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Oscillation analysis of a forced fractional order sum-difference equations

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Abstract

This paper contributes some new outcomes about the oscillation of a forced fractional order sum-difference equation of the form

$$\Delta^{\beta} y(\iota) + \sum_{\varkappa=a}^{\iota-1} \mathbb{T}(\iota,\varkappa)\Psi(\varkappa, y(\varkappa)) = \varepsilon(\iota),\ 0 < \beta < 1,\ \iota \in \mathbb{N}_a,$$

with $\Delta^{\beta-1}y(0) = y_0 \in \mathbb{R}$. Here $\mathbb{T}, \Psi, \varepsilon$ are well-defined functions along with continuity and Δ^{β} and $\Delta^{\beta-1}$ represent the Riemann-Liouville (R-L) fractional order difference and sum operators, respectively. Suitable examples are delivered to clarify the strength of the theoretical consequences.

Keywords: Fractional difference equations, oscillation, Riemann-Liouville fractional difference, Caputo fractional difference, forcing term.

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1. Introduction

Fractional calculus, which is an evolving field of applied mathematics, considers integrals and derivatives of an arbitrary order. Greater part of the mathematical theory pertinent to the study of fractional calculus was put forward before the commencement of 20^{th} century. Fractional derivative is not a local property, making it unique in its behaviour and thus opening new paths and avenues for exploration and application. But in the recent decades it has emerged as one of the significant interdisciplinary subjects both in Physical & Biological Sciences and Engineering. Moreover, fractional calculus has applications in several fields, including visco-elasticity, electrochemical dynamics, physics, porous media, control, electromagnetism, and so forth; see [3, 10, 11, 16, 23] and the references therein.

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Fractional order difference equations, the discrete version of fractional differential equations has started gaining popularity among researchers. The delta operator (Δ) is termed as the forward difference operator while the nabla operator (∇) is called as the backward difference operator. Much research has been pursued to develop the properties of discrete fractional operators, both delta and nabla operators [8, 12]. In 1956, Kuttner mentioned for the first time the differences of fractional order [21]. Diaz and Osler presented discrete fractional operator defined as an infinite series [15] in 1974. In 2007, Atici and Eloe formulated the Riemann-Liouville like fractional difference operator by means of the notion of fractional sum [7]. In 2011, Holm advanced further research in this area and employed the tools of discrete fractional calculus to the arena of fractional difference equations [19].

Fractional discrete models have a major advantage over their conventional counterparts due to the infinite memory. The applications using discrete fractional calculus have gained much attention during the last few years. In recent years the study of qualitative properties like stability, positive solution, dynamic equations on timescales, non-oscillatory of fractional difference equations has been paid much attention, especially oscillation theory regarding fractional difference equations became a very interesting topic [1, 4–6, 13, 14, 20, 22, 24–29] and the references therein.

In this paper, oscillation for forced discrete fractional order sum-difference equation of the form

$$\Delta^{\beta}y(\iota)+\sum_{\varkappa=a}^{\iota-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa))=\varepsilon(\iota),\ 0<\beta<1,\ \iota\in\mathbb{N}_a=\{a,a+1,a+2,\ldots\}, \tag{1.1}$$

with $\Delta^{\beta-1}y(0)=y_0\in\mathbb{R}$ is discussed. Here $\mathbb{T},\Psi,\varepsilon$ are well-defined continuous functions and Δ^{β} and $\Delta^{\beta-1}$ represent the Riemann-Liouville fractional order difference and sum operators, respectively. The design of equation (1.1) is so broad that it addresses a wide range of specific situations. Following assumptions are useful in the discussion.

- (A_1) $\varepsilon:[a,\infty)\to\mathbb{R}$ and $\mathbb{T}:[a,\infty)\times[a,\infty)\to\mathbb{R}$ are continuous functions with $\mathbb{T}(\iota,\varkappa)\geqslant 0$ for all $\iota>\varkappa$.
- (A_2) There exists continuous functions $\rho,\varphi:[a,\infty)\to[0,\infty)$ such that $\mathbb{T}(\iota,\varkappa)\leqslant\rho(\iota)\varphi(\iota)$ for all $\iota\geqslant\varkappa$.
- (A_3) $\Psi:[a,\infty)\times\mathbb{R}\to\mathbb{R}$ with $\Psi(\iota,y):=\Psi_1(\iota,y)-\Psi_2(\iota,y)$ is continuous and there exists continuous functions $\Psi_1,\Psi_2:[a,\infty)\times\mathbb{R}\to\mathbb{R}$ such that $y\Psi_i(\iota,y)>0,(i=1,2)$, for $y\neq 0$ and $\iota\geqslant a$.
- (A_4) There exists real constants λ,η and continuous functions $\rho_1,\rho_2:[a,\infty)\to[0,\infty)$ such that $\Psi_1(\iota,y)\geqslant\rho_1(\iota)y^{\lambda}$ and $\Psi_2(\iota,y)\geqslant\rho_2(\iota)y^{\eta}$, $y\neq 0$, $\iota\geqslant a$.

The rest of the paper is structured as follows. Section 2 contains the basic definitions and lemmas which are the foundation for the work supported in this paper. Oscillation results for fractional order sum-difference forced equations are established by using properties of R-L & Caputo difference operators and Hardy inequalities in Section 3. Suitable examples of the fractional order sum-difference equation (1.1) are demonstrated in Section 4.

2. Preliminaries

The growth of fractional discrete calculus is quite rapid and it is only during the past decade, that the researchers have been shaping a complete framework for the subject. This section holds basic foundation of definitions and lemmas for the work sustained in this paper.

Definition 2.1 ([17])**.** A solution $\{y(\iota)\}$ is said to be *oscillatory* if the terms of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is called *nonoscillatory*.

Definition 2.2 ([7])**.** Let $\beta>0$. The β-th fractional sum $\Delta^{-\beta}:N_a\to N_{a+\beta}$ of y is defined by

$$\Delta^{-\beta}y(\iota)=\frac{1}{\Gamma(\beta)}\sum_{\varkappa=a}^{\iota-\beta}(\iota-\varkappa-1)^{(\beta-1)}y(\varkappa),\ \text{ for }\ \iota\in\mathbb{N}_{a+\beta},$$

where y is defined for $\varkappa \equiv a \bmod(1)$ and $\Delta^{-\beta}y$ is defined for $\iota \equiv (a+\beta) \bmod(1)$ and the falling function is

$$\iota^{(\beta)} = \frac{\Gamma(\iota+1)}{\Gamma(\iota-\beta+1)},$$

where Γ is the Gamma function.

Definition 2.3 ([7])**.** The RL β^{th}-order fractional difference Δ^{β} is expressed by

$$\Delta^{\beta}y(\iota) = \Delta^{\alpha}\Delta^{-(\alpha-\beta)}y(\iota),\ \iota \in \mathbb{N}_{\iota+\alpha-\beta},$$

and so

$$\Delta^{\beta}y(\iota) = \frac{\Delta^{\alpha}}{\Gamma(\alpha-\beta)}\sum_{\varkappa=a}^{\iota-\alpha+\beta}(\iota-\varkappa-1)^{(\alpha-\beta-1)}y(\varkappa),\ \iota \in \mathbb{N}_{\iota+\alpha-\beta}.$$

Also for the fractional sum, the law of exponent is

$$\Delta^{-\beta}[\Delta^{-\nu}y(\iota)] = \Delta^{-(\beta+\nu)}y(\iota) = \Delta^{-\nu}[\Delta^{-\beta}y(\iota)].$$

Lemma 2.4 ([9])**.** *For the fractional sum and difference operators, commutative property is as follows. For any* $\beta > 0$*, the below equality holds*

$$\Delta^{-\beta}\Delta y(\iota) = \Delta\Delta^{-\beta}y(\iota) - \frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}y(a).$$

Lemma 2.5 ([18])**.** *If* U *and* V *are non-negative, then*

(i) $U^{\lambda} - \lambda UV^{\lambda-1} - (1-\lambda)V^{\lambda} \geqslant 0,\ \lambda > 1;$
(ii) $U^{\lambda} - \lambda UV^{\lambda-1} - (1-\lambda)V^{\lambda} \leqslant 0,\ 0 < \lambda < 1,$

where the equality holds iff $U = V$.

3. Oscillation results of R-L difference

This section forms a new condition for the oscillation of fractional order sum-difference equation (1.1) based on the properties of R-L derivatives and Hardy inequalities.

Theorem 3.1. *Let conditions* (A_1)-(A_3) *be hold with* $\Psi_2 = 0$*. If for every constant* $\delta > 0$ *such that*

$$\limsup_{\iota\to\infty}\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)] = +\infty,$$
$$\liminf_{\iota\to\infty}\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)] = -\infty, \tag{3.1}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $y(\iota)$ be non-oscillatory solution of (1.1) with $\Psi_2 = 0$. Suppose that $y(\iota) > 0$ for $\iota \geqslant \iota_1$ and $\iota_1 \geqslant a$. From equation (1.1), we have

$$\Delta^{\beta}y(\iota) = \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa)) = \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa)) - \sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa))$$
$$\leqslant \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1}\mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa)) + \Psi_2(\varkappa,y(\varkappa))].$$

Since $\Psi_2 = 0$, we get

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1} \mathbb{T}(\iota,\varkappa)\Psi_1(\varkappa, y(\varkappa)). \tag{3.2}$$

Letting

$$\sigma = \min\left\{\Psi_j(\iota, y(\iota)); j = 1,2,\ \iota \in [a, \iota_1]\right\} \leqslant 0$$

and

$$\delta = -\sigma \sum_{\varkappa=a}^{\iota_1-1} \varphi(\varkappa) \geqslant 0.$$

Using condition (A_2), equation (3.2) can be rewriten as

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1} \rho(\varkappa)\varphi(\varkappa)\Psi_1(\varkappa, y(\varkappa)) \leqslant \varepsilon(\iota) - \sigma \sum_{\varkappa=a}^{\iota_1-1} \rho(\varkappa)\varphi(\varkappa), \quad \Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota) + \delta\rho(\iota).$$

Therefore applying $\Delta^{-\beta}$ on both-sides leads to

$$\Delta^{-\beta}\Delta^{\beta} y(\iota) \leqslant \Delta^{-\beta}[\varepsilon(\iota) + \delta\rho(\iota)].$$

Using Lemma 2.4 yields

$$\Delta^{\beta}\Delta^{-\beta} y(\iota) - \frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)} y(a) \leqslant \Delta^{-\beta}[\varepsilon(\iota) + \delta\rho(\iota)], \quad y(\iota) \leqslant \frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)} b_1 + \Delta^{-\beta}[\varepsilon(\iota) + \delta\rho(\iota)].$$

Taking limit as $\iota \to \infty$, we obtain

$$\liminf_{\iota\to\infty} y(\iota) \leqslant \liminf_{\iota\to\infty} \left[\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)} b_1 + \Delta^{-\beta}[\varepsilon(\iota) + \delta\rho(\iota)]\right],$$

or

$$\liminf_{\iota\to\infty} y(\iota) \leqslant \liminf_{\iota\to\infty} \left[\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)} b_1 + \Delta^{-\beta}[\varepsilon(\iota) + \delta\rho(\iota)]\right] = -\infty,$$

which is obviously a contradiction to $y(\iota) > 0$ eventually. The case when $y(\iota) < 0$ eventually is similar, therefore concludes the proof. □

Theorem 3.2. *Let conditions* (A_1)-(A_4) *be valid with* $\lambda > 1$ *and* $\eta = 1$. *In addition to conditions of Theorem 3.1, if*

$$\lim_{\iota\to\infty} \frac{1}{\Gamma(\beta)} \sum_{\ell=a}^{\iota-\beta} (\iota-\ell-1)^{(\beta-1)} \sum_{\varkappa=\iota_1}^{\ell-1} \mathbb{T}(\ell,\varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\rho_2^{\frac{\lambda}{\lambda-1}}(\varkappa) < \infty, \tag{3.3}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $y(\iota)$ be non-oscillatory solution of (1.1) with $y(\iota) > 0$ eventually for $\iota \geqslant \iota_1$. From conditions (A_3)-(A_4) with $\lambda > 1$ and $\eta = 1$, we get

$$\Delta^{\beta} y(\iota) = \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota-1} \mathbb{T}(\iota,\varkappa)\Psi(\varkappa, y(\varkappa)) = \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1} \mathbb{T}(\iota,\varkappa)\Psi(\varkappa, y(\varkappa)) - \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)\Psi(\varkappa, y(\varkappa)).$$

From Theorem 3.1, we have

$$\begin{aligned}\Delta^{\beta} y(\iota) &\leqslant \varepsilon(\iota)+\delta\rho(\iota)-\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa))-\Psi_2(\varkappa,y(\varkappa))]\\ &\leqslant \varepsilon(\iota)+\delta\rho(\iota)-\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_1(\varkappa)y^{\lambda}-\rho_2(\varkappa)y^{\eta}\right]\\ &\leqslant \varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y^{\eta}-\rho_1(\varkappa)y^{\lambda}\right],\end{aligned}$$

i.e.,

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y-\rho_1(\varkappa)y^{\lambda}\right]. \tag{3.4}$$

Using Lemma 2.5 leads to

$$\rho_2 y-\rho_1 y^{\lambda} \leqslant (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\rho_2^{\frac{\lambda}{\lambda-1}}.$$

Applying the above inequality in (3.4), we have

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\rho_2^{\frac{\lambda}{\lambda-1}}. \tag{3.5}$$

Multiply β^{th} fractional sum on both-sides to (3.5), we obtain

$$\Delta^{-\beta}\Delta^{\beta} y(\iota) \leqslant \Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\rho_2^{\frac{\lambda}{\lambda-1}}\right].$$

Using Lemma 2.4 yields

$$\Delta^{\beta}\Delta^{-\beta} y(\iota)-\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}y(a) \leqslant \Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]+\Delta^{-\beta}\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\rho_2^{\frac{\lambda}{\lambda-1}}\right],$$

or

$$\begin{aligned}y(\iota) \leqslant{}& \frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]\\ &+\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\rho_2^{\frac{\lambda}{\lambda-1}}(\varkappa).\end{aligned} \tag{3.6}$$

Taking limit inferior on both-sides of (3.6) as $\iota \to \infty$ and using (3.1) and (3.3), we have

$$\begin{aligned}\liminf_{\iota\to\infty} y(\iota) \leqslant{}& \liminf_{\iota\to\infty}\left[\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]\right]\\ &+\liminf_{\iota\to\infty}\left[\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\rho_2^{\frac{\lambda}{\lambda-1}}(\varkappa)\right],\end{aligned}$$

i.e.,

$$\liminf_{\iota\to\infty} y(\iota) = -\infty,$$

a contradiction with $y(\iota) > 0$ eventually. The case when $y(\iota) < 0$ eventually is similar, therefore concludes the proof. □

Theorem 3.3. *Let conditions* (A_1)-(A_4) *and Theorem 3.1 hold with* $\lambda = 1$ *and* $\eta < 1$, *if*

$$\lim_{\iota\to\infty} \frac{1}{\Gamma(\beta)} \sum_{\ell=a}^{\iota-\beta} (\iota-\ell-1)^{(\beta-1)} \sum_{\varkappa=\iota_1}^{\ell-1} \mathbb{T}(\ell,\varkappa)\rho_1^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}(\varkappa) < \infty, \tag{3.7}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $y(\iota)$ be non-oscillatory solution of (1.1) with $y(\iota) > 0$ eventually for $\iota \geqslant \iota_1$. Form conditions (A_3)-(A_4) with $\lambda = 1$ and $\eta < 1$, we get

$$\begin{aligned}\Delta^{\beta} y(\iota) =& \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota-1} \mathbb{T}(\iota,\varkappa)\Psi(\varkappa, y(\varkappa))\\ =& \varepsilon(\iota) - \sum_{\varkappa=a}^{\iota_1-1} \mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa)) - \Psi_2(\varkappa,y(\varkappa))] - \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa)) - \Psi_2(\varkappa,y(\varkappa))].\end{aligned}$$

From Theorem 3.1, we have

$$\begin{aligned}\Delta^{\beta} y(\iota) &\leqslant \varepsilon(\iota) + \delta\rho(\iota) - \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa)) - \Psi_2(\varkappa,y(\varkappa))]\\ &\leqslant \varepsilon(\iota) + \delta\rho(\iota) - \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)\left[\rho_1(\varkappa)y^{\lambda} - \rho_2(\varkappa)y^{\eta}\right]\\ &\leqslant \varepsilon(\iota) + \delta\rho(\iota) + \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y^{\eta} - \rho_1(\varkappa)y^{\lambda}\right],\end{aligned}$$

since $\lambda = 1$ leads to

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota) + \delta\rho(\iota) + \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y^{\eta} - \rho_1(\varkappa)y\right]. \tag{3.8}$$

Using Lemma 2.5 leads to

$$\rho_2(\varkappa)y^{\eta} - \rho_1(\varkappa)y \leqslant (1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}.$$

Applying the above inequality in (3.8), we have

$$\Delta^{\beta} y(\iota) \leqslant \varepsilon(\iota) + \delta\rho(\iota) + \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}. \tag{3.9}$$

Multiply β^{th} fractional sum on both-sides to (3.9), we obtain

$$\Delta^{-\beta}\Delta^{\beta} y(\iota) \leqslant \Delta^{-\beta}\left[\varepsilon(\iota) + \delta\rho(\iota) + \sum_{\varkappa=\iota_1}^{\iota-1} \mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}\right].$$

Using Lemma 2.4 yields

$$\Delta^{\beta}\Delta^{-\beta}y(\iota)-\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}y(a)\leqslant\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]+\Delta^{-\beta}\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}\right],$$

or

$$y(\iota)\leqslant\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]+\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}.$$

Taking limit inferior on both-sides of (3.6) as $\iota\to\infty$ and using (3.1) and (3.7), we have

$$\begin{aligned}\liminf_{\iota\to\infty}y(\iota)\leqslant&\liminf_{\iota\to\infty}\left[\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]\right]\\&+\liminf_{\iota\to\infty}\left[\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}\right],\end{aligned}$$

i.e.,

$$\liminf_{\iota\to\infty}y(\iota)=-\infty,$$

a contradiction with $y(\iota)>0$ eventually. The case when $y(\iota)<0$ eventually is similar, therefore concludes the proof. □

Theorem 3.4. *Let conditions* (A_1)-(A_4) *and Theorem 3.1 be hold with* $\lambda>1$ *and* $\eta<1$, *suppose that there exists a continuous function* $\chi:\mathbb{R}\to[0,\infty)$ *such that*

$$\lim_{\iota\to\infty}\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)<\infty, \tag{3.10}$$

and

$$\lim_{\iota\to\infty}\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}(\varkappa)<\infty, \tag{3.11}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $y(\iota)$ be non-oscillatory solution of (1.1) with $y(\iota)>0$ eventually for $\iota\geqslant\iota_1$. Form conditions (A_3)-(A_4) with $\lambda>1$ and $\eta<1$, we get

$$\begin{aligned}\Delta^{\beta}y(\iota)&=\varepsilon(\iota)-\sum_{\varkappa=a}^{\iota-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa))\\&=\varepsilon(\iota)-\sum_{\varkappa=a}^{\iota_1-1}\mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa))-\Psi_2(\varkappa,y(\varkappa))]-\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa))-\Psi_2(\varkappa,y(\varkappa))].\end{aligned}$$

From Theorem 3.1, we have

$$\Delta^{\beta}y(\iota)\leqslant\varepsilon(\iota)+\delta\rho(\iota)-\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)[\Psi_1(\varkappa,y(\varkappa))-\Psi_2(\varkappa,y(\varkappa))].$$

Using Theorems 3.2 and 3.3 leads to

$$\Delta^{\beta}y(\iota)\leqslant\varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y-\rho_1(\varkappa)y^{\lambda}\right]+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y^{\eta}-\rho_1(\varkappa)y\right]. \tag{3.12}$$

Taking $\rho_2(\varkappa)=\rho_1(\varkappa)=\chi(\varkappa)$, (3.12) yields

$$\Delta^{\beta}y(\iota)\leqslant\varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[y(\varkappa)\chi(\varkappa)-\rho_1(\varkappa)y^{\lambda}\right]+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)\left[\rho_2(\varkappa)y^{\eta}-y(\varkappa)\chi(\varkappa)\right].$$

Using Lemma 2.4, we obtain

$$\Delta^{\beta}y(\iota)\leqslant\varepsilon(\iota)+\delta\rho(\iota)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)+\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}. \tag{3.13}$$

Multiplying β^{th} fractional sum on both-sides to (3.13), we obtain

$$\begin{aligned}\Delta^{-\beta}\Delta^{\beta}y(\iota)\leqslant\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]+\Delta^{-\beta}&\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)\right]\\&+\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}\right].\end{aligned}$$

Using Lemma 2.4 yields

$$\begin{aligned}\Delta^{\beta}\Delta^{-\beta}y(\iota)-\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}y(a)\leqslant\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]+\Delta^{-\beta}&\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)\right]\\&+\Delta^{-\beta}\left[\sum_{\varkappa=\iota_1}^{\iota-1}\mathbb{T}(\iota,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\rho_1^{\frac{\eta}{\eta-1}}\rho_2^{\frac{1}{1-\eta}}\right],\end{aligned}$$

or

$$\begin{aligned}y(\iota)\leqslant&\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]\\&+\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)\\&+\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}.\end{aligned} \tag{3.14}$$

Taking limit inferior on both-sides of (3.14) as $\iota\to\infty$ and using (3.1), (3.10), and (3.11), we have

$$\begin{aligned}\liminf_{\iota\to\infty}y(\iota)\leqslant\liminf_{\iota\to\infty}&\left[\frac{(\iota-a)^{(\beta-1)}}{\Gamma(\beta)}b_1+\Delta^{-\beta}\left[\varepsilon(\iota)+\delta\rho(\iota)\right]\right]\\&+\liminf_{\iota\to\infty}\left[\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\rho_1^{\frac{1}{1-\lambda}}\chi^{\frac{\lambda}{\lambda-1}}(\varkappa)\right]\end{aligned}$$

$$+\liminf_{\iota\to\infty}\left[\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)(1-\eta)\eta^{\frac{\eta}{1-\eta}}\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}\right],$$

i.e.,

$$\liminf_{\iota\to\infty} y(\iota)=-\infty,$$

a contradiction with $y(\iota)>0$ eventually. The case when $y(\iota)<0$ eventually is similar, therefore concludes the proof. □

3.1. Results for Caputo-like fractional difference operator

If we interchange the Caputo-like fractional difference operator by Reimann-Liouville fractional difference operator well-defined by

$${}^{C}\Delta^{\beta}y(\iota)={}^{C}\Delta^{-(n-\beta)}\Delta^{n}y(\iota)=\frac{1}{\Gamma(n-\beta)}\sum_{\varkappa=a}^{1-(n-\beta)}(\iota-\varkappa-1)^{(n-\beta-1)}\Delta^{n}y(\varkappa),\ \iota\in\mathbb{N}_{a+n-\beta},$$

equation (1.1) changes into

$${}^{C}\Delta^{\beta}y(\iota)+\sum_{\varkappa=a}^{\iota-1}\mathbb{T}(\iota,\varkappa)\Psi(\varkappa,y(\varkappa))=\varepsilon(\iota),\ n-1<\beta<n,\ \iota\in\mathbb{N}_a, \tag{3.15}$$

with $\Delta^{i}y(a)=b_i\in\mathbb{R}, i=0,1,2,\dots,n-1$. The following lemma is essential for providing the result for (3.15).

Lemma 3.5 ([2]). *Assume $\beta>0$ and y from $\mathbb{N}_a$ and $\mathbb{N}_b$. Then*

$$\Delta^{-\beta\,C}\Delta^{\beta}y(\iota)=y(\iota)-\sum_{\varkappa=0}^{n-1}\frac{(\iota-a)}{\varkappa!}\Delta^{\varkappa}y(a).$$

Theorem 3.6. *Let conditions (A_1)-(A_4) hold with $\Psi_2=0$. If for every constant $\delta>0$ such that*

$$\limsup_{\iota\to\infty}\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)]=+\infty,$$

and

$$\liminf_{\iota\to\infty}\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)]=-\infty, \tag{3.16}$$

then every solution of (3.15) is oscillatory.

Proof. Let $y(\iota)$ be non-oscillatory solution of (3.15) with $\Psi_2=0$. Suppose that $y(\iota)>0$ for $\iota\geqslant\iota_1$ and $\iota_1\geqslant a$. From equation (3.15) and proceeding as in the proof of Theorem 3.1, we have

$${}^{C}\Delta^{\beta}y(\iota)\leqslant\varepsilon(\iota)+\delta\rho(\iota).$$

Multiplying $\Delta^{-\beta}$ on both-sides of inequality (3.16) leads to

$$\Delta^{-\beta\,C}\Delta^{\beta}y(\iota)\leqslant\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)].$$

Using Lemma 3.5 yields

$$y(\iota)-\sum_{\varkappa=0}^{n-1}\frac{(\iota-a)}{\varkappa!}\Delta^{\varkappa}y(a)\leqslant\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)],\quad y(\iota)\leqslant\sum_{\varkappa=0}^{n-1}\frac{(\iota-a)}{\varkappa!}\Delta^{\varkappa}y(a)+\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)].$$

Taking limit as $\iota \to \infty$, we obtain

$$\liminf_{\iota\to\infty} y(\iota) \leqslant \liminf_{\iota\to\infty}\left[\sum_{\varkappa=0}^{n-1}\frac{(\iota-a)}{\varkappa!}\Delta^{\varkappa}y(a)+\Delta^{-\beta}[\varepsilon(\iota)+\delta\rho(\iota)]\right]=-\infty,$$

which is clearly contradiction to $y(\iota) > 0$ eventually. The case when $y(\iota) < 0$ eventually is similar, therefore concludes the proof. □

The proofs of the remaining theorems are removed since their arguments are similar to those of the R-L difference operator.

4. Examples

In this section, appropriate examples are presented to clarify the validity of the above results attained in Section 3. It is noteworthy to notice that no results in the previous literature can explain the oscillation of equations (4.1) and (4.2).

Example 4.1. In equation (1.1), we set $\mathbb{T}(\iota,\varkappa)=\iota\varkappa$, $\Psi_1(\iota,y)=y$, $\Psi_2(\iota,y)=\frac{y}{\iota}$, $\varepsilon(\iota)=\frac{\iota^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)}+\frac{\iota^4}{3}$, $a=0$, and fractional order $\beta=\frac{1}{2}$. Thus the discrete fractional order sum-difference equation (1.1) is replaced by

$$\Delta^{\frac{1}{2}}y(\iota)+\sum_{\varkappa=0}^{\iota-1}\iota\varkappa\left[y(\varkappa)+\frac{y(\varkappa)}{\iota}\right]=\frac{\iota^{\frac{2}{3}}}{\Gamma\left(\frac{5}{3}\right)}+\frac{\iota^4}{3},\ 0<\beta<1,\ \iota\in\mathbb{N}_0, \tag{4.1}$$

with $\Delta^{-\frac{1}{2}}y(0)=b_1=0$. Obviously assumptions (A_1)-(A_4) hold. Moreover, $\lambda=2$, $\eta=1$, $\rho_1(\iota)=\frac{1}{\iota^3}$, and $\rho_2(\iota)=\iota$. Now,

$$(\iota-\ell-1)^{\beta-1}=(\iota-\ell-1)^{(-\frac{1}{2})}\leqslant\sqrt{\pi},\ell=1,2,\dots,\iota-1,$$

and

$$\mathbb{T}(\ell,\varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\rho_2^{\frac{\lambda}{\lambda-1}}(\varkappa)=\ell\varkappa\times\varkappa^3\times\varkappa^2=\ell\varkappa^6.$$

Hence

$$\lim_{\iota\to\infty}\frac{1}{\Gamma(\beta)}\sum_{\ell=a}^{\iota-\beta}(\iota-\ell-1)^{(\beta-1)}\sum_{\varkappa=\iota_1}^{\ell-1}\mathbb{T}(\ell,\varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\rho_2^{\frac{\lambda}{\lambda-1}}(\varkappa)\leqslant\lim_{\iota\to\infty}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{\ell=0}^{\iota-\frac{1}{2}}\sqrt{\pi}\sum_{\varkappa=\iota_1}^{\ell-1}\ell\varkappa^6<\infty.$$

Thus condition (3.3) is satisfied. We deduce that every solution of equation (4.1) is oscillatory from Theorem 3.2.

Example 4.2. In equation (1.1), we set $\mathbb{T}(\iota,\varkappa)=\frac{\iota}{\varkappa}$, $\Psi_1(\iota,y)=y^2$, $\Psi_2(\iota,y)=\frac{y}{\iota^2}$, $\varepsilon(\iota)=\frac{2\iota^{\frac{1}{2}}}{\sqrt{\pi}}+\frac{\iota^5}{4}$, $a=0$, and fractional order $\beta=\frac{3}{2}$. Thus the discrete fractional order sum-difference equation (1.1) is replaced by

$$\Delta^{\frac{3}{2}}y(\iota)+\sum_{\varkappa=0}^{\iota-1}\frac{\iota}{\varkappa}\left[y^2(\varkappa)+\frac{y(\varkappa)}{\iota^2}\right]=\frac{2\iota^{\frac{1}{2}}}{\sqrt{\pi}}+\frac{\iota^5}{4},\ 0<\beta<1,\ \iota\in\mathbb{N}_0, \tag{4.2}$$

with $\Delta^{\frac{1}{2}}y(0)=b_1=0$. Obviously assumptions (A_1)-(A_4) hold. Moreover, $\lambda=2$, $\eta=\frac{1}{2}$, $\rho_1(\iota)=\frac{1}{\iota}$,

$\rho_2(\iota) = \iota^2$, and the continuous function $\chi(\iota) = \iota$. Now,

$$(\iota - \ell - 1)^{\beta - 1} = (\iota - \ell - 1)^{(\frac{1}{2})} \leqslant \Gamma\left(\frac{3}{2}\right), \ell = 1, 2, \dots, \iota - 1,$$

and

$$\mathbb{T}(\ell, \varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\chi^{\frac{\lambda}{\lambda-1}}(\varkappa) = \frac{\ell}{\varkappa} \times \varkappa \times \varkappa^3 = \ell\varkappa^3,$$
$$\mathbb{T}(\ell, \varkappa)\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}(\varkappa) = \frac{\ell}{\varkappa} \times \frac{1}{\varkappa} \times \varkappa^4 = \ell\varkappa^2,$$
$$\lim_{\iota\to\infty} \frac{1}{\Gamma(\beta)} \sum_{\ell=a}^{\iota-\beta} (\iota - \ell - 1)^{(\beta-1)} \sum_{\varkappa=\iota_1}^{\ell-1} \mathbb{T}(\ell, \varkappa)\rho_1^{\frac{1}{1-\lambda}}(\varkappa)\chi^{\frac{\lambda}{\lambda-1}}(\varkappa) \leqslant \lim_{\iota\to\infty} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sum_{\ell=0}^{\iota-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{\varkappa=\iota_1}^{\ell-1} \ell\varkappa^3 < \infty,$$

and

$$\lim_{\iota\to\infty} \frac{1}{\Gamma(\beta)} \sum_{\ell=a}^{\iota-\beta} (\iota - \ell - 1)^{(\beta-1)} \sum_{\varkappa=\iota_1}^{\ell-1} \mathbb{T}(\ell, \varkappa)\chi^{\frac{\eta}{\eta-1}}(\varkappa)\rho_2^{\frac{1}{1-\eta}}(\varkappa) \leqslant \lim_{\iota\to\infty} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sum_{\ell=0}^{\iota-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{\varkappa=\iota_1}^{\ell-1} \ell\varkappa^2 < \infty.$$

Thus conditions (3.10) and (3.11) are satisfied. We deduce that every solution of equation (4.2) is oscillatory from Theorem 3.4.

5. Conclusion

This work presents oscillation theorems for fractional order sum-difference forced equations, utilizing R-L and Caputo difference operators and Hardy Inequalities. The main equation is structured in a broad manner, thus it encompasses many of the specific circumstances. Suitable examples are obtained to demonstrate the validity of the theoretical results. Our future research will focus on the forced oscillation criteria for fractional order partial difference equations.

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