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Some Remarks on Manifolds with Vanishing Bochner Tensor

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Abstract

In this paper, we present our results on Einsteinian, almost Hermitian manifolds with Bochner tensor B = 0. It shall be shown that under some conditions, these Bochner flat manifolds are complex space forms. Moreover, they are also Kähler manifolds with a constant holomorphic sectional curvature. We also present an identity for the Riemannian curvature of a generalized complex space form.

Keywords: Complex space forms, Kähler manifolds, Bochner tensor, generalized complex space forms
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1 Preliminaries

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with the almost complex structure J and Riemannian metric g. Let ∇ be the Levi-Civita connection on M and R the Riemannian curvature tensor defined by

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$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for X, Y and $Z \in \Gamma(M)$, where $\Gamma(M)$ denotes the Lie algebra of all smooth vector fields of M.

The *Ricci tensor* ρ on M is a symmetric bilinear function on $\Gamma(M) \times \Gamma(M)$ defined as the trace of the mapping T: $\Gamma(M) \to \Gamma(M)$ given by $T(Z) = R(X,Z)Y$. Equivalently,

$$\begin{aligned}\rho(X,Y) &= trace(T)\\ &= \sum_{i=1}^{n} R(X,e_i,Y,e_i)\end{aligned}$$

where $\{e_i | i = 1, \ldots, n\}$ is an orthonormal basis of $T_p(M)$, the tangent space of M at the point p. The *Ricci tensor transformation* $Q = Q(R)$ is given by $\rho(X,Y) = g(QX,Y)$ and the trace of Q is called the *scalar curvature* $\tau = \tau(R)$ of R. Note that since ρ is symmetric, then for vectors X, $Y \in \Gamma(M)$,

$$\rho(X,Y) = \rho(Y,X).$$

Furthermore, we denote by ρ^* and τ^* the *Ricci *-tensor* and the **-scalar curvature* on M, respectively. The tensor ρ^* is defined pointwisely by

$$\begin{aligned}\rho^*(X,Y) &= trace(Z \mapsto R(JZ,X)JY)\\ &= -\sum_{i=1}^{2n} R(X,e_i,JY,Je_i)\\ &= -\frac{1}{2}\sum_{i=1}^{2n} R(X,JY,e_i,Je_i)\end{aligned}$$

where X, Y and $Z \in T_p(M)$, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\{e_i\}$ is an orthonormal basis of $T_p(M)$. We also define analogously the *Ricci *-operator*, denoted by Q^*, by $\rho^*(X,Y) = g(Q^*X,Y)$ for X and $Y \in T_p(M)$. The trace of Q^* is the **-scalar curvature* τ^* on M. We note that ρ^* satisfies $\rho^*(JX,JY) = \rho^*(Y,X)$ but is not symmetric in general.

A manifold M is called an *Einstein space* or *Einsteinian* if $\rho = \lambda g$ for some constant λ. It is easy to check that the Einstein constant λ is equal to $\frac{\tau}{m}$, where m is the dimension of the manifold and so $\rho = \frac{\tau}{m} g$. A space of constant curvature is necessarily an Einstein space. In fact, for m-dimensional space of constant curvature λ, $g = \lambda(m - 1)\,\rho$. An almost Hermitian manifold M is called a *weakly *-Einstein manifold if* $\rho^* = \frac{\tau^*}{2n} g$ (dim $M = 2n$) holds. In addition, if τ^* is constant-valued, then M is called **-Einstein manifold*.

We now define the Bochner tensor B. Let R_J denote the curvature tensor given by

$$R_J(X,Y,Z,W) = \mathrm{R}(JX,JY,JZ,JW)$$

π_1 and π_2 the tensors defined by

$$\pi_1(X,Y,Z,W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W)$$
$$\pi_2(X,Y,Z,W) = 2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W)$$

and for $S \in (\Gamma(M)\mathrm{x}\Gamma(M))^*$, we define the functions $\varphi(S)$ and $\psi(S)$ by

$$\begin{aligned}\varphi(S)(X,Y,Z,W) &= g(X,Z)S(Y,W) + g(Y,W)S(X,Z)\\ &\quad - g(X,W)S(Y,Z) - g(Y,Z)S(X,W)\\ \psi(S)(X,Y,Z,W) &= 2g(X,JY)S(Z,JW) + 2g(Z,JW)S(X,JY)\\ &\quad + g(X,JZ)S(Y,JW) + g(Y,JW)S(X,JZ)\\ &\quad - g(X,JW)S(Y,JZ) - g(Y,JZ)S(X,JW)\end{aligned}$$

where $\Gamma^*(M)$ denotes the dual space of $\Gamma(M)$. The *Bochner tensor* associated with the curvature tensor R is given by

$$\begin{aligned}B(R)(X,Y,Z,W) = R(X,Y,Z,W) &- \frac{1}{16(n+2)}(\varphi+\psi)(\rho+3\rho^*)(R+R_J)(X,Y,Z,W)\\ &- \frac{1}{16(n-2)}(3\varphi-\psi)(\rho-\rho^*)(R+R_J)(X,Y,Z,W)\\ &- \left\{\frac{1}{4(n+1)}(\psi \circ \rho^*) + \frac{1}{4(n-1)}(\varphi \circ \rho)\right\}(R-R_J)(X,Y,Z,W)\\ &+ \frac{1}{16(n+1)(n+2)}(\tau+3\tau^*)(R)(\pi_1+\pi_2)(X,Y,Z,W)\\ &+ \frac{1}{16(n-1)(n-2)}(\tau-\tau^*)(R)(3\pi_1-\pi_2)(X,Y,Z,W)\end{aligned}$$

for $n \geq 3$, and by

$$\begin{aligned}B(R)(X,Y,Z,W) = R(X,Y,Z,W) &- \frac{1}{2}\varphi\left(\rho(R) - \frac{1}{4}\tau(R)g\right)(X,Y,Z,W)\\ &- \frac{1}{12}\psi\left(\rho^*(R-R_J)\right)(X,Y,Z,W)\\ &- \frac{1}{96}(\tau+3\tau^*)(R)(\pi_1+\pi_2)(X,Y,Z,W)\\ &- \frac{1}{32}(\tau-\tau^*)(R)(3\pi_1-\pi_2)(X,Y,Z,W)\end{aligned} \tag{1}$$

for $n = 2$.

If the manifold has a vanishing Bochner tensor, i.e., $B = 0$, then it is called a *Bochner-flat manifold*.

The concept of a generalized complex space form is a natural generalization of a complex space form which has been discussed by Tricerri and Vanhecke in [9]. The authors stated the following theorem:

Theorem 1. *An almost Hermitian manifold M is a generalized complex space form if and only if M is Einsteinian, weakly *-Einsteinian and Bochner-flat, i.e.,* $B = 0$. *Furthermore, a* $2n$ $(n \geq 3)$*-dimensional generalized complex space form is a real space form or a complex space form.*

A $2n(n \geq 2)$-dimensional almost Hermitian manifold $M = (M,J,g)$ is a *generalized complex space form* if the curvature tensor R takes the following form:

(2)
$$\begin{aligned} R &= \frac{\tau + 3\tau^*}{16n(n+1)}(\pi_1 + \pi_2) + \frac{\tau - \tau^*}{16n(n-1)}(3\pi_1 - \pi_2) \\ &= \frac{(2n+1)\tau - 3\tau^*}{8n(n-1)(n+1)}\pi_1 + \frac{(2n-1)\tau^* - \tau}{8n(n-1)(n+1)}\pi_2. \end{aligned}$$

Observe that the results of Tricerri-Vanhecke's Theorem holds only for $2n \geq 6$. In [3], Lemence extended Theorem 1 to the 4-dimensional case. He had the following results:

Theorem 2. *Let* $M = (M,J,g)$ *be a* 4*-dimensional generalized complex space form. Then M is locally a real space form or globally conformal Kähler manifold. In the latter case,* (M,J,g^*) *with* $g^* = (3\tau^* - \tau)^{\frac{2}{3}} g$ *is a Kähler manifold, where* τ *and* τ^* *are the scalar curvature and the *-scalar curvature of* M*, respectively.*

Theorem 3. *Let* $M = (M,J,g)$ *be a compact* 4*-dimensional generalized complex space form. Then M is a real space of constant non-positive sectional curvature or compact complex space form.*

We now give our results on Bochner flat manifolds.

2 Main Results

Theorem 4. *If* $M = (M,J,g)$ *be a* 4*-dimensional Bochner-flat, Einstein, weakly *-Einstein manifold, then M is a generalized complex space form.*

Proof. When n=2, the expression for the Bochner tensor when M is Einstein and weakly *-Einstein in (1) is reduced to

$$B(X,Y,Z,W) = R(X,Y,Z,W) - \frac{5\tau - 3\tau^*}{48}\pi_1(X,Y,Z,W) - \frac{3\tau^* - \tau}{48}\pi_2(X,Y,Z,W).$$

Since M is Bochner-flat, then

$$R(X,Y,Z,W)=\frac{5\tau-3\tau^*}{48}\pi_1(X,Y,Z,W)+\frac{3\tau^*-\tau}{48}\pi_2(X,Y,Z,W),$$

which satisfies (2) for n=2. Thus, M is a generalized complex space form. □

In [9], Tricerri and Vanhecke gave another expression for the Bochner tensor. They stated that if the curvature tensor R of an almost Hermitian manifold satisfies $R(X,Y,Z,W)= R(X,Y,JZ,JW)$, then its corresponding Bochner tensor is given by

$$B(R)=R-\frac{1}{2(n+2)}(\varphi+\psi)(\rho(R))+\frac{1}{4(n+1)(n+2)}\tau(R)(\pi_1+\pi_2).$$

This type of manifold is called an *F-space*. With this expression for B, we have the following characterization:

Theorem 5. *Let M = (M,J,g) be a connected Bochner –flat, Einsteinian F-space with real dimension $n\geq 3$. Then M is a complex space form.*

Proof. Observe that $\rho=\frac{\tau}{2n}g$ since M is Einsteinian. Therefore,

$$\begin{aligned}
&(\varphi+\psi)(\rho(R))(X,Y,Z,W)\\
&=\frac{\tau}{2n}(\varphi+\psi)g(X,Y,Z,W)\\
&=\frac{\tau}{2n}\varphi(g)(X,Y,Z,W)+\frac{\tau}{2n}\psi(g)(X,Y,Z,W)\\
&=\frac{\tau}{2n}[g(X,Z)g(Y,W)+g(Y,W)g(X,Z)-g(X,W)g(Y,Z)-g(Y,Z)g(X,W)\\
&\quad+2g(X,JY)g(Z,JW)+2g(Z,JW)g(X,JY)+g(X,JZ)g(Y,JW)\\
&\quad+g(Y,JW)g(X,JZ)-g(X,JW)g(Y,JZ)-g(Y,JZ)g(X,JW)]\\
&=\frac{\tau}{n}[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\\
&\quad+2g(X,JY)g(Z,JW)+g(X,JZ)g(Y,JW)-g(Y,JZ)g(X,JW)]\\
&=\frac{\tau}{n}[\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\}\\
&\quad+\{2g(JX,Y)g(JZ,W)+g(JX,Z)g(JY,W)-g(JY,Z)g(JX,W)\}]\\
&=\frac{\tau}{n}\left[\pi_1(X,Y,Z,W)+\pi_2(X,Y,Z,W)\right].
\end{aligned}$$

Thus, the Bochner tensor will be reduced to

$$B = R - \frac{1}{2(n+2)}\frac{\tau}{n}(\pi_1 + \pi_2) + \frac{1}{4(n+1)(n+2)}\tau(\pi_1 + \pi_2)$$
$$= R - \frac{\tau}{4n(n+1)}(\pi_1 + \pi_2).$$

Since M is Bochner-flat, then the curvature tensor R takes the form

(3)
$$R(X,Y,Z,W) = \frac{\tau}{4n(n+1)}(\pi_1 + \pi_2)(X,Y,Z,W)$$
$$= \frac{\tau}{4n(n+1)}\pi_1(X,Y,Z,W) + \frac{\tau}{4n(n+1)}\pi_2(X,Y,Z,W).$$

In [9], it was stated that if M is a connected almost Hermitian manifold with real dimension *with real dimension* $n \geq 3$ and Riemannian curvature tensor R of the following form:

$$R = f\pi_1 + h\pi_2,$$

Where f and h are C^∞ functions on M such that h is not identically zero, then M is a complex space form. This still holds even if f and h are both constant functions. Hence, our assertion is proved. ☐

Corollary 1. *A connected Bochner-flat, Einstein F-space of real dimension* $n \geq 3$ *is a Kähler manifold with constant holomorphic sectional curvature* $\frac{\tau}{4n(n+1)}$, *where* τ *is the scalar curvature.*

Finally, we state an identity for the Riemannian curvature of a generalized complex space form.

Theorem 6. *Let M = (M,J,g) be a generalized complex space form. Then the curvature tensor R satisfies the following curvature identity:*

(4) $$R(X,Y,Z,W) = R(X,Y,JZ,JW) + R(X,JY,Z,JW) + R(X,JY,JZ,W).$$

Proof. Let $\alpha = \frac{(2n+1)\tau - 3\tau^*}{8n(n^2-1)}$ and $\beta = \frac{(2n-1)\tau^* - \tau}{8n(n^2-1)}$. Then

(5)
$$R(X,Y,Z,W) = \alpha\pi_1(X,Y,Z,W) + \beta\pi_2(X,Y,Z,W)$$
$$= \alpha\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}$$
$$+ \beta\{2\, g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W)\}.$$

The right-hand side of (4) can be expressed as

(6)
$$R(X,Y,JZ,JW) + R(X,JY,Z,JW) + R(X,JY,JZ,W)$$
$$= \alpha\{g(X,JZ)g(Y,JW) - g(Y,JZ)g(X,JW)\}$$
$$+ \beta\{-2\, g(JX,Y)g(Z,JW) + g(JX,JZ)g(JY,JW) - g(JY,JZ)g(JX,JW)\}$$

$$+ \alpha\{g(X,Z)g(JY,JW) - g(JY,Z)g(X,JW)\}$$
$$+ \beta\{2\, g(JX,JY)g(JZ,JW) - g(JX,Z)g(Y,JW) + g(Y,Z)g(JX,JW)\}$$
$$+ \alpha\{g(X,JZ)g(JY,W) - g(JY,JZ)g(X,W)\}$$
$$+ \beta\{-2\, g(JX,JY)g(Z,W) - g(JX,JZ)g(Y,W) + g(Y,JZ)g(JX,W)\}.$$

We know that for an almost Hermitian manifold, $g(JX,JY) = g(X,Y)$ and since $J^2 = -1$, where 1 denotes the identity,

$$g(JX,Y) = g(J^2X\,,JY) = -\,g(X,JY).$$

Using these properties, we be able to identify the following and eliminate similar terms in (6).

$$g(X,JZ)\, g(Y,JW) = -\,g(X,JZ)\, g(JY,W)$$
$$g(Y,JZ)\, g(X,JW) = -\,g(JY,Z)\, g(X,JW)$$
$$g(JX,JZ)\, g(JY,JW) = g(JX,JZ)\, g(Y,W)$$
$$g(JY,JZ)\, g(JX,JW) = -\,g(Y,Z)\, g(JX,JW)$$
$$2g(JX,JY)\, g(JZ,JW) = 2g(JX,JY)\, g(Z,W).$$

Equation (6) now reduces to

(7) $$R(X,Y,JZ,JW) + R(X,JY,Z,JW) + R(X,JY,JZ,W)$$
$$= \alpha\{g(X,Z)g(JY,JW) - g(JY,JZ)g(X,W)\}$$
$$+ \beta\{-2\, g(JX,Y)g(Z,JW) - g(JX,Z)g(Y,JW) + g(JY,JZ)g(JX,\, W)\}$$

Using the same mentioned properties about g and J in (6), finally we obtained

(8) $$R(X,Y,JZ,JW) + R(X,JY,Z,JW) + R(X,JY,JZ,W)$$
$$= \alpha\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}$$
$$+ \beta\{2\, g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W)\}$$

Finally, combining equations (5) and (8), we have the desired identity. ☐

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