)}}\Big)\right],\\
s^{qj+1}Resy_j &= \sum_{i=1}^{j}\frac{y_i}{s^{q(i-j)}}-\frac{1}{s^{\gamma_2 q}}\left[(24-4\rho)\sum_{i=0}^{j}\frac{x_i}{s^{q(i-j)}}-s^{qj+1}L\left\{L^{-1}[\sum_{i=0}^{j}\frac{x_i}{s^{qi+1}}]L^{-1}[\sum_{i=0}^{j}\frac{z_i}{s^{qi+1}}]\right\}\right.\\
&\quad\left.+\rho\sum_{i=0}^{j}\frac{y_i}{s^{q(i-j)}}\right]\\
&= \sum_{i=1}^{j}\frac{y_i}{s^{q(i-j)}}-\frac{1}{s^{\gamma_2 q}}\left[(24-4\rho)\sum_{i=0}^{j}\frac{x_i}{s^{q(i-j)}}+\rho\sum_{i=0}^{j}\frac{y_i}{s^{q(i-j)}}\right.\\
&\quad\left.-\sum_{i=0}^{2j}\frac{\Gamma(qi+1)}{s^{q(i-j)}}\sum_{n=\max[0,n-j]}^{\min[i,j]}\frac{x_n z_{i-n}}{\Gamma(\alpha n+1)(\Gamma(q(i-n)+1))}\right],\\
s^{qj+1}Resz_j &= \sum_{i=1}^{j}\frac{z_i}{s^{q(i-j)}}-\frac{1}{s^{\gamma_3 q}}\left[s^{qj+1}L\left\{L^{-1}[\sum_{i=0}^{j}\frac{x_i}{s^{q(i+1}}]L^{-1}[\sum_{i=0}^{j}\frac{y_i}{s^{qi+1}}]\right\}-\beta\sum_{i=0}^{j}\frac{z_i}{s^{q(i-j)}}\right]\\
&= \sum_{i=1}^{j}\frac{z_i}{s^{q(i-j)}}-\frac{1}{s^{\gamma_3 q}}\left[\sum_{i=0}^{2j}\frac{\Gamma(qi+1)}{s^{q(i-j)}}\sum_{n=\max[0,n-j]}^{\min[i,j]}\frac{x_n y_{i-n}}{\Gamma(\alpha n+1)(\Gamma(q(i-n)+1))}-\beta\sum_{i=0}^{j}\frac{z_i}{s^{q(i-j)}}\right].
\end{aligned}$$

By taking

$$\lim_{s\to\infty}s^{qj+1}Resx_j=\lim_{s\to\infty}s^{qj+1}Resy_j=\lim_{s\to\infty}s^{qj+1}Resz_j=0,$$

for $j=0,1,2,\dots$, the initial conditions give $x_0=\beta_1, y_0=\beta_2, z_0=\beta_3$. And $x_i=0$ for $i=1,2,\dots,\gamma_1-1$, $y_i=0$ for $i=1,2,\dots,\gamma_2-1$, $z_i=0$ for $i=1,2,\dots,\gamma_3-1$. The recurrence relation is given by

$$\begin{aligned}
x_j &= \sigma(y_{j-\gamma_1}-x_{j-\gamma_1}),\\
y_j &= -\Gamma(q(j-\gamma_2)+1)\sum_{k=0}^{j-\gamma_2}\frac{x_k z_{j-\gamma_2-k}}{\Gamma(kq+1)\Gamma((j-k-\gamma_2)q+1)}+(24-4\rho)x_{j-\gamma_2}+\rho y_{j-\gamma_2},\\
z_j &= \Gamma(q(j-\gamma_3)+1)\sum_{k=0}^{j-\gamma_3}\frac{x_k y_{j-\gamma_3-h}}{\Gamma(kq+1)\Gamma((j-k-\gamma_3)q+1)}-\beta z_{j-\gamma_3},
\end{aligned}$$

for $j=\min(\gamma_1,\gamma_2,\gamma_3),\min(\gamma_1,\gamma_2,\gamma_3)+1,\min(\gamma_1,\gamma_2,\gamma_3)+2,\dots,M$. Now, the M-th order of approximation for the system is

$$\begin{aligned}
x(u) &= \sum_{i=0}^{M}L^{-1}\frac{x_i}{s^{qi+1}}=x(t-t_{k-1}),\\
y(u) &= \sum_{i=0}^{M}L^{-1}\frac{y_i}{s^{qi+1}}=y(t-t_{k-1}),
\end{aligned}$$

$$z(u) = \sum_{i=0}^{M} L^{-1} \frac{z_i}{s^{qi+1}} = z(t - t_{k-1}).$$

Using the same manner in Application 4.1, interval $[0, T]$ can be divided into equal sub-intervals, and applying the algorithm to each k sub-interval. By fixing $\rho = -1$, $\sigma = 10$, $\beta = \frac{8}{3}$, $(\beta_1, \beta_2, \beta_3) = (10, 0, 10)$,$(\alpha_1, \alpha_2, \alpha_3) = (0.99, 0.98, 0.97)$, $q = \frac{1}{100}$, $h = 0.001$, and $M = 600$, the numerical values of the solutions and its residual errors are given in Table 3. It is notable that, the error converges to zero. Moreover, we plot the solution attractor in Figure 2. One of the new algorithm's privileges is to determine which fractional derivative value makes the system non-chaotic. To answer that we fix $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, $T = 20, h = 0.001, M = 3000$, and $q = \frac{1}{1000}$. The experimental results show that the solution at $\alpha = 0.694$ will change the attractors from chaotic to non-chaotic behaviors. The solution attractor at $\alpha = 0.695$ and 0.694 are presented in Figure 4. Moreover, as $\rho = 5$ with initial conditions $(0.1, 0.2, 0.3)$, the critical α that makes the system chaotic is 0.45, which appears in Figure 5. These results follow the published one using the Adomian decomposition method [30, 44].

Table 3: Numerical simulation and absolute residual error for Lorenz system with $(\alpha_1, \alpha_2, \alpha_3) = (0.99, 0.98, 0.97), q = \frac{1}{100}, h = 0.001, \rho = -1$, and $M = 700$.

t	$x(t)$	$y(t)$	$z(t)$	$\lvert Res_x(t)\rvert$	$\lvert Res_y(t)\rvert$	$\lvert Res_z(t)\rvert$
1.	-3.97798	1.79595	28.5376	-2.4016×10^{-12}	2.0203×10^{-11}	5.3290×10^{-12}
3.	-2.37833	0.949494	24.7188	1.3998×10^{-12}	2.3315×10^{-12}	4.9454×10^{-12}
5.	2.83	1.92956	21.3675	-5.7909×10^{-13}	1.3145×10^{-13}	7.6739×10^{-13}
7.	1.30623	3.29885	19.8692	6.8567×10^{-13}	8.8640×10^{-13}	7.8160×10^{-14}
9.	1.35643	2.68333	16.6818	1.2967×10^{-13}	2.2560×10^{-13}	1.4211×10^{-14}
11.	3.05452	4.23683	16.8376	8.5265×10^{-14}	6.8212×10^{-13}	3.1974×10^{-14}
13.	3.22217	5.43176	16.811	3.0198×10^{-13}	5.5422×10^{-13}	3.2330×10^{-13}
15.	-4.24308	-6.9871	16.5961	-4.6896×10^{-13}	8.9528×10^{-13}	1.4051×10^{-12}

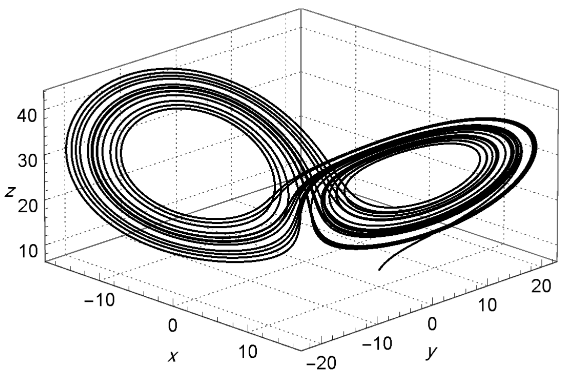

Figure 2: Solution attractor for Lorenz system with $(\alpha_1, \alpha_2, \alpha_3) = (0.99, 0.98, 0.97), q = \frac{1}{100}, h = 0.001$, and $M = 600$.

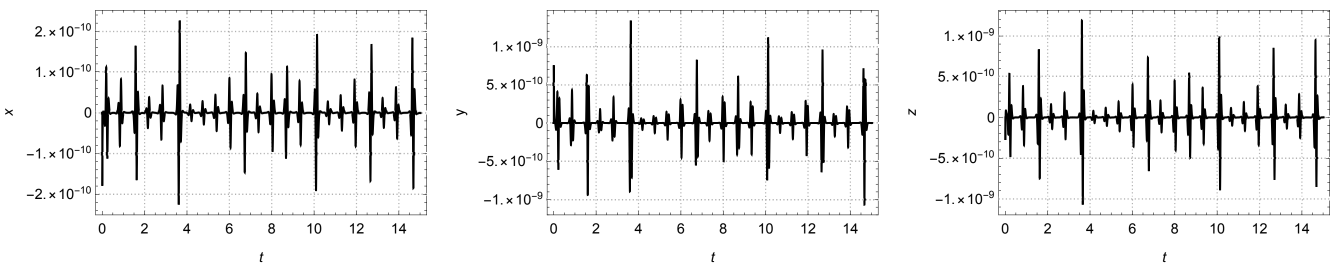

Figure 3: Residual error for the MLPS method with $(\alpha_1, \alpha_2, \alpha_3) = (0.99, 0.98, 0.97), q = \frac{1}{100}, h = 0.001, \rho = -1$, and $M = 600$ along x, y and z.

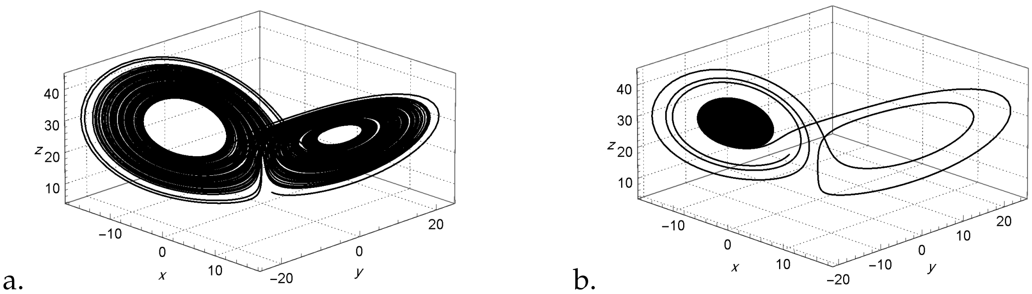

Figure 4: Attractors for Lorenz system with $\rho = -1$, a. for $\alpha = 0.695$; and b. for $\alpha = 0.694$.

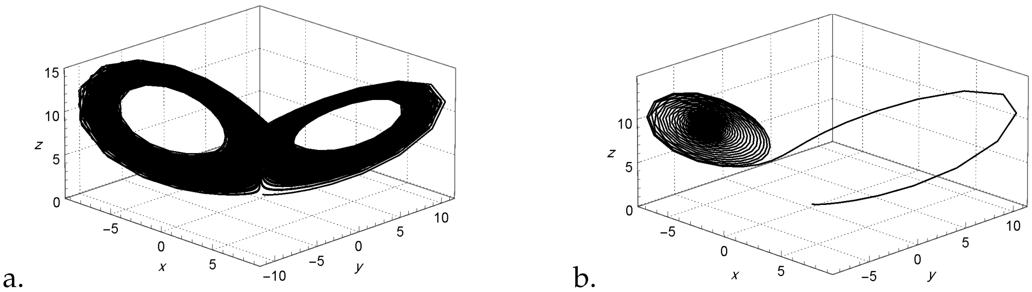

Figure 5: Attractors for Lorenz system with $\rho = 5$, a. for $\alpha = 0.45$; and b. for $\alpha = 0.44$.

5. Conclusions

This work presents a fast convergent algorithm for a long domain interval approximate continuous solution of a certain class of non-linear FDEs under the Caputo FD operator, including Riccati and Chaotic Lorenz models. A simple algorithm with easy computational terms is successfully built with the help of Laplace transformation and dividing the general domain into subdomains that make the series solution converge in each one without assuming any unsanctified limitations. Two attractive applications of posed models are examined to demonstrate the recommended algorithm's accuracy and convergence. The accuracy and the convergence are based on the residual error that comes from substituting the solution in the equation. In all cases, we reach a higher accuracy and converge to zero. Also, we were able to determine whether the system is chaotic or non-chaotic. Numeric and graphic simulations were carried out based on the results gained. In this context, it is obvious that the performed approximation scheme is a significant contribution to computational intents, it is computer-oriented, a simple methodology that requires low computational cost to acquire accurate approximate solutions in terms of FOPS, whose coefficients are established by a recursive formula. In future studies, the recommended algorithm can be employed to investigate long-domain interval approximate continuous solutions for many fractional problems related to the propagation of nonlinear phenomena in light of utilizing different kinds of fractional operators.

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