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Control analysis of Nilaparvata Lugens with Wolbachia using sterile insect techniques

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Abstract

The ecologist and farmers have both greatly benefited from the modeling and control of *Nilaparvata Lugens* (N. Lugens) populations in rice fields. This work describes a sex-structured wild Lugens population and male Lugens infected with w-Stri (Wolbachia). The w-Stri type Wolbachia can naturally control wild Lugens, as shown in a test done in the laboratory. Male Lugens infected with w-Stri produce larger amounts of cytoplasmic incompatibility, when they mate with wild female Lugens. Using the time-dependent control parameter of the continuous releasing rate of male Lugens infected with w-Stri, we create an optimal control problem. By analyzing the necessary and sufficient conditions, we investigate the optimality. Furthermore, we explore the sufficient conditions for the elimination of wild Lugens via periodic impulsive releases of male Lugens infected with w-Stri. Numerical simulations validate the theoretical conclusions.

Keywords: Feedback control, optimal control, sterile insect technique, Lugens, Wolbachia.

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1. Introduction

Plant viruses are transmitted by insects, which cause substantial damage to agricultural crops. One of the most important crops, Oryza sativa (Rice), is susceptible to nearly 800 different insect and herbicide species. Among them are Lugens, Bollworms, Aphids, and others ([4, 25, 29]). Lugens is a monophagous bug species that transmits *Rice Ragged Stunt Virus* (RRSV) [5] throughout India, China, and other Asian countries. The hoppers' burns are evidence of the direct damage that these insects cause by sucking the phloem sap from rice stalks and depleting the nutrients, especially when there are a lot of rice plants. For controlling these types of insects, synthetic chemical pesticides have been used. The authors of [6, 17] studied a series of experiments that examine chemotaxis models involving attraction and repulsion. These models consider various factors, including nonlinear diffusion, sensitivities, logistic sources, and cell density dynamics. Additionally, they explore the consequences of consumption and/or production on the chemicals involved. Increased pesticide use in agriculture can lead to insecticide resistance, pest

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resurgence, harm to natural enemies, and environmental degradation. The use of practical, efficient, and environmentally friendly pest control methods is therefore essential. Basir et al. [1] investigated the impact of public knowledge on crop pest management using a mathematical model that included plant biomass, pests, and aware populations. Biological control methods, also known as integrated pest management, were also discussed.

In recent years, two biological control methods have been developed to control or eradicate these types of insects and mosquito-borne diseases, namely, the *Incompatible Insect Technique* (IIT) and the *Sterile Insect Technique* (SIT). IIT is associated with a bacterium (intracellular) called Wolbachia, which is typically found in insects like mosquitoes. Using a population replacement strategy involves releasing insects that contain Wolbachia to replace wild mosquitoes and insects. A female infected with Wolbachia will also transmit the Wolbachia to her offspring. The *Cytoplasmic Incompatibility* (CI) of Wolbachia occurs when infected males mate with uninfected females. Insects or mosquitoes will be gradually reduced by the IIT. Using feedback control strategies, Bliman [3] discussed control of Wolbachia-infected Aedes aegypti mosquito populations. The stabilizing effects of several feedback control law are studied in order to illustrate these concepts. In [18], the authors examined the qualitative behaviour of Aedes Aegypti mosquitoes infected with Wolbachia, which interrupts dengue virus transmission by virtue of its CI and maternal transmission. In recent years, some authors studied the abstract theory of impulsive control methods ([16, 22, 26]).

By disrupting mosquito and other insect reproduction, SIT creates sterile male populations by using chemical, physical, and radical tactics. Males are sterilized and reintroduced into the wild to mate with existing wild populations. Insects and mosquitoes could be reduced or eliminated if sterile males are released multiple times into the environment [10]. Dumont et al. [9] presented the modelling and analysis of SIT techniques to Aedes Albopictus populations and derived the conditions for constant, periodic impulsive sterile male releases, which are helpful for disease prevention, reduction, and elimination. To release impulsive sterile male mosquitoes into the wild population, the authors discussed both open-loop and closed-loop control strategies in [2]. In [24, 33], the authors analysed the control of mosquitoes populations using incompatible and sterile insect techniques. Du et al. [8] examined the comparative efficacy of irradiation with X-rays as a mosquito SIT. Xue et al. [30] discussed the SIT into the mosquitoes populations with allee effects and the release of sterile insects with different strategies and control analysis used find the releasing strategy as well as eliminating the mosquitoes population level. Strugarek et al. [28] investigated a rudimentary mathematical model that was developed for the reduction of the Aedes mosquito population model using SIT and IIT, showed elimination conditions for steady, periodic, and impulsive releases and predict treatment times.

In the natural environment of Lugens, Wolbachia variant w-Lug is present [31]. According to some experimental findings, the Wolbachia w-Lug increases the fecundity of Lugens by naturally laying more eggs than uninfected Lugens in all temperatures [14], and in order for Laodelphax striatellus and Lugens to thrive, Wolbachia must supply the essential nutrients biotin and riboflavin. Wild Lugens populations were not controlled by Wolbachia strain w-Lug in this case. Researchers found another Wolbachia strain as w-Stri from a small planthopper Laodelphax Striatellus [32]. According to [23], this strain can cause CI and interfere with maternal transmission. In a laboratory experiment, Gong et al. [11] transferred w-Stri from Striatellus to Lugens, and found that the w-Stri reduced rice ragged stunt viral infection, transmission, and disease prevention. By utilizing the differential equation method, the authors in [20] analyzed w-Stri and w-Lug Wolbachia spreading dynamics in populations of N. Lugens using imperfect maternal transmission and incomplete CI. Dai et al. [7] framed the population replacement and suppression Lugens model with periodic impulsive releases of N. Lugens infected with w-Stri and control analysis.

In this study, a mathematical model is developed for wild Lugens and male Lugens infected with w-Stri. A continuous releasing size of male Lugens infected with w-Stri is used as a control parameter in an optimal control problem. To arrive at the effective control procedures, Pontryagin's maximum principle is applied. Periodic impulsive sterile emissions are studied and eradication criteria are developed based on their frequency and magnitude. Male Lugens infected with w-Stri must be released frequently and in sufficient quantities in order to ensure population replacement and suppression. Our model suppresses

the Lugens population with a release period shorter than the sexual lifetime of male Lugens infected with w-Stri. This paper is organized as follows. Section 2 defines the model and steady states. The impulsive control and optimality conditions are derived in Section 3. The numerical examples in Section 4 verify the theoretical results. Conclusions are provided in the last section.

2. Model formulation

According to Liu et al. [19], the following replacement model is applicable to Lugens infected with w-Stri:

$$\begin{aligned}
\dot{S}_M(t) &= \frac{bS_F(t)}{2}\frac{S_M(t)}{S_M(t)+I_M(t)} + \frac{bS_F(t)}{2}\frac{(1-\xi)I_M(t)}{S_M(t)+I_M(t)} - d_1 S_M(t)N(t),\\
\dot{S}_F(t) &= \frac{bS_F(t)}{2}\frac{S_M(t)}{S_M(t)+I_M(t)} + \frac{bS_F(t)}{2}\frac{(1-\xi)I_M(t)}{S_M(t)+I_M(t)} - d_1 S_F(t)N(t),\\
\dot{I}_M(t) &= \frac{bI_F(t)}{2} - d_2 I_M(t)N(t),\\
\dot{I}_F(t) &= \frac{bI_F(t)}{2} - d_2 I_F(t)N(t).
\end{aligned} \tag{2.1}$$

Here, S_M, S_F, and I_M, I_F are the numbers of male and female Lugens uninfected and infected with w-Stri, receptively. Here, b is the birth rate and d_1, d_2 are the decay rate constants. The CI intensity of infected males against uninfected females is $\xi \in (0,1)$. From model (2.1), we include the individual competition effect [13, 21] and continuous releases of male Lugens infected with w-Stri, so the model becomes,

$$\begin{aligned}
\dot{M} &= \Big(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\Big)\vartheta\alpha e^{-\gamma(M+F)} - \mu_M M,\\
\dot{F} &= \Big(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\Big)\vartheta(1-\alpha) e^{-\gamma(M+F)} - \mu_F F,\\
\dot{M}_I &= u(t) - \mu_{M_I} M_I.
\end{aligned} \tag{2.2}$$

M and F are the population densities of wild male and female Lugens, respectively, and t is the time in days. Male Lugens infected with w-Stri are denoted as M_I, and $u(t) : \mathbb{R}_+ \to [0, u_{max}]$ represents the releasing rate of M_I. Assume that $\frac{\alpha}{1-\alpha}$ is the primary sex ratio and ϑ is the average number of eggs laid by each female every day. An individual's competition effect is characterized by γ ([13, 21]). In comparison to wild male fitness, β represents the relative reproductive fitness or effectiveness, which is typically less than one. μ_M, μ_F, and μ_{M_I} are the death rates of wild male and female Lugens, male Lugens infected with w-Stri, respectively. Here this is an explicit solution to the last equation in (2.2):

$$M_I(u(\cdot), t) = e^{-\mu_{M_I} t}\Big(M_I(0) + \int_0^t e^{\mu_{M_I} s} u(s)\,ds\Big).$$

According to the above equation, $M_I(u(\cdot), t)$ is bounded and non-negative from $u \in [0, u_{max}]$. The right side of the first two equations of (2.2) is decreasing in $M_I(t)$. The authors in [19, 20] showed the wild Lugens model constrained trajectories from an absorbing set, so (2.2) computes Lipschitz continuous solutions, and (2.2) computes a bounded solution.

For each wild male and female Lugens, define the basic off-spring numbers as $\mathcal{N}_M = \frac{\vartheta\alpha}{\mu_M}$ and $\mathcal{N}_F = \frac{\vartheta(1-\alpha)}{\mu_F}$. The model (2.2) without M_I populations has two steady states. i) if $\mathcal{N}_F \leqslant 1$, trivial steady state $E_0(M_0^* = 0, F_0^* = 0)$; ii) $\mathcal{N}_F > 1$, endemic steady state $E_1(M_1^* = \frac{1}{\gamma}\frac{\mathcal{N}_M}{\mathcal{N}_M+\mathcal{N}_F}\ln\mathcal{N}_F, F_1^* = \frac{1}{\gamma}\frac{\mathcal{N}_F}{\mathcal{N}_M+\mathcal{N}_F}\ln\mathcal{N}_F)$. For the stability analysis of these two steady states, the reader may be referred [2, 19, 20] and so is omitted here. Now, we consider $u(t) = b$ as constant ([2]), the third equation of (2.2) entirely decouples from the model and its value $M_I^* = \frac{b}{\mu_{M_I}}$ can be replaced in first two equations of (2.2), we obtain the endemic

equilibrium $E_2(M_2^*, F_2^*, M_{I,2}^* = \frac{b}{\mu_{M_I}})$, it is difficult to get the values for M_2^* and F_2^* analytically, so better will get numerically.

3. Main results

3.1. Optimal control technique

The proposed control method seeks to minimize the number of wild Lugens. We define control function $u(t), t \in [0, t_f]$ as time-dependent release rate of male Lugens infected with w-Stri with minimal time t_f. Objective function includes: i) controlling the wild population within a specified time period; ii) reducing the total duration of control action; iii) minimizing the cost of control action. We consider the objective function as follows:

$$\min_{0<t_f<\infty, 0\leqslant u(t)\leqslant u_{max}} J(t_f, u) = B_1(F(t_f) - \epsilon)^2 + \int_0^{t_f} [B_2F(t) + B_3 + \frac{B_4}{2}u^2(t)]dt. \tag{3.1}$$

The cost coefficient B_1 describes the highest priority as achieving elimination in a limited amount of time, B_2 defines the significance of wild female elimination intensity during the release operation, B_3 represents the appreciation for time, and B_4 refers the control effort costs (i.e., mass-rearing of male Lugens infected with w-Stri). The objective is to minimise the cost function (3.1) and to determine the functions of optimal control $\left(u^*, t_f^*\right)$ such that

$$J(u^*, t_f^*) = \min_{0\leqslant u(t)\leqslant u_{max}, 0<t_f<\infty} \{J(u, t_f)\},$$

depending on (2.2) and suitable initial conditions given at $t = 0$. We demonstrate the model's optimal control existence and explain its optimality conditions.

Theorem 3.1 ([15]). *If* $\mathcal{N}_F > 1$, *then the model* (2.2) *has a solution* (u^*, t_f^*) *such that* $J(u^*, t_f^*) = \min_{0<t_f<\infty, 0\leqslant u(t)\leqslant u_{max}}\{J(t_f, u)\}$.

Proof. State and control variables are non-negative in the control system (2.2). To minimize the problem, the objective function in $u(t)$ must be proven to be convex. In addition to being convex and closed, the control variable $u(t) \in [0, u_{max}], t \geqslant 0$ is also closed. The optimal system must be bounded in order for optimal control to occur. We assure the existence of an optimal control $u(t)$ which minimize (3.1) with the help of (2.2). □

To find the optimal control, we determine the Hamiltonian and Lagrangian for the model (2.2). The Lagrangian is given by $L = B_2F + B_3 + \frac{1}{2}B_4u^2$. Then, we describe a Hamiltonian function H for (2.2) where $\lambda_i, i = 1, 2, 3$ are the adjoint variables:

$$\begin{aligned} H(X, u, \lambda) = &-B_2F - B_3 - \frac{B_4}{2}u^2 + \lambda_1\left\{\left(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\right)\vartheta\alpha e^{-\gamma(M+F)} - \mu_M M\right\} \\ &+ \lambda_2\left\{\left(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\right)\vartheta(1-\alpha)e^{-\gamma(M+F)} - \mu_F F\right\} \\ &+ \lambda_3\left\{u(t) - \mu_{M_I} M_I\right\}. \end{aligned} \tag{3.2}$$

Let us use Pontryagin's maximum principle from [15] to generate the necessary conditions. If (X, u) is an optimal solution for the model (2.2), where $X = (M, F, M_I)$ and $u \in [0, u_{max}]$, then there exists a non-trivial vector function $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfying the inequalities:

$$\frac{dX}{dt} = \frac{\partial H}{\partial \lambda}, \quad \frac{\partial H}{\partial u} = 0, \quad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial X}.$$

In order to derive the necessary condition for the optimal control problem, the adjoint system and control characterisation are presented in the following theorem.

Theorem 3.2. *Given an optimal control* $u^* \in [0, u_{max}]$ *and a solution* $X^* = (M^*, F^*, M_I^*)$ *of the corresponding model* (2.2)*, there exist adjoint variables* $\lambda_i, i = 1, 2, 3$ *satisfying*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial M} = \lambda_1\Big\{\mu_M - \vartheta\alpha e^{-\gamma(M+F)}F\Big(\frac{\beta M_I}{(M+\beta M_I)^2} - (1-\xi)\frac{M_I}{(M+\beta M_I)^2}\Big)\\
&\quad + \gamma\Big(\frac{M}{M+\beta M_I} - (1-\xi)\frac{M_I}{M+\beta M_I}\Big)\Big\} - \lambda_2\Big\{\vartheta(1-\alpha)e^{-\gamma(M+F)}F\Big(\frac{\beta M_I}{(M+\beta M_I)^2}\\
&\quad - (1-\xi)\frac{M_I}{(M+\beta M_I)^2}\Big) - \gamma\Big(\frac{M}{M+\beta M_I} + (1-\xi)\frac{M_I}{M+\beta M_I}\Big)\Big\},\\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial F} = B_2 - \vartheta e^{-\gamma(M+F)}(1-F\gamma)\Big\{\frac{M}{M+\beta M_I} + (1-\xi)\frac{M_I}{M+\beta M_I}\Big\}(\lambda_1\alpha + \lambda_2(1-\alpha)) + \mu_F\lambda_2,\\
\frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial M_I} = \lambda_3\mu_2 - F\vartheta e^{-\gamma(M+F)}\frac{(1-\xi-\beta)M}{(M+\beta M_I)^2}\Big(\lambda_1\alpha + \lambda_2(1-\alpha)\Big),
\end{aligned}$$

with transversality conditions $\lambda_i(t_f) = 0$, $i = 1, 2, 3$. *For* $t \in [0, t_f]$*, then there exists an optimal control function*

$$u^* = \max\Big\{0, \min\{\frac{1}{B_4}\lambda_3(t), u_{max}\}\Big\}.$$

Proof. To find the adjoint equations and the criteria for transversality, we employ the Hamiltonian (3.2). The adjoint system results from Pontryagin's maximum principle, which is described in [15],

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial M}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial F}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial M_I};$$

with $\lambda_i(t_f) = 0$. Solving the equations to obtain the characterization of the optimal control, $\frac{\partial H}{\partial u} = 0$, on the interior of the control set and utilising the control space $[0, u_{max}]$ property, we can derive the desired characterization $u^* = \max\Big\{0, \min\{\frac{1}{B_4}\lambda_3(t), u_{max}\}\Big\}$. □

3.2. Impulsive releases of male Lugens with infected w-Stri

We insert the periodic impulsive release of male Lugens infected with w-Stri and the model becomes

$$\begin{aligned}
\dot{M} &= \Big(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\Big)\vartheta\alpha e^{-\gamma(M+F)} - \mu_M M,\\
\dot{F} &= \Big(F\frac{M}{M+\beta M_I} + (1-\xi)F\frac{M_I}{M+\beta M_I}\Big)\vartheta(1-\alpha)e^{-\gamma(M+F)} - \mu_F F,\\
\dot{M}_I &= -\mu_{M_I}M_I, \quad \text{in any case } t \in \bigcup_{n\in\mathbb{N}}\Big(n\tau, (n+1)\tau\Big),\\
M_I(n\tau^+) &= \tau b_n + M_I(n\tau^-), \quad n = 1, 2, 3, \ldots.
\end{aligned} \tag{3.3}$$

Make b_n a constant, drop sub-index n. $M_I(n\tau^{\pm})$ denotes the left and right limits of $M_I(t)$ at $t = n\tau$. If $t \to \infty$, the function M_I approaches the periodic solution as follows

$$M_I^{per}(t) = \frac{\tau b}{1 - e^{-\mu_{M_I}\tau}}e^{-\mu_{M_I}(t-n\tau)}.$$

Describe the periodic model

$$\begin{aligned}\dot{M} &= \Big(F\frac{M}{M+\beta M_I^{per}} + (1-\xi)F\frac{M_I^{per}}{M+\beta M_I^{per}}\Big)\vartheta\alpha e^{-\gamma(M+F)} - \mu_M M,\\ \dot{F} &= \Big(F\frac{M}{M+\beta M_I^{per}} + (1-\xi)F\frac{M_I^{per}}{M+\beta M_I^{per}}\Big)\vartheta(1-\alpha)e^{-\gamma(M+F)} - \mu_F F.\end{aligned} \tag{3.4}$$

The model (3.3) has the same Lugens free steady state E_0. We will now investigate the asymptomatic stability of E_0. So, we describe the mean value of $\frac{1}{M_I^{per}}$ ([2]),

$$\Big\langle \frac{1}{M_I^{per}} \Big\rangle := \frac{2(\cosh(\mu_{M_I}\tau)-1)}{b\mu_{M_I}\tau^2}.$$

Theorem 3.3. *Assume that*

$$b \geqslant b_{per}^{crit} = \frac{2\cosh(\mu_{M_I}\tau)-1}{\mu_{M_I}\tau^2(e\gamma)}\min\Big\{\frac{\beta}{\mathcal{N}_F}+\xi-1, \frac{\beta}{\mathcal{N}_M}\Big\}, \quad \tau>0.$$

Then, the solution of the model (3.4) *converges globally exponentially to the Lugens free equilibrium* E_0.

Proof. From (3.4),

$$\dot{F} = \Big(\frac{FM}{M+\beta M_I^{per}} + (1-\xi)\frac{FM_I^{per}}{M+\beta M_I^{per}}\Big)(1-\alpha)\vartheta e^{-\gamma(M+F)} - \mu_F F.$$

For any $t \geqslant 0, M \geqslant 0, F \geqslant 0$, use $\alpha_1 = \max\{e^{-\gamma x}x; \quad x \geqslant 0\} = \frac{1}{e\gamma}$,

$$\frac{M}{M+\beta M_I^{per}}e^{-\gamma(M+F)} \leqslant \frac{Me^{-\gamma M}}{\beta M_I^{per}} \leqslant \frac{\alpha_1}{\beta M_I^{per}} \quad \text{and} \quad \frac{M_I^{per}}{M+\beta M_I^{per}}e^{-\gamma(M+F)} \leqslant \frac{1}{\beta}.$$

Applying the integration with $n\tau < t$ from $n\tau$ to t,

$$\begin{aligned}F(t) &\leqslant e^{\int_{n\tau}^{t}\left(\left(\frac{M}{M+\beta M_I^{per}}+(1-\xi)\frac{M_I^{per}}{M+\beta M_I^{per}}\right)(1-\alpha)\vartheta e^{-\gamma(M+F)}-\mu_F\right)ds}F(n\tau),\\ F(t) &\leqslant e^{\int_{n\tau}^{t}\left(\left(\frac{\alpha_1}{\beta M_I^{per}}+\frac{(1-\xi)}{\beta}\right)(1-\alpha)\vartheta-\mu_F\right)ds}F(n\tau),\\ F((n+1)\tau) &\leqslant \Big[e^{\left(\frac{\alpha_1(1-\alpha)\vartheta}{\beta M_I^{per}}+(1-\alpha)\vartheta(\frac{1-\xi}{\beta})-\mu_F\right)\tau}\Big]F(n\tau).\end{aligned}$$

The above sequences decrease towards 0, we deduce that

$$\Big\langle \frac{1}{M_I^{per}} \Big\rangle < \frac{1}{\alpha_1}\Big(\frac{\beta}{\mathcal{N}_F}+\xi-1\Big) = e\gamma\Big(\frac{\beta}{\mathcal{N}_F}+\xi-1\Big).$$

In the first equation of (3.4), we similarly demonstrate that $\Big\langle \frac{1}{M_I^{per}} \Big\rangle < e\gamma\frac{\beta}{\mathcal{N}_M}$. Providing the necessary conditions $\Big\langle \frac{1}{M_I^{per}} \Big\rangle$ leads to sufficient conditions for the asymptotic stability at E_0,

$$\Big\langle \frac{1}{M_s^{per}} \Big\rangle = \frac{(2\cosh(\mu_{M_I})\tau)-1}{\mu_{M_I}\tau^2 b} < e\gamma\max\Big\{\frac{\beta}{\mathcal{N}_F}+(\xi-1), \frac{\beta}{\mathcal{N}_M}\Big\}$$

$$b \geqslant \frac{2\cosh(\mu_{M_I}\tau)-1}{\mu_{M_I}\tau^2(e\gamma)} \min\left\{\frac{\beta}{\mathcal{N}_F}+\xi-1, \frac{\beta}{\mathcal{N}_M}\right\}.$$

□

Lemma 3.4. *Let* k_1 *and* k_2 *be two real numbers such that* $0 < k_1, k_2 < \frac{1}{\mathcal{N}_F}$. *Then, every solution of* (2.2) *such that* $\frac{M}{M+\beta M_I} \leqslant k_1$ *and* $\frac{M_I}{M+\beta M_I} \leqslant k_2, t \geqslant 0$, *converges exponentially to* E_0.

Proof. By using the assumptions $\frac{M}{M+\beta M_I} \leqslant k_1$ and $\frac{M_I}{M+\beta M_I} \leqslant k_2$, the model (2.2) becomes

$$\dot{M} \leqslant \Big((k_1+(1-\xi)k_2)\alpha\vartheta - \mu_M\Big)M, \qquad \dot{F} \leqslant \Big((k_1+(1-\xi)k_2)(1-\alpha)\vartheta - \mu_F\Big)F.$$

The autonomous linear system

$$\begin{pmatrix} \dot{\mathbb{M}} \\ \dot{\mathbb{F}} \end{pmatrix} = \begin{pmatrix} -\mu_M & (k_1+(1-\xi)k_2)\alpha\vartheta \\ 0 & (k_1+(1-\xi)k_2)(1-\alpha)\vartheta - \mu_F \end{pmatrix} \begin{pmatrix} \mathbb{M} \\ \mathbb{F} \end{pmatrix} \tag{3.5}$$

is monotone ([27]). It is used as a method of comparison for evolution of (2.2). Then, it can be concluded that

$$0 \leqslant M(t) \leqslant \mathbb{M}(t), \quad 0 \leqslant F(t) \leqslant \mathbb{F}(t), \qquad t \geqslant 0.$$

Here, $(\mathbb{M}, \mathbb{F})$ be the solution of (3.5) attained by same initial values as the solution (M, F) of (2.2). Then, the linear system (3.5) is asymptotically stable if $0 < k_1, k_2 < \frac{1}{\mathcal{N}_F}$, i.e., $(\mathbb{M}, \mathbb{F})$ asymptotically converges to E_0. Based on this, (M, F) also asymptotically converges to E_0. □

Here, we want to verify the condition $\frac{M}{M+\beta M_I} \leqslant k_1$, according to the adequate sterile impulse releases b_n. Before, the value of M_I on $(n\tau, (n+1)\tau]$ derived,

$$M_I(t) = M_I(n\tau^+)e^{-\mu_{M_I}(t-n\tau)} = (\tau b_n + M_I(n\tau))e^{-\mu_{M_I}(t-n\tau)}. \tag{3.6}$$

We impose the stronger condition instead of $\frac{M}{M+\beta M_I} \leqslant k_1$, on $(n\tau, (n+1)\tau]$,

$$\beta M_I(t) \geqslant \Big(\frac{1}{k_1}-1\Big)\mathbb{M}(t), \quad t \geqslant 0, \tag{3.7}$$

where $\mathbb{M}(t)$ means to super solution of $M(t)$.

Lemma 3.5. *The solution of* (3.5) *on* $(n\tau, (n+1)\tau]$ *with initial values* $(\mathbb{M}(n\tau), \mathbb{F}(n\tau)) = (M(n\tau), F(n\tau))$ *is defined by*

$$\begin{pmatrix} \mathbb{M}(n\tau) \\ \mathbb{F}(n\tau) \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \\ 0 & q_3 \end{pmatrix} \begin{pmatrix} M(n\tau) \\ F(n\tau) \end{pmatrix}, \tag{3.8}$$

where $q_1 = e^{-\mu_M(t-n\tau)}$, $q_2 = \frac{(k_1+(1-\xi)k_2)\alpha\vartheta}{\mu_M-\mu_F+(k_1+(1-\xi)k_2)(1-\alpha)\vartheta} e^{-\Big(\mu_F-(1-\alpha)\vartheta(k_1+(1-\xi)k_2)\Big)(t-n\tau)} - e^{-\mu_M(t-n\tau)}$, $q_3 = e^{-\Big(\mu_F-(1-\alpha)\vartheta(k_1+(1-\xi)k_2)\Big)(t-n\tau)}$.

Here, the feedback control analysis is defined as, on any $(n\tau, (n+1)\tau]$, substituting the values of (3.6) and (3.8) into (3.7), thus

$$\begin{aligned} \beta M_I(t) &\geqslant \Big(\frac{1}{k_1}-1\Big)\mathbb{M}(t), \beta\Big(b_n\tau + M_I(n\tau)\Big)e^{-\mu_{M_I}(t-n\tau)} \\ &\geqslant \frac{1-k_1}{k_1}\Big(e^{-\mu_M(t-n\tau)}M(n\tau) + \Big\{\frac{(k_1+(1-\xi)k_2)\alpha\vartheta}{\mu_M-\mu_F+(k_1+(1-\xi)k_2)(1-\alpha)\vartheta} \end{aligned}$$

$$\times e^{-\left(\mu_F-(1-\alpha)\vartheta(k_1+(1-\xi)k_2)\right)(t-n\tau)} - e^{-\mu_M}(t-n\tau)\Big\}F(n\tau)\Big),$$

$$b_n\tau \geqslant -M_I(n\tau) + \frac{1-k_1}{\beta k_1}e^{(\mu_{M_s}-\mu_M)s}\Big(M(n\tau)+F(n\tau)$$
$$\times \frac{(k_1+(1-\xi)k_2)\alpha\vartheta}{\mu_M-\mu_F+(k_1+(1-\xi)k_2)(1-\alpha)\vartheta}e^{\mu_M-\mu_F+(1-\alpha)\vartheta(k_1+(1-\xi)k_2)s-1}\Big), \quad s\in[0,\tau].$$

Theorem 3.6. *For a given* $k_1 \in \left(0, \frac{1}{\mathcal{N}_{F_u}}\right)$*, assume, for* $n \in \mathbb{N}$*,*

$$\tau b_n \geqslant \left|\Gamma\begin{pmatrix} M(n\tau) \\ F(n\tau)\end{pmatrix} - M_I(n\tau)\right|_+,$$
$$\Gamma = \begin{pmatrix} \frac{1-k_1}{\beta k_1}e^{(\mu_{M_I}-\mu_M)\tau} \\ \frac{(1-k_1)}{\beta k_1}\frac{(k_1+(1-\xi)k_2)\alpha\vartheta}{\mu_M-\mu_F+(k_1+(1-\xi)k_2)(1-\alpha)\vartheta}(e^{(\mu_{M_I}-(\mu_F+(1-\alpha)\vartheta(k_1+(1-\xi)k_2))\tau} - e^{(\mu_{M_I}-\mu_M)\tau}) \end{pmatrix}^T. \tag{3.9}$$

Then, every solution of (3.3) *exponentially converges to* E_0 *with a rate of convergence restricted from below by a value unrelated to the initial condition. Furthermore,*

$$\tau b_n \leqslant \Gamma\begin{pmatrix} M(n\tau) \\ F(n\tau)\end{pmatrix},$$

then the series $\sum_{n=0}^{+\infty} b_n$ *also converges.*

The proof of the above theorem is similar to that of Theorem 6 in [2] and Theorem 3 in [24], hence it is omitted. Here, $|x|_+ := \max\{0, x\}$ defines the non-negative part of the real number x. The row vector Γ described in below the equation (3.9) has positive components.

4. Numerical discussions

The numerical simulation split into two parts. For both the initial condition and parameter values are based on [31]. The parameters are as follows: $\alpha = 0.5$, $\beta = 1$, $\gamma = 3.57 * 10^{-4}$, $\vartheta = 4.55$, $\mu_M = 0.04$, $\mu_F = 0.03$, $\mu_{M_I} = 0.084$, $\xi = 0.45$. Define

$$M^* := \frac{\mathcal{N}_M}{\mathcal{N}_F+\mathcal{N}_M}\frac{1}{\gamma}\ln\mathcal{N}_F \quad \text{and} \quad F^* := \frac{\mathcal{N}_F}{\mathcal{N}_F+\mathcal{N}_M}\frac{1}{\gamma}\ln\mathcal{N}_F.$$

Let $\mathcal{N}_M = 56.87$ and $\mathcal{N}_F = 75.83$. The initial values for the model (2.2) are taken as $M(0) = M^*$, $F(0) = F^*$, $M_I(0) = 1$.

4.1. Optimal control

The optimality system (2.2) described in the above section has to be solved numerically in addition to the conditions for time optimality for a free terminal time. The scaling factors for all quantities included in the objective functional (3.1) are $B_i, i = 1, 2, 3, 4$. Here, we use the subsequent scaling for $B_i, i = 1, 2, 3, 4$;

$$B_1 = \frac{\mathbb{P}_1}{\mathbb{F}}, \quad B_2 = \frac{\mathbb{P}_2}{\mathbb{F}}, \quad B_3 = \frac{\mathbb{P}_3}{365}, \quad B_4 = \frac{\mathbb{P}_4}{u_{max}}.$$

We are most concerned with eliminating wild Lugens in the shortest time period. So, the highest value given to $\mathbb{P}_1$, in order to achieve our primary objective. The rearing (unit) cost of one sterile male might be given the lowest priority, which is denoted by $\mathbb{P}_4$ in the priority ranking system. Having a high elimination intensity (measured by $\mathbb{P}_2$ in the objective function (3.1)) will allow us to keep eliminating Lugens even as $F(t)$ decreases. The part of minimizing time $\mathbb{P}_3$ must be explored, additional considerations are the

following options: i) $\mathbb{P}_3 = 0$; ii) $\mathbb{P}_4 < \mathbb{P}_3 < \mathbb{P}_2$; iii) $\mathbb{P}_3 > \mathbb{P}_2$. Taking all of the above into account, we can find the coefficients as $\mathbb{P}_1 = 10^{10}$, $\mathbb{P}_2 = 10^4$, $\mathbb{P}_3 \in \{0, 10^3, 10^5\}$, $\mathbb{P}_4 = 1$. The value of u^* needs to be increased such that it is higher than $b_n = 1.4 * 10^3$. Also we should prevent the large releases of u^* and keep the total control intervention cost within reasonable bounds. u^* needs to be sufficiently large in order to facilitate a more rapid reduction in the numbers of wild Lugens. For this reason, in all subsequent simulations, we will use a maximum release capacity of $u^* = 2.5 * 10^3$. Figures 1-6 represent the numerical results of the model (2.2) using three different sets of coefficients.

Figures 1, 3, and 5 show with and without optimal control $u^*(t), t \in [0, T]$ of the model (2.2) with three scenarios: $\mathbb{P}_3 = 0, 10^3, 10^5$, respectively. Although it displays a similar basic structure, their tangible implementation may be feasible due to the practicality of continuous time releases.

According to Figures 2, 4, and 6, the left side shows the optimal release rate, while the right side shows the daily changes in release rate, a technical variable used to quantitatively explain control actions. w-Stri carriers should increase rapidly during the first 25 days after release, and then gradually decrease over the next 250 days. These instances will undoubtedly be dominated by w-Stri-infected Lugens populations by the end of this period. Furthermore, all optimal release programs have similar structures, which naturally illustrates their robustness. Nevertheless, as $\mathbb{P}_3$ rises, the total numbers of sterile insects required for the implementation of the program's increases, while the control intervention duration gradually decreases. In making decisions, this information tells us that there is a trade-off between the elimination effort's overall cost and time duration.

A plot of the optimal state for wild male, female Lugens, and male Lugens infected with w-Stri can be found on the right side of Figures 2, 4, and 6. We can see that when $u^*(t)$ is applied, the number of wild male and female Lugens keeps going down as $t \to T^*$ goes on, and both wild populations will finally go extinct.

A more practical release program based on sterile males impulsive releases will be discussed in the next subsection.

4.2. *Impulsive system*

Here, we propose a numerical solution of impulsive release of males Lugens infected with w-Stri (3.3). Using the following parameters, we can get the numerical trajectories of the model (3.3) as $\alpha = 0.074$, $\beta = 0.5$, $\gamma = 3.57 * 10^{-1}$, $\vartheta = 0.0005$, $\mu_M = 0.004$, $\mu_F = 0.003$, $\mu_{M_I} = 0.014$, $\xi = 0.65$, $\mathcal{N}_M = 56.87$, $\mathcal{N}_F = 75.83$. The initial values for the model 3.3 are follows: $M(0) = M^*$, $F(0) = F^*$, $M_I(0) = 1$.

The quantity of male Lugens infected with w-Stri released can be lowered by using the closed-loop technique. Reducing the overall numbers of males Lugens infected with w-Stri released is demonstrated in Theorem 3.6. In the examination of feedback method, we take the wild population into account every τ days. To illustrate the trade off between treatment duration and control effort, we additionally consider the values of k_1. Convergence to E_0 occurs more quickly and with more control effort for smaller values of k_1. Figure 7 displays diagrams indicating that the wild population is nearly extinct with the help of SIT treatment for $k_1\mathcal{N}_F = 0.3, \tau = 7$, and $k_1\mathcal{N}_F = 0.3, \tau = 14$. For larger values of k_1, the control effort is reduced, and convergence should be delayed. The n^{th} release's size, b_n is equal to the value on the right side of (3.9). It is evident that k_1 and τ significantly affect the convergence of the Lugens population to E_0.

Remark 4.1. We present the continuous (optimal control) and periodic (feedback control) releases of male Lugens infected with w-Stri in Figures 1-6 and 7 . Due to the difficulty of implementing continuous time releases, their tangible realization is likely to become impossible inevitably. Feedback control begins with frequent releases, which gradually decrease in intensity as the wild Lugens population decreases. When the system is free of wild Lugens, the releases cease completely. A closed-loop control approach requires evaluating the current magnitude of the untamed population, such as through MRR experiments [12].

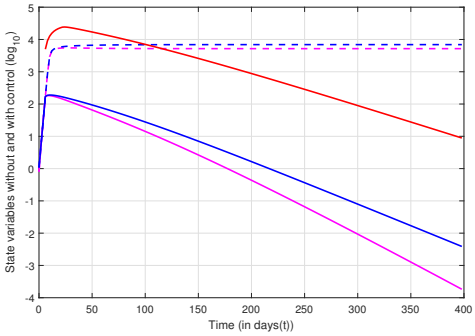

Figure 1: Numerical trajectories of the state variables with $\mathbb{P}_3 = 0$. Here purple line indicates controlled state $M(t)$, purple dot is state $M(t)$, blue line describes controlled state $F(t)$, dot line is state $F(t)$, and red line is state $M_I(t)$.

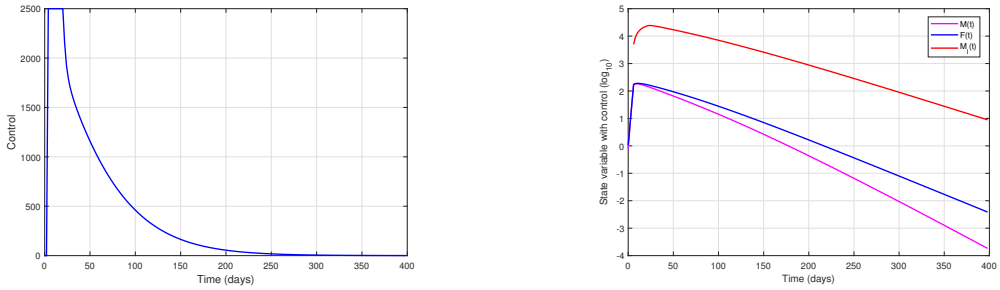

Figure 2: Optimal trajectory of the state variables with $\mathbb{P}_3 = 0$; control trajectory (left) and controlled state trajectories (right).

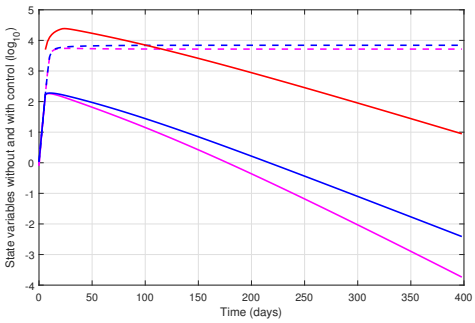

Figure 3: Optimal trajectories of the state variables with $\mathbb{P}_3 = 10^3$. Here purple line indicates controlled state $M(t)$, purple dot is state $M(t)$, blue line describes controlled state $F(t)$, dot line is state $F(t)$, and red line is state $M_I(t)$.

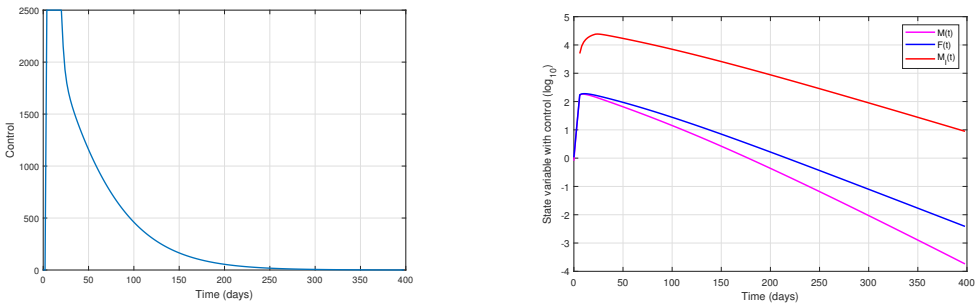

Figure 4: Optimal trajectory of the state variables with $\mathbb{P}_3 = 10^3$. Control trajectory (left) and controlled state trajectories (right).

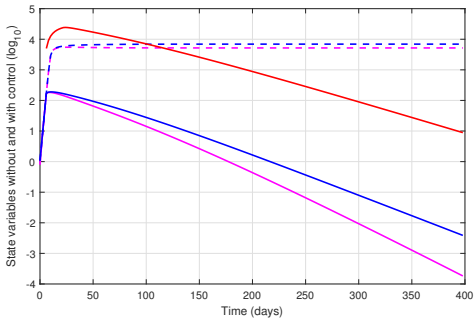

Figure 5: Optimal trajectories of the state variables with $\mathbb{P}_3 = 10^5$. Here purple line indicates controlled state $M(t)$, purple dot states $M(t)$, blue line describes controlled state $F(t)$, dot line is state $F(t)$, and red line is state $M_I(t)$.

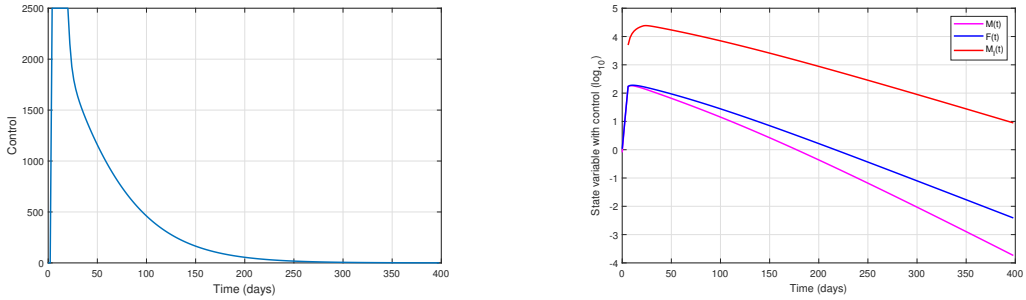

Figure 6: Optimal trajectory of the state variables with $\mathbb{P}_3 = 10^5$. Control trajectory (left) and controlled state trajectories (right).

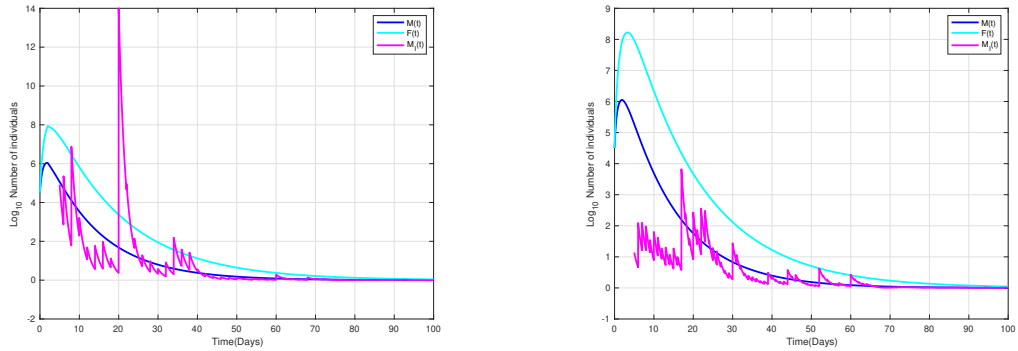

Figure 7: Periodic impulsive system of (3.3) with $k_1\mathcal{N}_F = 0.3, \tau = 7$ days (left) and $k_1\mathcal{N}_F = 0.3, \tau = 14$ days (right).

5. Concluding remarks

In this study, w-Stri-infected male Lugens and sex-structured wild Lugens were examined. Wild Lugens populations were reduced using a model based on the releasing size of male Lugens infected with w-Stri as a time-dependent control parameter. An optimal control solution is derived for wild Lugens cases. We first achieve continuous-time release programs through an optimal control strategy. To replace the population in the shortest time possible and with the least effort, we consider SIT techniques. The method we presented allows this Wolbachia strain to be dispersed throughout natural Lugens populations.

Further, we investigated the impulsive periodic release of male Lugens infected with w-Stri into wild Lugens populations, as well as the elimination of wild Lugens populations. To reduce the prevalence of wild Lugens, two control methods are introduced, namely optimal control and impulsive control. These two control methods are economically viable based on numerical simulations of wild Lugens populations. This study proposes a practical method for quickly replacing wild Lugens populations with w-Stri infection.

In the future, we will incorporate a delay factor into the model to gain insight into the spatio-temporal dynamics of Lugens models and sparse measurements.

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