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The Appell sequences of fractional type

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Abstract

In the article, we explore a form of generalization of Appell polynomials stemming from fractional differential operators within the classical sense of Caputo and Riemann-Liouville. To ascertain its generating function, we used the Mittag-Leffler function. Additionally, we propose a determinant form for this novel sequence family and derive general properties thereof.

Keywords: The Appell polynomials, Caputo operator, Riemann-Liouville operator, Mittag-Leffler function.

2020 MSC: 11B83, 26A33, 11B68.

1. Introduction

The Appell polynomials, denoted as $\{\mathscr{A}_n(x)\}_{n\in\mathbb{N}_0}$, constitute a significant mathematical sequence introduced by the esteemed French mathematician Paul Appell (see [3]). These polynomials satisfy the differential equation:

$$\frac{d}{dx}\mathscr{A}_n(x) = n\mathscr{A}_{n-1}(x),\ n \in \mathbb{N}, \tag{1.1}$$

with $A_0(x)$ being a non-zero constant. Alternatively, the sequence can be elegantly expressed through the generating function:

$$f(t)e^{xt} = \sum_{n=0}^{\infty} \mathscr{A}_n(x)\frac{t^n}{n!},$$

where f is a formal power series in t.

The Appell polynomials exhibit diverse properties that render them invaluable in the realm of mathematical analysis, particularly within the study of differential equations and related fields, as documented by Adel, Khan et al., and Nemati et al., [9, 11]. Prominent instances of polynomial sequences satisfying equation (1.1), or equivalently the recursive relations, encompass the well-known polynomials of Bernoulli and Euler. The exponential generating functions for the geometric polynomials of Bernoulli and Euler are expressed as follows (refer to [2]):

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi. \tag{1.2}$$

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Recently, Caratelli et al. [6], extended the classical Bernoulli and Euler polynomials substituting the fractional exponential E_α of order $0 < \alpha \leqslant 1$ (a particular case of the two-parameters Mittag-Leffler function $E_{\alpha,\beta}$) in place of the ordinary exponential in the corresponding classical generating function:

$$\frac{t^\alpha E_\alpha(x^\alpha t^\alpha)}{E_\alpha(t^\alpha)-1} = \sum_{n=0}^{\infty} B_n^\alpha(x)\frac{t^{\alpha n}}{\Gamma(n\alpha+1)} \quad \text{and} \quad \frac{2E_\alpha(x^\alpha t^\alpha)}{E_\alpha(t^\alpha)+1} = \sum_{\nu=0}^{\infty} E_n^\alpha(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{1.3}$$

We recall $E_\alpha(x^\alpha t^\alpha) \equiv e^{xt}$, for $\alpha = 1$. This new generalization is called by the authors of [6] as fractional Appell-type polynomials of order α. It's important to note that the designations Appell-type polynomials actually represent functions rather than conforming strictly to the polynomial definition. However, because they consist of combinations of monomials with fractional powers, for the authors it is more suitable to refer to them as polynomials, serving as a convenient shorthand for fractional power polynomials.

In [6, 13], some examples of these generalized mathematical entities were demonstrated. Specifically, they focused on and provided numerical examples of Bernoulli and Euler numbers (i.e., for $x = 0$ in (1.3)), as well as Laguerre-type Bernoulli and Euler numbers. However, many theories and results of classical polynomials remained unextended. For instance, the theory is not presented abstractly, and there is no indication regarding the specific type of differential equation, such as equation (1.1), that satisfies this family of polynomials, among other aspects. Furthermore, they do not present definitions based on the other specific cases of the Mittag-Leffler function.

In that sense, the objective of this study is to embark on an exploration of fractional Appell-type polynomials in abstract form, building upon the interesting work initiated by researchers Diego Caratelli and Paolo Emilio Ricci. Our focus lies particularly on incorporating differential operators of arbitrary order, namely, the Caputo ${}_CD_x^\alpha$ and Riemann-Liouville ${}_RD_x^\alpha$ operators into our analysis. This extension significantly enhances the versatility of our approach, allowing us to introduce novel polynomials termed the Appell-Caputo sequence $\{\mathscr{C}_n(x)\}_{n\in\mathbb{N}_0}$ and the Appell-Riemann sequence $\{\mathscr{R}_n(x)\}_{n\in\mathbb{N}_0}$. These sequences are defined by the following equations:

$${}_CD_x^\alpha \mathscr{C}_n^\alpha(x) = n\mathscr{C}_{n-1}^\alpha(x), \qquad {}_RD_x^\alpha \mathscr{R}_n^\alpha(x) = n\mathscr{R}_{n-1}^\alpha(x),\ n \in \mathbb{N}.$$

We shall demonstrate that the Apple-Caputo and Appell-Riemann type sequences possess captivating generating functions:

$$a(t)E_\alpha(x^\alpha t) = \sum_{n=0}^{\infty} \mathscr{C}_n^\alpha(x)\frac{t^n}{n!} \quad \text{and} \quad b(t)x^{\alpha-1}E_{\alpha,\alpha}(x^\alpha t) = \sum_{n=0}^{\infty} \mathscr{R}_n^\alpha(x)\frac{t^n}{n!},$$

where $a(t) := \sum_{r\geqslant 0} a_r \frac{t^r}{r!}$ and $b(t) := \sum_{r\geqslant 0} b_r \frac{t^r}{r!}$. Furthermore, we derive various algebraic and differential properties of these sequences, predominantly relying on generating function methods. This comprehensive exploration enhances our understanding and opens avenues for further mathematical inquiries.

The article is structured as follows. In Section 2, we delve into the definitions and properties of fractional differential operators, alongside an exploration of the Mittag-Leffler function, and in Section 3 we introduce the Appell sequences of fractional type, presenting their properties and various representations.

2. Fractional calculus and Mittag Leffler function

To introduce the fractional derivative, we begin by defining the function $g_\alpha(x) := \dfrac{x^{\alpha-1}}{\Gamma(\alpha)}$ for $x > 0$ and $\alpha > 0$, where Γ is the Euler gamma function. Additionally, we define $g_0 := \delta_0$, representing the Dirac measure concentrated at 0. The family $(g_\alpha)_{\alpha>0}$ adheres to the semigroup property:

$$g_{\alpha+\beta} = g_\alpha * g_\beta,\ \alpha, \beta \geqslant 0.$$

The Riemann-Liouville fractional integral of order $0 < \alpha < 1$ for a locally integrable function $u : [0, \infty) \to X$ is expressed as:

$$I_x^{\alpha} u(x) := (g_{\alpha} * u)(x) := \int_0^x g_{\alpha}(t-s)u(s)ds.$$

For the Caputo fractional derivative of order $0 < \alpha < 1$ of a function u, we use the formula:

$${}_C D_t^{\alpha} u(x) := I_x^{m-\alpha} u^{(m)}(x) = \int_0^x g_{m-\alpha}(x-s)u^{(m)}(s)ds,$$

where $m := \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. When $\alpha = n$ is a natural number, we recover the classical derivative ${}_C D_x^n := \frac{d^n}{dx^n}$. On the other hand, the Riemann-Liouville fractional derivative ${}_R D_x^{\alpha}$ of u of order $\alpha > 0$ is given by

$${}_R D_x^{\alpha} u(x) := \frac{d}{dx^m}\left(I_x^{m-\alpha} u(x)\right), \ x \geqslant 0.$$

For $k \in \mathbb{R}$ and $0 < \alpha < 1$, we recall that

$${}_C D_x^{\alpha} x^{\gamma} = {}_R D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \ \gamma > 0. \tag{2.1}$$

As a particular case, we have that

$${}_C D_x^{\alpha} k = 0 \quad \text{and} \quad {}_R D_x^{\alpha} x^{\alpha-1} = 0. \tag{2.2}$$

For more details about fractional calculus, see [1, 4, 10].

The Mittag-Leffler function, as described in various references (e.g., [12]), is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{\nu=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}, \ z \in \mathbb{C}. \tag{2.3}$$

We write $E_{\alpha,1}(z) \equiv E_{\alpha}(z)$. The Mittag-Leffler function, an entire function, serves as a straightforward extension of the exponential function $E_1(z) = e^z$ and the cosine function $E_2\left(-z^2\right) = \cos(z)$.

The Laplace transform of the Mittag-Leffler function is expressed as (see [5, 8]):

$$\int_0^{\infty} e^{-\lambda x} x^{\beta-1} E_{\alpha,\beta}\left(\pm z x^{\alpha}\right) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} \mp z}, \quad \mathrm{Re}(\lambda) > |z|^{1/\alpha}.$$

Utilizing this crucial formula, we derive the following expression for $0 < \alpha \leqslant 2$:

$${}_C D_x^{\alpha} E_{\alpha}\left(z x^{\alpha}\right) = z E_{\alpha}\left(z x^{\alpha}\right), \quad x > 0, z \in \mathbb{C},$$

and

$${}_R D_x^{\alpha} \mathcal{E}_{\alpha,\alpha}\left(z x^{\alpha}\right) = z \mathcal{E}_{\alpha,\alpha}\left(z x^{\alpha}\right), \quad x > 0, z \in \mathbb{C},$$

where $\mathcal{E}_{\alpha,\alpha}\left(z x^{\alpha}\right) := x^{\alpha-1} E_{\alpha,\alpha}\left(z x^{\alpha}\right)$.

3. Appell sequences of fractional type.

Initially, we establish an exploration of the Appell sequences of fractional type within the framework of the Caputo fractional operator. Through detailed analysis and definition, we delve into the intricate characteristics and properties of these polynomials, their significance, and applications within fractional calculus. Let

$$\begin{bmatrix} n \\ r \end{bmatrix}_{\alpha,\mu} := \frac{n!}{(n-r)!\Gamma(\alpha r+\mu)}.$$

We write

$$\begin{bmatrix} n \\ r \end{bmatrix}_{\alpha,1} := \begin{bmatrix} n \\ r \end{bmatrix}_{\alpha}.$$

Definition 3.1. Let $0 < \alpha \leqslant 1$. Sequences of functions of order α, denoted as $\{\mathscr{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$, are termed Appell-Caputo sequences when they adhere to the relationship:

$$_C D_x^{\alpha} \mathscr{C}_n^{\alpha}(x) = n\mathscr{C}_{n-1}^{\alpha}(x), \ n \in \mathbb{N}. \tag{3.1}$$

Remark 3.2. Observe that equation (3.1), when $\alpha = 1$, yields (1.1) due to the definition of the Caputo derivative. In other words, the classical Appell polynomials are a particular case of the Appell-Caputo sequences.

An alternative approach to establish the Appell-Caputo sequences is as follows.

Theorem 3.3. *Let $0 < \alpha \leqslant 1$ and $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence of arbitrary numbers. The sequence of functions $\{\mathscr{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ give by*

$$\mathscr{C}_n^{\alpha}(x) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} a_{n-j} x^{\alpha j}, \ x \geqslant 0, \tag{3.2}$$

satisfying the relation (3.1).

Proof. By (2.1) and (2.2), we have

$$\begin{aligned} _C D_x^{\alpha} \mathscr{C}_n^{\alpha}(x) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} a_{n-j}\, {_C D_x^{\alpha}} x^{\alpha j} \\ &= \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} a_{n-j} \frac{\Gamma(\alpha j+1)}{\Gamma(\alpha j-\alpha+1)} x^{\alpha j-\alpha} \\ &= \sum_{j=0}^{n-1} \frac{n!}{(n-1-j)!\Gamma(\alpha j+1)} a_{n-1-j} x^{\alpha j} \\ &= n \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{\alpha} a_{n-1-j} x^{\alpha j} = n\mathscr{C}_{n-1}^{\alpha}(x). \end{aligned}$$

□

Now, we establish the generating function of the Appell-Caputo polynomials.

Theorem 3.4. *Given the power series*

$$a(t) := \sum_{j=0}^{\infty} a_j \frac{t^j}{j!}, \ a_0 \neq 0, \tag{3.3}$$

with a_j, $j = 0, 1, \ldots$, real coefficients, a Appell-Caputo sequences $\{\mathscr{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ is determined by the power series expansion of the product $a(t)E_{\alpha}(x^{\alpha}t)$, i.e.,

$$a(t)E_{\alpha}(x^{\alpha}t) = \sum_{n=0}^{\infty} \mathscr{C}_n^{\alpha}(x) \frac{t^n}{n!}. \tag{3.4}$$

Proof. By (2.3), (3.3), and using the following identity (see [7, p. 18, Eq. 0.36]):

$$\left(\sum_{\nu=0}^{\infty} a_n\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}b_k, \tag{3.5}$$

we can recognize polynomials $\mathscr{C}_n^\alpha(x)$, expressed in form (3.2), as coefficients of $t^j/j!$. □

Remark 3.5. Observe that in [6], the fractional Appell-type polynomials of order α are defined by the following generating function:

$$G_\alpha(x,t) = A(t)E_\alpha(x^\alpha t^\alpha) = \sum_{n=0}^{\infty} R_{n\alpha}(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

By implementing certain modifications, both generating functions can coincide.

Example 3.6. In (3.4), note that

1. if $a(t) := \frac{t}{e^t-1}$ and $\alpha = 1$, we have Bernoulli polynomials classic;
2. if $a(t) := \frac{2}{e^t+1}$ and $\alpha = 1$, we have Euler polynomials classic.

See the generating functions (1.2).

Example 3.7. Let $a(t) := \frac{t}{e^t-1}$ and $0 < \alpha \leqslant 1$. Then,

$$\begin{aligned}
\mathscr{C}_0^\alpha(x) &= 1,\\
\mathscr{C}_1^\alpha(x) &= \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2},\\
\mathscr{C}_2^\alpha(x) &= \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{1}{6},\\
\mathscr{C}_3^\alpha(x) &= \frac{6x^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{3x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{1}{2}\frac{x^\alpha}{\Gamma(\alpha+1)},\\
\mathscr{C}_4^\alpha(x) &= \frac{24x^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{12x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{1}{30}.
\end{aligned}$$

From the generating function, we derive the determinantal form of the Appell-Caputo sequences.

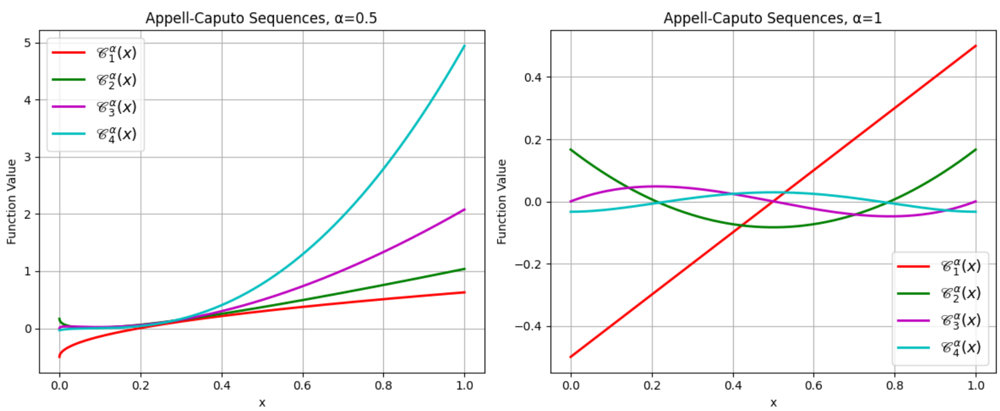

Figure 1: Appell-Caputo sequence of Example 3.7.

Theorem 3.8. *The sequences of Appell-Caputo sequences has the following determinantal representation:*

$$\mathscr{C}_0^{\alpha}(x) = \frac{1}{\gamma_0},$$
$$\mathscr{C}_n^{\alpha}(x) = \frac{(-1)^n}{\gamma_0^{n+1}} \begin{vmatrix} Q_0^{\alpha}(x) & Q_1^{\alpha}(x) & Q_2^{\alpha}(x) & \cdots & \cdots & Q_{n-1}^{\alpha}(x) & Q_n^{\alpha}(x) \\ \gamma_0 & \gamma_1 & \frac{1}{2}\gamma_2 & \cdots & \cdots & \frac{1}{(n-1)!}\gamma_n & \frac{1}{n!}\gamma_n \\ 0 & \gamma_0 & \gamma_1 & \cdots & \cdots & \frac{1}{(n-2)!}\gamma_{n-2} & \frac{1}{(n-1)!}\gamma_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \gamma_0 & \gamma_1 \end{vmatrix}, \tag{3.6}$$

where $Q_n^{\alpha}(x) := \dfrac{x^{\alpha n}}{\Gamma(\alpha n+1)}$ *and* $[a(t)]^{-1} = \sum_{k=0}^{\infty} \gamma_k \frac{t^k}{k!}$.

Proof. Inverse of $a(t)$ applied to both sides of the generating function (3.4) yields

$$E_{\alpha}(x^{\alpha}t) = \sum_{k=0}^{\infty} \gamma_k \frac{t^k}{k!} \sum_{n=0}^{\infty} \mathscr{C}_n^{\alpha}(x) \frac{t^n}{n!}.$$

Applying (3.5), we get

$$\sum_{n=0}^{\infty} Q_n^{\alpha}(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\gamma_k \mathscr{C}_{n-k}^{\alpha}(x)}{(n-k)!k!} t^n.$$

Equating the coefficients of t^n, we have

$$Q_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\gamma_k \mathscr{C}_{n-k}^{\alpha}(x)}{(n-k)!k!}, \ n \in \mathbb{N}_0.$$

As a result, the system of equations in the unknown $\mathscr{C}_n^{\alpha}(x)$ is as follows:

$$\begin{aligned} Q_0^{\alpha}(x) &= \gamma_0 \mathscr{C}_0^{\alpha}(x), \\ Q_1^{\alpha}(x) &= \gamma_0 \mathscr{C}_1^{\alpha}(x) + \gamma_1 \mathscr{C}_0^{\alpha}(x), \\ Q_2^{\alpha}(x) &= \frac{1}{2}\gamma_0 \mathscr{C}_2^{\alpha}(x) + \gamma_1 \mathscr{C}_1^{\alpha}(x) + \frac{1}{2}\gamma_2 \mathscr{C}_0^{\alpha}(x), \\ &\vdots \\ Q_n^{\alpha}(x) &= \frac{\gamma_0 \mathscr{C}_n^{\alpha}(x)}{n!} + \frac{\gamma_1 \mathscr{C}_{n-1}^{\alpha}(x)}{(n-1)!} + \frac{\gamma_2 \mathscr{C}_{n-2}^{\alpha}(x)}{2(n-1)!} + \cdots + \frac{\gamma_n \mathscr{C}_0^{\alpha}(x)}{n!}. \end{aligned}$$

By using Cramer's rule and the properties of the determinant of a triangular matrix, we have

$$\mathscr{C}_n^{\alpha}(x) = \frac{1}{\gamma^{n+1}} \begin{vmatrix} \gamma_0 & 0 & 0 & \cdots & Q_0^{\alpha}(x) \\ \gamma_1 & \gamma_0 & 0 & \cdots & Q_1^{\alpha}(x) \\ \frac{1}{2}\gamma_2 & \gamma_1 & \frac{1}{2}\gamma_0 & \cdots & Q_2^{\alpha}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\gamma_n}{n!} & \frac{\gamma_{n-1}}{(n-1)!} & \frac{\gamma_{n-2}}{(n-2)!} & \cdots & Q_n^{\alpha}(x) \end{vmatrix}.$$

Finally, by properties of the transpose of a matrix, we get the specified result. □

Utilizing Theorems 3.3, 3.4, and 3.8, we can deduce the subsequent circular theorem.

Theorem 3.9. *For the Appell-Caputo sequence, the following statements are equivalent.*

1. *$\{\mathcal{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ is a the Appell-Caputo sequence.*
2. *The Appell-Caputo sequence $\{\mathcal{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ possess a generating function given by (3.4).*
3. *The Appell-Caputo sequence $\{\mathcal{C}_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ are expressed in a determinantal form given by (3.6).*

In the next result, let us consider the following definition:

$$(x \oplus_{\alpha} y)^n := \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_{\alpha} x^r y^{n-r},$$

where

$$\left(\begin{array}{c} n \\ r \end{array} \right)_{\alpha} := \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-r)+1)\Gamma(\alpha r+1)}.$$

Proposition 3.10. *The following identity is hold*

$$\mathscr{C}_n\left(x \oplus_{\alpha} y\right) = \sum_{r=0}^{n} \left[\begin{array}{c} n \\ r \end{array} \right]_{\alpha} \mathscr{C}_{n-r}(x) y^{\alpha r}.$$

Proof. Observe that, $t^n\left(x \oplus_{\alpha} y\right) = \left(tx \oplus_{\alpha} ty\right)$ and $E_{\alpha}\left(x \oplus_{\alpha} y\right) = E_{\alpha}\left(x\right) E_{\alpha}\left(y\right)$. Then,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathscr{C}_n^{\alpha}\left(x \oplus_{\alpha} y\right) \frac{t^n}{n!} &= a(t) E_{\alpha}(x^{\alpha} t) E_{\alpha}\left(y^{\alpha} t\right) \\ &= \sum_{n=0}^{\infty} \mathscr{C}_n^{\alpha}\left(x\right) \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{t^r y^{\alpha r}}{\Gamma(\alpha r+1)} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathscr{C}_{n-r}^{\alpha}\left(x\right) \frac{t^n}{(n-r)!} \frac{y^{\alpha r}}{\Gamma(\alpha r+1)}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result. □

We now introduce the fractional Appell sequences defined under the Riemann-Liouville fractional derivative operator of order α, where $0 < \alpha < 1$.

Definition 3.11. Sequences of functions of order α, denoted as $\{\mathscr{R}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$, are termed Appell-Riemann sequences when they adhere to the relationship:

$${}_{R}D_x^{\alpha} \mathscr{R}_n^{\alpha}(x) = n \mathscr{R}_{n-1}^{\alpha}(x), \ n \in \mathbb{N}. \tag{3.7}$$

Theorem 3.12. *Let $0 < \alpha < 1$ and $\{b_n\}_{n\in\mathbb{N}_0}$ be a sequence of arbitrary numbers. The sequence of polynomials $\{\mathscr{R}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ give by*

$$\mathscr{R}_n^{\alpha}(x) := \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array} \right]_{\alpha,\alpha} b_{n-j} x^{\alpha j+\alpha-1}, \ x \geqslant 0, \tag{3.8}$$

satisfies the relation (3.7).

Proof. By (2.1) and (2.2), we have

$$\begin{aligned}{}_{R}D_x^{\alpha} \mathscr{R}_n^{\alpha}(x) &= \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array} \right]_{\alpha,\alpha} b_{n-j}\, {}_{R}D_x^{\alpha} x^{\alpha j+\alpha-1} \\ &= \sum_{j=1}^{n} \left[\begin{array}{c} n \\ j \end{array} \right]_{\alpha,\alpha} b_{n-j} \frac{\Gamma(\alpha j+\alpha)}{\Gamma\left(\alpha j\right)} x^{\alpha j-1}\end{aligned}$$

$$= \sum_{j=0}^{n-1} \frac{n!}{(n-1-j)!\Gamma(\alpha j+\alpha)} b_{n-1-j} x^{\alpha j+\alpha-1}$$
$$= n \sum_{j=0}^{n-1} \left[\begin{array}{c} n-1 \\ j \end{array} \right]_{\alpha,\alpha} b_{n-1-j} x^{\alpha j+\alpha-1} = n\mathscr{R}_{n-1}^{\alpha}(x).$$

□

Now, we establish the generating function of the Appell-Riemann sequences.

Theorem 3.13. *Let* $0 < \alpha < 1$. *Given the power series*

$$b(t) := \sum_{j=0}^{\infty} b_j \frac{t^j}{j!}, \ a_0 \neq 0, \tag{3.9}$$

with b_j, $j = 0, 1, \ldots$, *real coefficients, the Appell-Riemann sequences* $\{\mathscr{R}_n^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ *are determined by the power series expansion of the product* $b(t)\mathcal{E}_{\alpha,\alpha}(x^{\alpha}t)$, *i.e.,*

$$b(t)\mathcal{E}_{\alpha,\alpha}(x^{\alpha}t) = \sum_{n=0}^{\infty} \mathscr{R}_n^{\alpha}(x)\frac{t^n}{n!}. \tag{3.10}$$

Proof. By (2.3), (3.9), and (3.5), we can recognize polynomials $\mathscr{R}_n^{\alpha}(x)$, expressed in form (3.8), as coefficients of $t^j/j!$. □

Example 3.14. Let $a(t) := \frac{t}{e^t - 1}$, $x > 0$, and $0 < \alpha < 1$. By (3.10), we have

$$\mathscr{R}_0^{\alpha}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \qquad \mathscr{R}_1^{\alpha}(x) = \frac{x^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{1}{2}\frac{x^{\alpha-1}}{\Gamma(\alpha)},$$
$$\mathscr{R}_2^{\alpha}(x) = \frac{2x^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{x^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{6}\frac{x^{\alpha-1}}{\Gamma(\alpha)}, \qquad \mathscr{R}_3^{\alpha}(x) = \frac{6x^{4\alpha-1}}{\Gamma(4\alpha)} - \frac{3x^{3\alpha-1}}{\Gamma(3\alpha)} + \frac{1}{2}\frac{x^{2\alpha-1}}{\Gamma(2\alpha)},$$
$$\mathscr{R}_4^{\alpha}(x) = \frac{24x^{5\alpha-1}}{\Gamma(5\alpha)} - \frac{12x^{4\alpha-1}}{\Gamma(4\alpha)} + \frac{2x^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{30}\frac{x^{\alpha-1}}{\Gamma(\alpha)}.$$

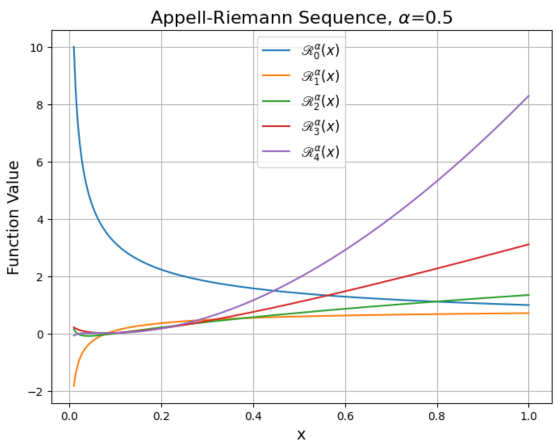

Figure 2: Appell-Riemann sequence of Example 3.14.

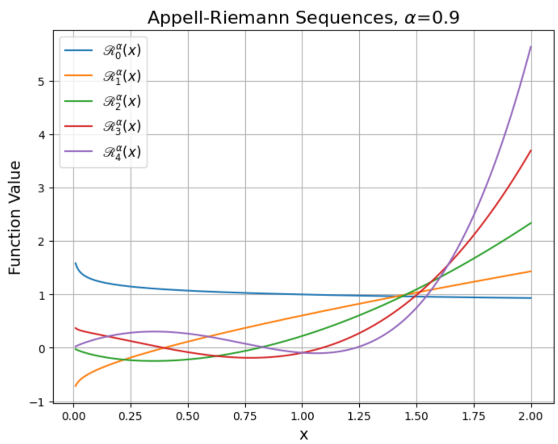

Figure 3: Appell-Riemann sequence of Example 3.15.

Example 3.15. Let $a(t) := \frac{2}{e^t - 1}$, $x > 0$, and $0 < \alpha < 1$. By (3.10), we have

$$\mathscr{R}_0^\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \qquad \mathscr{R}_1^\alpha(x) = \frac{x^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{1}{2}\frac{x^{\alpha-1}}{\Gamma(\alpha)},$$

$$\mathscr{R}_2^\alpha(x) = \frac{2x^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{x^{2\alpha-1}}{\Gamma(2\alpha)}, \qquad \mathscr{R}_3^\alpha(x) = \frac{6x^{4\alpha-1}}{\Gamma(4\alpha)} - \frac{3x^{3\alpha-1}}{\Gamma(3\alpha)} + \frac{1}{4}\frac{x^{\alpha-1}}{\Gamma(\alpha)},$$

$$\mathscr{R}_4^\alpha(x) = \frac{24x^{5\alpha-1}}{\Gamma(5\alpha)} - \frac{12x^{4\alpha-1}}{\Gamma(4\alpha)} + \frac{x^{2\alpha-1}}{\Gamma(2\alpha)}.$$

Theorem 3.16. *The sequences of Appell-Caputo sequences has the following determinantal representation:*

$$\mathscr{R}_0^\alpha(x) = \frac{1}{\gamma_0}\frac{x^{\alpha-1}}{\Gamma(\alpha)},$$

$$\mathscr{R}_n^\alpha(x) = \frac{(-1)^n}{\gamma_0^{n+1}} \begin{vmatrix} Q_0^\alpha(x) & Q_1^\alpha(x) & Q_2^\alpha(x) & \cdots & \cdots & Q_{n-1}^\alpha(x) & Q_n^\alpha(x) \\ \gamma_0 & \gamma_1 & \frac{1}{2}\gamma_2 & \cdots & \cdots & \frac{1}{(n-1)!}\gamma_n & \frac{1}{n!}\gamma_n \\ 0 & \gamma_0 & \gamma_1 & \cdots & \cdots & \frac{1}{(n-2)!}\gamma_{n-2} & \frac{1}{(n-1)!}\gamma_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \gamma_0 & \gamma_1 \end{vmatrix},$$

where $Q_n^\alpha(x) := \frac{x^{\alpha(n+1)-1}}{\Gamma(\alpha n + \alpha)}$ *and* $[b(t)]^{-1} = \sum_{k=0}^{\infty} \gamma_k \frac{t^k}{k!}$.

Proof. The steps of the proof of Theorem 3.8 are followed. □

4. Conclusion

The article explores the generalization of Appell polynomials through fractional differential operators, particularly Caputo and Riemann-Liouville operators. It introduces the concept of Appell-Caputo and Appell-Riemann sequences, showing their recursive relations and providing generating functions for these novel sequences. Additionally, the article derives determinantal representations for the sequences, shedding light on their structural properties.

Through the presented theorems and examples, the article illustrates the versatility and significance of these fractional Appell-type sequences in fractional analysis. It bridges classical polynomial theory with fractional calculus, offering a broader framework for studying differential equations of arbitrary orden and related fields.

In summary, the article contributes to the advancement of fractional calculus by extending classical polynomial sequences to fractional orders, opening up new avenues for research and applications in various mathematical disciplines.

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