J. Math. Computer Sci., 37 (2025), 201–213

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Ordered constraint semigroups: a novel class of algebraic systems

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Abstract

The distinguishing factor between semigroups and ordered semigroups lies in compatibility. Compatibility is a specific form of polymorphisms, an essential tool in clone theory. In this paper, we apply a generalized version of polymorphisms, known as constraint, to define a novel algebraic structure called ordered constraint semigroups. It turns out that ordered constraint semigroups are generalizations of semigroups. We define various ideals in ordered constraint semigroups and examine their fundamental properties. Specifically, we investigate their generated forms and explore the relationships among these ideals. Moreover, we focus on the intersection property of quasi-ideals in ordered constraint semigroups.

Keywords: Ordered semigroups, constraints, ordered constraint semigroups, ideals.

2020 MSC: 06F05, 08A05, 20M12.

1. Introduction

A technique used to study clones theory is a study of Galois connection between the set of finitary operations $\mathcal{O}_A$ defined on a nonempty set A and the set of finitary relations $\mathcal{R}_A$ on A. The concept of polymorphisms is the main idea applying in such study. Let $f\colon A^n \to A$ be a function, and $\sigma \subseteq A^k$ be a k-ary relation on A. Then f is said to be a *polymorphism* of σ, denoted by $f \triangleright \sigma$, if

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1k} \end{pmatrix}, \dots, \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nk} \end{pmatrix} \in \sigma \Longrightarrow \begin{pmatrix} f(a_{11}, a_{21}, \dots, a_{n1}) \\ f(a_{12}, a_{22}, \dots, a_{n2}) \\ \vdots \\ f(a_{1k}, a_{2k}, \dots, a_{nk}) \end{pmatrix} \in \sigma.$$

Sometimes we said that f *preserves* the relation σ. The $\triangleright$ preservation relation induces a Galois connection $(\mathrm{Pol}, \mathrm{Inv})$ between the set of finitary operations and relations on a nonempty set. In fact, it is used to characterize the Galois closed classes of $(\mathrm{Pol}, \mathrm{Inv})$ (see [13]). The preservation relation is a concept that is

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doi: 10.22436/jmcs.037.02.05

Received: 2023-09-04 *Revised:* 2024-08-01 *Accepted:* 2024-08-19

not only important in clone theory but also plays a critical role in defining an important algebraic structure widely used in various mathematics and computer science areas, such as formal language theory, automata theory, and algebraic geometry, so-called ordered semigroups. An algebraic structure $\langle S; \cdot, \leqslant\rangle$ is called an *ordered semigroup* if $\langle S; \cdot\rangle$ is a semigroup, $\langle S; \leqslant\rangle$ is a partially ordered set, and $\cdot \triangleright \leqslant$, that is, for any $a, b \in S$, if $a \leqslant b$, then $a \cdot c \leqslant b \cdot c$ and $c \cdot a \leqslant c \cdot b$ for all $c \in S$.

The study of ideals is a well-known and important topic in investigating ordered semigroups. Ideals are substructures of ordered semigroups that play a significant role in understanding the structure and properties of ordered semigroups. Over two decades, many authors have introduced and studied different kinds of ideals in ordered semigroups, contributing to a deep and broad understanding of this area of research. The study of ideals has been pursued in several directions. While ideals have been a subject of much attention in the study of ordered semigroups, researchers have also explored using more generalized concepts of sets to define ideals. These concepts include fuzzy sets, intuitionistic fuzzy sets, bipolar fuzzy sets, tripolar fuzzy sets, soft sets, and hybrid structures. By applying these more flexible definitions, researchers have expanded their investigations' scope and opened up a new approach to research in ordered semigroups (see [2, 6, 7, 9, 14–16, 20–23]).

This paper introduces a novel extension of ordered semigroups, offering a more generalized approach. We introduce a new concept of algebraic systems called ordered constraint semigroups by considering the preservation relation hidden in the definition of ordered semigroups. We investigate the fundamental properties of this new concept. One of the key contributions of this paper is the introduction of ideals in ordered constraint semigroups. We study the fundamental properties of these ideals and investigate how they are related. In addition, we examine the intersection property of quasi-ideals in ordered constraint semigroups. Extending ordered semigroups to ordered constraint semigroups represents a significant development in the algebraic systems theory.

This paper is organized as follows. In Section 2, we recall the related notions of constraints and satisfaction relations, which we use to define and study ordered constraint semigroups. We introduce the concept of ideals in ordered constraint semigroups and explore some of their fundamental properties. In Section 3, we investigate the connections between different types of ideals in ordered constraint semigroups. By studying the relationships between these ideals, we gain a deeper understanding of the behavior and properties of these algebraic systems. Section 4 focuses on a specific class of ordered constraint semigroups and presents a sufficient and necessary condition for a quasi-ideal to have the intersection property. This condition offers a new perspective on quasi-ideals' behavior in ordered constraint semigroups and provides a useful tool for analyzing their properties and relationships. Lastly, we focus on examining the coincidence of different ideals in regular duo ordered constraint semigroups.

2. Ordered constraint semigroups

We begin by recalling the related concepts of constraints and satisfaction relations. These relations offer a natural extension of the preservation relation commonly studied in clone theory. By understanding satisfaction relations, we can better understand the relationship between ordered semigroups and our new algebraic system, ordered constraint semigroups.

Let A and B be nonempty sets. Suppose that R and S are k-ary relations on A and B, respectively. Then the ordered pair (R, S) is said to be an k*-ary* A*-to-*B *relational constraint*, or simply *constraint*. An n-ary function $f\colon A^n \to B$ is said to *satisfy* k-ary A-to-B relational constraint (R, S) if

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1k} \end{pmatrix}, \dots, \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nk} \end{pmatrix} \in R \quad \text{implies} \quad \begin{pmatrix} f(a_{11}, a_{21}, \dots, a_{n1}) \\ f(a_{12}, a_{22}, \dots, a_{n2}) \\ \vdots \\ f(a_{1k}, a_{2k}, \dots, a_{nk}) \end{pmatrix} \in S$$

for any k-ary tuple $(a_{11}, a_{12}, \dots, a_{1k}), \dots, (a_{n1}, a_{n2}, \dots, a_{nk})$ on A. This means that we obtain a satisfaction relation $\approx$ between the class of n-ary functions and a constraint. Let f be an n-ary function from A into B satisfying a constraint (S, T). We can observe that if $A = B$ and $S = T$, then f is just a polymorphism of S. This means that the satisfaction relation extends the preservation relation. The concept of satisfaction relation $\approx$ was introduced by Pippenger in 2002 (see [17]). It was also extensively studied by many researchers (see [3–5, 18]).

We can define a new class of algebraic systems by utilizing the satisfaction relation as follows. Let S be a nonempty set. A binary constraint $(\leqslant_1, \leqslant_2)$ on S is called a *binary ordered constraint* on S if both $\leqslant_1$ and $\leqslant_2$ are partial order on S. If the underlying set and the arity are clear from the context, we simply say $(\leqslant_1, \leqslant_2)$ an *ordered constraint*.

Definition 2.1. An algebraic system $\langle S; \cdot, \leqslant_1, \leqslant_2\rangle$ of type $(2; 2, 2)$ is called an *ordered constraint semigroup* if $\langle S; \cdot, \leqslant_1\rangle$ is an ordered semigroup, $(\leqslant_1, \leqslant_2)$ is an ordered constraint, and $\cdot \approx (\leqslant_1, \leqslant_2)$, that is, for any $a, b \in S$, $a \leqslant_1 b$ implies $a \cdot c \leqslant_2 b \cdot c$ and $c \cdot a \leqslant_2 c \cdot b$ for all $c \in S$.

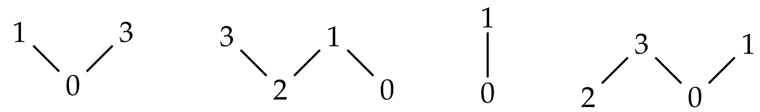
We see that any semigroup $\langle S; \cdot\rangle$ can be regarded as an ordered constraint semigroup $\langle S; \cdot, \Delta_S, \Delta_S\rangle$, where Δ_S is the equality relation. Similarly, any ordered semigroup $\langle S; \cdot, \leqslant\rangle$ can be considered as an ordered constraint semigroup $\langle S; \cdot, \leqslant, \leqslant\rangle$. This means that ordered constraint semigroups can be thought as a generalization of semigroups and ordered semigroups.

From now on, we denote an ordered constraint semigroup $\langle S; \cdot, \leqslant_1, \leqslant_2\rangle$ by $\mathbf{S}$ the boldface letter of its underlying set, and we write the product $a \cdot b$ of a and b by ab. Moreover, we denote by $\mathbf{S} \setminus (\leqslant_1, \leqslant_2)$ and $\mathbf{S} \setminus \leqslant_2$ the semigroup $\langle S; \cdot\rangle$ and the ordered semigroup $\langle S; \cdot, \leqslant_1\rangle$, respectively. Let $\mathbf{S}$ be an ordered constraint semigroup, and A and B nonempty subsets of S. Then we define $AB := \{ab : a \in A \text{ and } b \in B\}$. Let $a \in S$ and B be a nonempty subset of S. We write aB and Ba instead of $\{a\}B$ and $B\{a\}$, respectively.

Example 2.2. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	0

Moreover, we define binary relations $\leqslant_1$, $\leqslant_2$, $\leqslant_3$, and $\leqslant_4$ on S, respectively, by the following illustrations.



Then we obtain the following.

1. $\langle S; \cdot, \leqslant_1, \leqslant_2\rangle$ is an ordered constraint semigroup, but $\langle S; \cdot, \leqslant_2\rangle$ is not an ordered semigroup. Moreover, $\leqslant_1$ and $\leqslant_2$ do not contain each other.
2. $\langle S; \cdot, \leqslant_1, \leqslant_3\rangle$ is an ordered constraint semigroup such that $\leqslant_3 \subseteq \leqslant_1$.
3. $\langle S; \cdot, \leqslant_1, \leqslant_4\rangle$ is an ordered constraint semigroup. We can see that $\leqslant_1 \subseteq \leqslant_4$, but $\langle S; \cdot, \leqslant_4\rangle$ is not an ordered semigroup.

We can observe that if $\langle S; \circ, \leqslant_1\rangle$ is an ordered semigroup, then $\langle S; \circ, \leqslant_1, \preceq\rangle$ is an ordered constraint semigroup for any ordered constraint $(\leqslant_1, \preceq)$ such that $\leqslant_1 \subseteq \preceq$. It is known that any ordered semigroup can be derived from a semigroup. Building upon the previous discussion, we can construct an ordered constraint semigroup using an ordered semigroup. To further clarify this observation, it is necessary to introduce the following notion. Let $\langle S; \cdot, \leqslant_1\rangle$ be an ordered semigroup. Define the set

$$\leqslant_{1,\text{compatible}} := \{(ac, bc) : a \leqslant_1 b \text{ and } c \in A\} \cup \{(ca, cb) : a \leqslant_1 b \text{ and } c \in A\}.$$

Then we can construct an ordered constraint semigroup by an ordered semigroup shown as follows.

Theorem 2.3. *Let $\langle S; \cdot, \leqslant_1 \rangle$ be an ordered semigroup, and $\leqslant_2$ a partial order on S. Then the structure $\langle S; \cdot, \leqslant_1, \leqslant_2 \rangle$ is an ordered constraint semigroup if and only if $\leqslant_{1,\text{compatible}} \subseteq \leqslant_2$.*

Proof. Suppose that $\leqslant_2$ contains $\leqslant_{1,\text{compatible}}$. It is clear that $(\leqslant_1, \leqslant_2)$ is an ordered constraint on S. We need to show that for any $a, b, c \in S$ such that $a \leqslant_1 b$, we have that $ac \leqslant_2 bc$ and $ca \leqslant_2 cb$. But this is clear since $\leqslant_{1,\text{compatible}} \subseteq \leqslant_2$. Conversely, suppose that $\langle S; \cdot, \leqslant_1, \leqslant_2 \rangle$ is an ordered constraint semigroup. Let $(x, y) \in \leqslant_{1,\text{compatible}}$. Without loss of generality, assume that $(x, y) \in \{(ac, bc) : a \leqslant_1 b \text{ and } c \in A\}$. Then $x = ac$ and $y = bc$ for some $a, b, c \in S$ with $a \leqslant_1 b$. Since $\langle S; \cdot, \leqslant_1, \leqslant_2 \rangle$ is an ordered constraint semigroup, $ac \leqslant_2 bc$. This means that $(x, y) \in \leqslant_2$. Thus, we complete the proof. □

The primary objective of this section is to define the concepts of ideals in ordered constraint semigroups. However, before establishing these definitions, we introduce an important fundamental component of our algebraic system. Furthermore, this component is pivotal in differentiating the concepts of ordered semigroups and ordered constraint semigroups.

Let **S** be an ordered constraint semigroup, and A a nonempty subset of S. We denote the sets $(A]_1 := \{x \in S : x \leqslant_1 a \text{ for some } a \in A\}$ and $(A]_2 := \{x \in S : x \leqslant_2 a \text{ for some } a \in A\}$. Let us denote $(A] := (A]_1 \cap (A]_2$. This means that $(A] = \{x \in S : x \leqslant_1 a_1 \text{ and } x \leqslant_2 a_2 \text{ for some } a_1, a_2 \in A\}$. It is clear that $A \subseteq (A]_i$ and $(A]_i = ((A]_i]_i$ for all $i \in \{1, 2\}$.

By the above definition, we obtain the following important lemma.

Lemma 2.4. *Let **S** be an ordered constraint semigroup, A, B nonempty subsets of S, and $\{A_i\}_{i \in I}$ a family of nonempty subsets of S. Then the following statements hold.*

1. $A \subseteq (A]$ *and* $(A] = ((A]]$.
2. *If* $A \subseteq B$, *then* $(A] \subseteq (B]$.
3. $(A](B] \subseteq (AB]$.
4. $(A]_1(B]_1 \subseteq (AB]_2$.
5. $(A] \cup (B] \subseteq (A \cup B]$.
6. $\left(\bigcap_{i \in I} A_i\right] \subseteq \bigcap_{i \in I} (A_i]$.

Proof.

(1) Since $\leqslant_1$ and $\leqslant_2$ are reflexive, it is clear that $A \subseteq (A]$. This means that $(A] \subseteq ((A]]$. Therefore, it remains to show that $((A]] \subseteq (A]$. Let $x \in ((A]]$. Then $x \leqslant_1 a_1$ and $x \leqslant_2 a_2$ for some $a_1, a_2 \in (A]$. Since $a_1, a_2 \in (A]$, $a_1 \leqslant_1 t_1$ and $a_2 \leqslant_2 t_2$ for some $t_1, t_2 \in A$. By the transitivity of $\leqslant_1$ and $\leqslant_2$, we obtain that $x \leqslant_1 t_1$ and $x \leqslant_2 t_2$. This means that $x \in (A]$. Therefore, $(A] = ((A]]$.

(2) If $x \in (A]$ under the presumption that $A \subseteq B$, then $x \leqslant_1 a_1$ and $x \leqslant_2 a_2$ for some $a_1, a_2 \in A \subseteq B$. This shows that $x \in (B]$. Hence, $(A] \subseteq (B]$.

(3) Let $x \in (A](B]$. Then $x = yz$ such that $y \leqslant_1 a_1$, $y \leqslant_2 a_2$, $z \leqslant_1 b_1$, and $z \leqslant_2 b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. By the compatibility and transitivity of $\leqslant_1$ and $\leqslant_2$, we obtain that $x = yz \leqslant_1 a_1 z \leqslant_1 a_1 b_1$ and $x = yz \leqslant_2 a_2 z \leqslant_2 a_2 b_2$. This shows that $x \in (AB]$. Therefore, $(A](B] \subseteq (AB]$.

(4) Let $x \in (A]_1(B]_1$. Then $x = yz$ such that $y \leqslant_1 a$ and $z \leqslant_1 b$ for some $a \in A$ and $b \in B$. Since $\cdot \approx (\leqslant_1, \leqslant_2)$, we obtain that $yz \leqslant_2 az$ and $az \leqslant_2 ab$. By the transitivity of $\leqslant_2$, it turns out that $x = yz \leqslant_2 ab$ and so $x \in (AB]_2$. Hence, $(A]_1(B]_1 \subseteq (AB]_2$.

(5) They are directly obtained by (2) that $A \subseteq A \cup B$ implies $(A] \subseteq (A \cup B]$ and $B \subseteq A \cup B$ implies $(B] \subseteq (A \cup B]$. Hence, $(A] \cup (B] \subseteq (A \cup B]$.

(6) Using (2), it is immediately obtained that $\left(\bigcap_{i \in I} A_i\right] \subseteq (A_i]$ for all $i \in I$ since $\bigcap_{i \in I} A_i \subseteq A_i$ for all $i \in I$. Therefore, $\left(\bigcap_{i \in I} A_i\right] \subseteq \bigcap_{i \in I} (A_i]$. □

The opposite conclusions of Lemma 2.4 (5) and (6) are not true in general as the following example shows.

Example 2.5. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

Moreover, we define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.

Then $\mathbf{S}$ is an ordered constraint semigroup. Consider $A = \{0\}$, $B = \{3\}$, $C = \{0, 3\}$, and $D = \{1, 3\}$. Then we have that $(A \cup B] = \{0, 1, 3\} \nsubseteq \{0, 3\} = (A] \cup (B]$ and $(C] \cap (D] = \{1, 3\} \nsubseteq \{3\} = (C \cap D]$.

Let $\mathbf{S}$ be an ordered constraint semigroup. A nonempty subset A of S is said to be a *subsemigroup* of $\mathbf{S}$ if $AA \subseteq A$. Verifying that any subsemigroup A of an ordered constraint semigroup $\mathbf{S}$ can be used as the underlying set of an ordered constraint semigroup $\langle A; \cdot|_{A\times A}, \leqslant_1|_{A\times A}, \leqslant_2|_{A\times A}\rangle$ is a straightforward process.

Now, we are ready to define the notions of ideals.

Definition 2.6. Let $\mathbf{S}$ be an ordered constraint semigroup. A nonempty subset A of S such that $(A] \subseteq A$ is called:

1. a *left ideal* of $\mathbf{S}$ if $SA \subseteq A$;
2. a *right ideal* of $\mathbf{S}$ if $AS \subseteq A$;
3. an (*two-sided*) *ideal* of $\mathbf{S}$ if it is both a left and a right ideal of $\mathbf{S}$;
4. a *quasi-ideal* of $\mathbf{S}$ if $(AS] \cap (SA] \subseteq A$;
5. a *bi-ideal* of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $ASA \subseteq A$;
6. an *interior ideal* of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $SAS \subseteq A$.

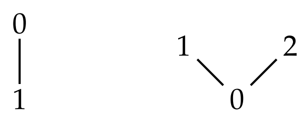
Sometimes the notions of left ideals and right ideals are called *one-sided* ideals. For any ordered constraint semigroup $\mathbf{S}$, we can reduce the definitions of the above ideals for an ordered semigroup $\mathbf{S} \smallsetminus \leqslant_2$ as follows. A nonempty subset A of S such that $(A]_1 \subseteq A$ is called: (1) a *left ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if $SA \subseteq A$; (2) a *right ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if $AS \subseteq A$; (3) an (*two-sided*) *ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if it is both a left and a right ideal of $\mathbf{S} \smallsetminus \leqslant_2$; (4) a *quasi-ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if $(AS]_1 \cap (SA]_1 \subseteq A$; (5) a *bi-ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if A is a subsemigroup of $\mathbf{S}$ and $ASA \subseteq A$; and (6) an *interior ideal* of $\mathbf{S} \smallsetminus \leqslant_2$ if A is a subsemigroup of $\mathbf{S}$ and $SAS \subseteq A$.

The following examples illustrate the existence of ideals we have presented.

Example 2.7. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	2	3

We define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.

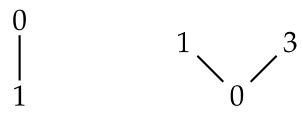


Then **S** is an ordered constraint semigroup. We obtain that $\{0, 3\}$ is a left ideal of **S**, but not a right ideal of **S**. Furthermore, $\{0, 3\}$ is not a left ideal of $\mathbf{S} \setminus \leqslant_2$.

Example 2.8. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	2	2	2	2
3	3	3	3	3

We define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.

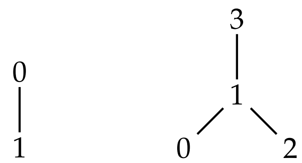


Then **S** is an ordered constraint semigroup. We obtain that $\{0, 3\}$ is a right ideal of **S**, but not a left ideal of **S**. Furthermore, $\{0, 3\}$ is not a right ideal of $\mathbf{S} \setminus \leqslant_2$.

Example 2.9. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	0
3	0	0	0	4

Moreover, we define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.

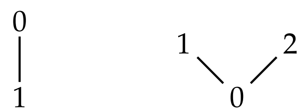


Then **S** is an ordered constraint semigroup. We obtain that $\{0\}$ is an ideal of **S**. Furthermore, $\{0\}$ is not an ideal of $\mathbf{S} \setminus \leqslant_2$.

Example 2.10. Let $S = \{0, 1, 2\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

Moreover, we define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.

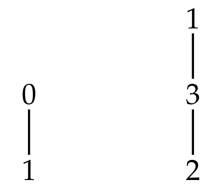


Then **S** is an ordered constraint semigroup. We obtain that $\{0\}$ is a quasi-ideal of **S**. Furthermore, $\{0\}$ is not a quasi-ideal of $\mathbf{S} \setminus \leqslant_2$.

Example 2.11. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Moreover, we define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.



Then $\mathbf{S}$ is an ordered constraint semigroup. We obtain that $\{0, 2\}$ is both a bi-ideal and an interior ideal of $\mathbf{S}$. Furthermore, $\{0, 2\}$ is neither a bi-ideal nor an interior ideal of $\mathbf{S} \setminus \leqslant_2$.

For any ordered constraint semigroup $\mathbf{S}$, based on the definition of ideals in ordered constraint semigroups, it is evident that any ideal A of $\mathbf{S} \setminus \leqslant_2$ is also an ideal of $\mathbf{S}$. However, the converse is not always true, as demonstrated in Examples 2.7, 2.8, 2.9, 2.10, and 2.11. Hence, it is reasonable to inquire about the conditions under which any ideal A of $\mathbf{S}$ is an ideal of $\mathbf{S} \setminus \leqslant_2$. The following theorem addresses this question.

Theorem 2.12. *Let $\mathbf{S}$ be an ordered constraint semigroup, and A a nonempty subset of S such that $(A]_1 \subseteq (A]_2$. Then the following statements are equivalent.*

1. *A is a left (resp., right, two-sided, quasi-, bi-, and interior) ideal of $\mathbf{S} \setminus \leqslant_2$.*
2. *A is a left (resp., right, two-sided, quasi-, bi-, and interior) ideal of $\mathbf{S}$.*

Proof. The proof can be obtained by observing that $(A] = (A]_1$. □

According to Theorem 2.12, we can construct an ordered constraint semigroup $\mathbf{S}$ in which its ideals differ from those of ordered semigroup $\mathbf{S} \setminus \leqslant_2$. Furthermore, since $\leqslant_1 \subseteq \leqslant_2$ implies $(A]_1 \subseteq (A]_2$ for any subset A of S, an ordered constraint semigroup $\mathbf{S}$ in which $\leqslant_1 \subseteq \leqslant_2$ is not considered interesting. Hence, we typically focus on ordered constraint semigroups where $\leqslant_2$ does not contain $\leqslant_1$.

The following lemma is useful for the subsequent study of ideals in ordered constraint semigroups.

Lemma 2.13. *Let $\mathbf{S}$ be an ordered constraint semigroup, and $\{A_i\}_{i \in I}$ a family of left (resp., right, two-sided, quasi-, bi-, and interior) ideals of $\mathbf{S}$. Then $\bigcap_{i \in I} A_i$ is a left (resp., right, two-sided, quasi-, bi-, and interior) ideal of $\mathbf{S}$ if $\bigcap_{i \in I} A_i \neq \emptyset$.*

Proof. We demonstrate the proof for the case of bi-ideals, and the other cases can be established similarly. Let $\{A_i\}_{i \in I}$ a family of bi-ideals of $\mathbf{S}$ such that $B := \bigcap_{i \in I} A_i \neq \emptyset$. Since $(B] = \left(\bigcap_{i \in I} A_i\right] \subseteq \bigcap_{i \in I} (A_i] \subseteq (A_i] \subseteq A_i$ for all $i \in I$, we have that $(B] \subseteq B$. Since $BB \subseteq A_i A_i \subseteq A_i$ for all $i \in I$, we have that $BB \subseteq B$. This means that B is a subsemigroup of $\mathbf{S}$. Since $BSB \subseteq A_i S A_i \subseteq A_i$ for all $i \in I$, we have that $BSB \subseteq B$. Therefore, B is a bi-ideal of $\mathbf{S}$. □

Let $\mathbf{S}$ be an ordered constraint semigroup. We define unary functions L, R, J, Q, B, and I on the set $\mathcal{P}^*(S)$ of all subsets of S without the empty set by:

1. $L(A) := \bigcap\{B : B \text{ is a left ideal of } \mathbf{S} \text{ such that } A \subseteq B\}$;
2. $R(A) := \bigcap\{B : B \text{ is a right ideal of } \mathbf{S} \text{ such that } A \subseteq B\}$;
3. $J(A) := \bigcap\{B : B \text{ is an ideal of } \mathbf{S} \text{ such that } A \subseteq B\}$;
4. $Q(A) := \bigcap\{B : B \text{ is a quasi-ideal of } \mathbf{S} \text{ such that } A \subseteq B\}$;

5. $B(A) := \bigcap\{B : B$ is a bi-ideal of **S** such that $A \subseteq B\}$;
6. $I(A) := \bigcap\{B : B$ is an interior ideal of **S** such that $A \subseteq B\}$;

for all $A \subseteq S$. By Lemma 2.13, the functions L, R, J, Q, B, and I are well-defined.

Example 2.14. According to an ordered constraint semigroup **S** defined in Example 2.7, we have that $\{\{0,3\},\{0,1,3\},\{0,2,3\},S\}$ are the set of all left ideals of **S** containing $\{3\}$. Then, $L(\{3\}) = \{0,3\} \cap \{0,1,3\} \cap \{0,2,3\} \cap S = \{0,3\}$.

The computation illustrated in the previous example becomes increasingly inconvenient when dealing with ordered constraint semigroups with a large cardinality. Hence, it is natural to find formulae to simplify the computation of $L(A)$, $R(A)$, $J(A)$, $Q(A)$, $B(A)$, and $I(A)$. The following theorem offers a more convenient method for calculation.

Theorem 2.15. *Let* **S** *be an ordered constraint semigroup, and* A *a nonempty subset of* S. *Then the following statements hold.*

1. $L(A) = (A \cup SA]$.
2. $R(A) = (A \cup AS]$.
3. $J(A) = (A \cup AS \cup SA \cup SAS]$.
4. $Q(A) = (A \cup ((AS] \cap (SA])]$.
5. $B(A) = (A \cup A^2 \cup ASA]$.
6. $I(A) = (A \cup A^2 \cup SAS]$.

Proof. We give a proof only for (3), (4), and (5).

(3) Clearly, $A \subseteq (A \cup AS \cup SA \cup SAS]$. We consider

$$\begin{aligned} S(A \cup AS \cup SA \cup SAS] &\subseteq (S](A \cup AS \cup SA \cup SAS] \\ &\subseteq (S(A \cup AS \cup SA \cup SAS)] \subseteq (AS \cup SA \cup SAS] \subseteq (A \cup AS \cup SA \cup SAS]. \end{aligned}$$

Similarly, $(A \cup AS \cup SA \cup SAS]S \subseteq (A \cup AS \cup SA \cup SAS]$. Obviously, $(A \cup AS \cup SA \cup SAS] = ((A \cup AS \cup SA \cup SAS]]$. Now, $(A \cup AS \cup SA \cup SAS]$ is an ideal of **S** containing A. Let K be an ideal of **S** such that $A \subseteq K$. We obtain that $(A \cup AS \cup SA \cup SAS] \subseteq (K \cup KS \cup SK \cup SKS] \subseteq (K \cup K \cup K \cup K] = (K] \subseteq K$. Therefore, $(A \cup AS \cup SA \cup SAS]$ is the smallest ideal of **S** containing A. That is, $J(A) = (A \cup AS \cup SA \cup SAS]$.

(4) Clearly, $A \subseteq (A \cup ((AS] \cap (SA])]$ and $((A \cup ((AS] \cap (SA])]] = (A \cup ((AS] \cap (SA])]$. Consider

$$\begin{aligned} ((A \cup ((AS] \cap (SA])]S] \cap (S(A \cup ((AS] \cap (SA])]] &\subseteq ((A \cup (AS]]S] \cap (S(A \cup (SA]]] \\ &\subseteq ((AS \cup (ASS]]] \cap ((SA \cup (SSA]]] \\ &\subseteq ((AS \cup (AS]]] \cap ((SA \cup (SA]]] \\ &\subseteq (((AS]]] \cap (((SA]]] = (AS] \cap (SA] \subseteq (A \cup ((AS] \cap (SA])]. \end{aligned}$$

Let K be a quasi-ideal of **S** such that $A \subseteq K$. Then $(A \cup ((AS] \cap (SA])] \subseteq (K \cup ((KS] \cap (SK])] \subseteq (K \cup K] = (K] \subseteq K$. Therefore, $(A \cup ((AS] \cap (SA])]$ is the smallest quasi-ideal of **S** containing A. That is, we obtain our claim.

(5) Clearly, $A \subseteq (A \cup A^2 \cup ASA]$ and $((A \cup A^2 \cup ASA]] = (A \cup A^2 \cup ASA]$. We obtain that

$$(A \cup A^2 \cup ASA](A \cup A^2 \cup ASA] \subseteq ((A \cup A^2 \cup ASA)(A \cup A^2 \cup ASA)] \subseteq (A^2 \cup ASA] \subseteq (A \cup A^2 \cup ASA]$$

and

$$\begin{aligned} (A \cup A^2 \cup ASA]S(A \cup A^2 \cup ASA] &\subseteq (A \cup A^2 \cup ASA](S](A \cup A^2 \cup ASA] \\ &\subseteq (AS](A \cup A^2 \cup ASA] \subseteq (ASA] \subseteq (A \cup A^2 \cup ASA]. \end{aligned}$$

If B is a bi-ideal of $\mathbf{S}$ such that $A \subseteq B$, then $(A \cup A^2 \cup ASA] \subseteq (B \cup B^2 \subseteq BSB] \subseteq (B \cup B \cup B] = (B] \subseteq B$. Therefore, $(A \cup A^2 \cup ASA]$ is the smallest bi-ideal of $\mathbf{S}$ containing A. That is, $B(A) = (A \cup A^2 \cup ASA]$. □

Example 2.16. Applying the above theorem to Example 2.14, we see that

$$L(\{3\}) = (\{3\} \cup S\{3\}] = (\{3\} \cup \{0,3\}] = (\{0,3\}] = \{0,3\}.$$

Theorem 2.15 proves to be highly advantageous in terms of convenience compared to Example 2.14, particularly when finding the smallest ideal containing a nonempty set. It provides significant assistance in simplifying the process, as seen in Example 2.16.

3. Connections among ideals in ordered constraint semigroups

In this section, we provide connections among various ideals introduced in Section 2. Let $\mathbf{S}$ be an ordered constraint semigroup. For our convenience, we use:

1. $\mathsf{L}(\mathbf{S})$ to denote the set of all left ideals of $\mathbf{S}$;
2. $\mathsf{R}(\mathbf{S})$ to denote the set of all right ideals of $\mathbf{S}$;
3. $\mathsf{J}(\mathbf{S})$ to denote the set of all ideals of $\mathbf{S}$;
4. $\mathsf{Q}(\mathbf{S})$ to denote the set of all quasi-ideals of $\mathbf{S}$;
5. $\mathsf{B}(\mathbf{S})$ to denote the set of all bi-ideals of $\mathbf{S}$;
6. $\mathsf{I}(\mathbf{S})$ to denote the set of all interior ideals of $\mathbf{S}$.

The following connection between one-sided ideals and ideals is straightforward to establish.

Theorem 3.1. *Let $\mathbf{S}$ be an ordered constraint semigroup. Then we have $\mathsf{J}(\mathbf{S}) \subseteq \mathsf{L}(\mathbf{S})$ and $\mathsf{J}(\mathbf{S}) \subseteq \mathsf{R}(\mathbf{S})$.*

However, it should be noted that the converse of the above theorem does not hold, as shown in Examples 2.7 and 2.8.

Below, we present the connections between one-sided ideals and quasi-ideals.

Theorem 3.2. *Let $\mathbf{S}$ be an ordered constraint semigroup. Then we have $\mathsf{L}(\mathbf{S}) \subseteq \mathsf{Q}(\mathbf{S})$ and $\mathsf{R}(\mathbf{S}) \subseteq \mathsf{Q}(\mathbf{S})$.*

Proof. We prove only that $\mathsf{L}(\mathbf{S}) \subseteq \mathsf{Q}(\mathbf{S})$. Let A be a left ideal of $\mathbf{S}$. It is clear that $(A] \subseteq A$. Consider $(AS] \cap (SA] \subseteq (SA] \subseteq (A] \subseteq A$. This means that A is a quasi-ideal of $\mathbf{S}$. Therefore, $\mathsf{L}(\mathbf{S}) \subseteq \mathsf{Q}(\mathbf{S})$. □

However, Example 2.10 demonstrates that the converse of Theorem 3.2 is not true. More precisely, $\{0\}$ is not a left and a right ideal of $\mathbf{S}$.

The following theorem establishes a connection between quasi-ideals and bi-ideals in ordered constraint semigroups.

Theorem 3.3. *Let $\mathbf{S}$ be an ordered constraint semigroup. Then we have $\mathsf{Q}(\mathbf{S}) \subseteq \mathsf{B}(\mathbf{S})$.*

Proof. Let A be a quasi-ideal of $\mathbf{S}$. It is clear that $(A] \subseteq A$. Since $AA \subseteq AS \subseteq (AS]$ and $AA \subseteq SA \subseteq (SA]$, we have that $AA \subseteq (AS] \cap (SA] \subseteq A$. This means that A is a subsemigroup of $\mathbf{S}$. Lastly, since $ASA \subseteq AS \subseteq (AS]$ and $ASA \subseteq SA \subseteq (SA]$, we obtain that $ASA \subseteq (AS] \cap (SA] \subseteq A$. This illustrates that A is a bi-ideal of $\mathbf{S}$. □

In general, the converse of the above result does not hold by Example 2.11. Indeed, $(\{0,2\}S] \cap (S\{0,2\}] = \{0,1\} \not\subseteq \{0,2\}$.

Theorem 3.4. *Let $\mathbf{S}$ be an ordered constraint semigroup. Then we have $\mathsf{J}(\mathbf{S}) \subseteq \mathsf{I}(\mathbf{S})$.*

Proof. Let A be an ideal of $\mathbf{S}$. It is clear that $(A] \subseteq A$ and A is a subsemigroup of $\mathbf{S}$. Since $SAS \subseteq AS \subseteq A$, we obtain that A is an interior ideal of $\mathbf{S}$. □

It can be stated that the converse of the above theorem does not hold, as illustrated by Example 2.11. Indeed, $S\{0,2\} = \{0,2\}S = \{0,1\} \not\subseteq \{0,2\}$. This shows that $\{0,2\}$ is neither a left ideal nor a right ideal of $\mathbf{S}$.

Now, we summarize connections among types of ideals in ordered constraint semigroups, as illustrated in Figure 1.

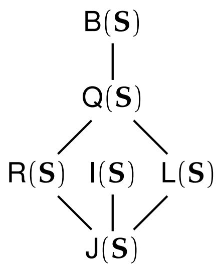


Figure 1: Relationships among ideals in an ordered constraint semigroup **S**.

Furthermore, a connection between one-sided ideals and quasi-ideals can be provided in terms of the intersection as follows.

Proposition 3.5. *The intersection of a left and a right ideal of an ordered constraint semigroup is a quasi-ideal.*

Proof. Let L and R be a left and a right ideal of an ordered constraint semigroup $\mathbf{S}$, respectively. Then $((L \cap R)S] \cap (S(L \cap R)] \subseteq (RS] \cap (SL] \subseteq (R] \cap (L] = R \cap L$. Since $(L \cap R] \subseteq (L] = L$ and $(L \cap R] \subseteq (R] = R$, $(L \cap R] \subseteq L \cap R$. Hence, $L \cap R$ is a quasi-ideal of $\mathbf{S}$. □

Generally, a quasi-ideal of an ordered constraint semigroup could not be the intersection of a left and a right ideal. We give the following notion to define the property that a quasi-ideal of an ordered constraint semigroup can be written in the form of the intersection of a left and a right ideal. Prior research on the algebraic structure of ordered semigroups can be found in various sources, including references [1, 12, 19].

Definition 3.6. A subsemigroup A of an ordered constraint semigroup $\mathbf{S}$ has *intersection property* if A is the intersection of a left ideal and a right ideal of $\mathbf{S}$.

The following theorem comprises a condition that a quasi-ideal of an ordered constraint semigroup satisfies the intersection property.

Theorem 3.7. *Let Q be a quasi-ideal of an ordered constraint semigroup* $\mathbf{S}$*. Then Q has the intersection property if and only if* $Q = L(Q) \cap R(Q)$.

Proof. Let Q be a quasi-ideal of an ordered constraint semigroup $\mathbf{S}$. If $Q = L(Q) \cap R(Q)$, it is clear that Q satisfies the intersection property. Hence, we remain to show that if Q has the intersection property, then $Q = L(Q) \cap R(Q)$. Assume that Q has the intersection property. Obviously, $Q \subseteq L(Q) \cap R(Q)$. By assumption, there exist a left ideal A and a right ideal R of $\mathbf{S}$ such that $Q = A \cap B$. So, $Q \subseteq A$ and $Q \subseteq B$. Consequently, $L(Q) = (Q \cup SQ] \subseteq (A \cup SA] \subseteq (A \cup A] = (A] = A$ and $R(Q) = (Q \cup QS] \subseteq (B \cup BS] \subseteq (B \cup B] = (B] = B$. Hence, $L(Q) \cap R(Q) \subseteq A \cap B = Q$. Therefore, $Q = L(Q) \cap R(Q)$. □

4. Regular ordered constraint semigroups

In the preceding section, we provide links between various types of ideals in ordered constraint semigroups. Now, we investigate the notions of regular ordered constraint semigroups. Furthermore, it is demonstrated in this study that the converse statements of various theorems and propositions presented in the preceding section can hold true within regular ordered constraint semigroups. Prior research on the algebraic structures of ordered semigroups can be found in various sources, including references [10, 11].

The regularity of ordered constraint semigroups can be defined as follows.

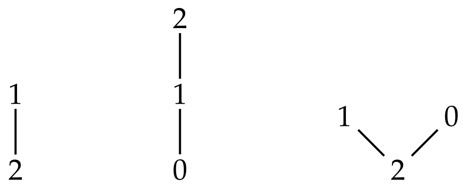
Definition 4.1. An ordered constraint semigroup $\mathbf{S}$ is *regular* if for any $a \in S$, we have $a \in (aSa]$.

We observe that if $\mathbf{S}$ is regular, then $\mathbf{S} \setminus \leqslant_2$ is also regular but not conversely.

Example 4.2. Let $S = \{0, 1, 2\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2
0	0	0	0
1	1	1	1
2	1	1	1

Moreover, we define binary relations $\leqslant_1$, $\leqslant_2$, and $\leqslant_3$, respectively, by the following illustrations.



Then $\langle S; \cdot, \leqslant_1, \leqslant_2\rangle$ and $\langle S; \cdot, \leqslant_1, \leqslant_3\rangle$ are ordered constraint semigroups. It is not difficult to calculate that the ordered semigroup $\langle S; \cdot, \leqslant_1\rangle$ and the ordered constraint semigroup $\langle S; \cdot, \leqslant_1, \leqslant_3\rangle$ are regular. However, $\langle S; \cdot, \leqslant_1, \leqslant_2\rangle$ is not regular because there does not exist $x \in S$ such that $c \leqslant_2 cxc$.

We know already that, from Theorem 3.3, any quasi-ideal is a bi-ideal. One may ask which class of ordered constraint semigroups makes such two concepts coincide. The following proposition addresses this question.

Theorem 4.3. *Let $\mathbf{S}$ be a regular ordered constraint semigroup. Then we have $\mathsf{Q}(\mathbf{S}) = \mathsf{B}(\mathbf{S})$.*

Proof. By Theorem 3.3, every quasi-ideal of $\mathbf{S}$ is a bi-ideal of $\mathbf{S}$. Let B be a bi-ideal of $\mathbf{S}$. If $x \in (BS] \cap (SB]$, then $x \in (xSx] \subseteq ((BS]S(SB]] \subseteq ((BSB]] = (BSB] \subseteq (B] \subseteq B$. Hence, B is a quasi-ideal of $\mathbf{S}$. □

The converse of Theorem 3.4 can be true in a regular ordered constraint semigroup as follows.

Theorem 4.4. *Let $\mathbf{S}$ be a regular ordered constraint semigroup. Then we have $\mathsf{J}(\mathbf{S}) = \mathsf{I}(\mathbf{S})$.*

Proof. By Theorem 3.4, every ideal of $\mathbf{S}$ is an interior ideal of $\mathbf{S}$. Let I be an interior ideal of $\mathbf{S}$. If $x \in SI$, then we obtain that $x \in (xSx] \subseteq (xS] \subseteq (SIS] \subseteq (I] = I$. So, $SI \subseteq I$. Similarly, we also get $IS \subseteq I$. Hence, I is an ideal of $\mathbf{S}$. □

We summarize connections among various types of ideals in regular ordered constraint semigroups, as illustrated in Figure 2.

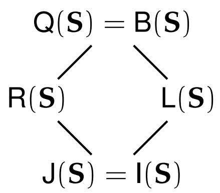


Figure 2: Relationships among ideals in a regular ordered constraint semigroup $\mathbf{S}$.

In the following results, we illustrate that the converse of Proposition 3.5 holds in the class of regular ordered constraint semigroups.

Lemma 4.5. *Let $\mathbf{S}$ be a regular ordered constraint semigroup, and A a nonempty subset of S. Then $L(A) = (SA]$ and $R(A) = (AS]$.*

Proof. Assume that $\mathbf{S}$ is regular. Obviously, $(SA] \subseteq (A \cup SA] = L(A)$. By assumption, we get that for any $a \in A$, $a \in (aSa] \subseteq (ASA] \subseteq (SA]$ implies $A \subseteq (SA]$. It follows that $L(A) = (A \cup SA] \subseteq ((SA] \cup SA] = (SA]$. The case of $R(A) = (AS]$ can be proved analogously. □

Proposition 4.6. *Let* $\mathbf{S}$ *be a regular ordered constraint semigroup. Then, every quasi-ideal of* $\mathbf{S}$ *is the intersection of a left and a right ideal of* $\mathbf{S}$.

Proof. Let Q be a quasi-ideal of $\mathbf{S}$ a regular ordered constraint semigroup. It is clear that $Q \subseteq L(Q) \cap R(Q)$. Using Lemma 4.5, we obtain that $L(Q) \cap R(Q) = (SQ] \cap (QS] \subseteq Q$. Hence, $Q = L(Q) \cap R(Q)$. □

Based on the result above, we can conclude that every quasi-ideal has the intersection property in regular ordered constraint semigroups.

An ordered constraint semigroup $\mathbf{S}$ is said to be *commutative* if $xy = yx$ for all $x, y \in S$. It is obvious that $\mathsf{J}(\mathbf{S}) = \mathsf{R}(\mathbf{S}) = \mathsf{L}(\mathbf{S})$ if $\mathbf{S}$ is commutative.

Nevertheless, it is possible for us to provide the notion of duo ordered constraint semigroups, which is a broader concept than a commutative ordered constraint semigroup, by utilizing the following definition.

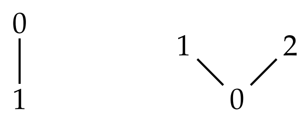
Definition 4.7. An ordered constraint semigroup $\mathbf{S}$ is said to be *duo* if every one-sided ideal is a two-sided ideal.

It is clear by the above definition that $\mathsf{J}(\mathbf{S}) = \mathsf{R}(\mathbf{S}) = \mathsf{L}(\mathbf{S})$ if $\mathbf{S}$ is a duo ordered constraint semigroup. Moreover, it is obvious that every commutative ordered constraint semigroup is duo. This illustration serves to demonstrate that there exists a duo ordered constraint semigroup which fails to exhibit the commutativity.

Example 4.8. Let $S = \{0, 1, 2, 3\}$. Define a binary operation $\cdot$ on S as follows.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	0	0	1	1

We define binary relations $\leqslant_1$ and $\leqslant_2$, respectively, by the following illustrations.



Then $\mathbf{S}$ is an ordered constraint semigroup. Clearly, $\mathbf{S}$ is not commutative. We obtain that $\{0\}$, $\{0, 1\}$, $\{0, 1, 3\}$, $\{0, 1, 2\}$, and S are all one-sided ideals of $\mathbf{S}$. It is not difficult to show that every one-sided ideal of $\mathbf{S}$ is a two-sided ideal. Hence, $\mathbf{S}$ is a duo ordered constraint semigroup.

In regular duo ordered constraint semigroups, all types of ideals mentioned in this work, namely, ideals, left ideals, right ideals, quasi-ideals, bi-ideals, and interior ideals are coincidence as the following corollary which is directly obtained by Theorem 4.3, Theorem 4.4, Proposition 4.6, and Definition 4.7.

Corollary 4.9. *Let* $\mathbf{S}$ *be a regular duo ordered constraint semigroup. Then* $\mathsf{J}(\mathbf{S}) = \mathsf{L}(\mathbf{S}) = \mathsf{R}(\mathbf{S}) = \mathsf{Q}(\mathbf{S}) = \mathsf{B}(\mathbf{S}) = \mathsf{I}(\mathbf{S})$.

5. Conclusion

This paper presents the concept of ordered constraint semigroups, which represents an extension of ordered semigroups. The introduction of this new algebraic system incorporates an expanded notion of compatibility. Various types of ideals in ordered constraint semigroups are introduced and examined, focusing on exploring their interconnections in specific classes of ordered constraint semigroups. Additionally, the paper considers the intersection property of quasi-ideals. The utilization of ideals in ordered constraint semigroups holds great promise for delving deeper into the study of this algebraic structure. By employing the notion of ideals, researchers can explore various aspects of ordered constraint semigroups, such as their characterization, the investigation of ideals purity, the analysis of primitive ideals, and the examination of radical ideals. These avenues of research have the potential to provide valuable insights into the properties and behavior of ordered constraint semigroups, contributing to a more comprehensive understanding of this mathematical framework.

Acknowledgment

The corresponding author wish to extend appreciation to Rajamangala University of Technology Isan, Khon Kaen Campus for providing research facilities. The authors are immensely grateful to the reviewers for their valuable suggestions, which have greatly helped to improve this work.

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